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On the elliptic equation $\Delta u=\phi(x)e^u$ in the plane

Hiroyuki USAMI

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1. Introduction

Recently Ni [4] has considered the elliptic equation

$$\Delta u = \phi(x)e^u, \qquad x \in R^2, \tag{1}$$

where $x=(x_1, x_2)$, $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$, and $\phi: R^2\to(0, \infty)$ is locally Hölder continuous, and presented conditions under which (1) has entire solutions with various orders of growth at infinity. By an entire solution of (1) [or another equation] we mean a function $u\in C^2(R^2)$ which satisfies (1) [or that equation] at every point of R^2.

The purpose of this paper is to obtain conditions guaranteeing the existence of entire solutions which are eventually positive and have logarithmic growth as $|x|=(x_1^2+x_2^2)^{1/2}\to\infty$. Our method is different from that of Ni [4]; we heavily rely on the results and techniques developed by Kawano, Kusano and Naito [2] in the study of the equation

$$\Delta u = \phi(x)u^\gamma, \qquad x \in R^2,$$

where γ is a positive constant.

We note that the equation (1) in higher dimensions has been studied by Kawano [1] and Ni [4].

2. Main result

In what follows we assume that ϕ: $R^2\to(0, \infty)$ is locally Hölder continuous with exponent $\theta\in(0, 1)$, and define the functions ϕ^*, ϕ_*: $[0, \infty)\to(0, \infty)$ by

$$\phi^*(t) = \max_{|x|=t} \phi(x), \qquad \phi_*(t) = \min_{|x|=t} \phi(x).$$

The main result of this paper is the following theorem.

THEOREM 1. *Suppose that there exists a positive constant c such that*

$$\int_0^\infty t^{c+1}\phi^*(t)dt < \infty. \tag{2}$$

Then, equation (1) *has an eventually positive entire solution u such that*

(3) $$k_1 \log|x| \leq u(x) \leq k_2 \log|x|, \quad |x| \geq R,$$

for some positive constants k_1, k_2 *and* R.

The proof of this theorem is done via the following result which asserts that equation (1) has a *positive* entire solution provided the value of the integral in (2) is small enough.

THEOREM 2. *Consider the equation*

(4) $$\Delta u = \lambda\phi(x)e^u, \quad x \in R^2$$

where λ *is a positive constant. If* (2) *holds for some* $c>0$ *and if* λ *is sufficiently small, then* (4) *has an entire solution* u *which is positive throughout* R^2 *and satisfies* (3) *for some positive constants* k_1, k_2 *and* R.

PROOF OF THEOREM 2. We show that there exists a constant $\lambda>0$ and positive functions v, $w \in C^{2+\theta}_{loc}(R^2)$ such that

(5) $$\Delta v \leq \lambda\phi(x)e^v, \qquad \Delta w \leq \lambda\phi(x)e^w,$$

and $w \leq v$ in R^2, with the additional requirement that v and w have logarithmic growth as $|x| \to \infty$. Then, the existence of an entire solution u lying between v and w follows from Theorem 2.10 of Ni [3].

We wish to construct v and w as solutions of the equations

(6) $$\Delta u = \lambda\phi_*(|x|)v^{1/2}, \quad x \in R^2,$$

and

(7) $$\Delta w = \lambda\phi^*(|x|)e^w, \quad x \in R^2,$$

respectively. It is easy to see that such v and w satisfy (5) in R^2. Furthermore we require that v and w depend only on $|x|$: $v(x)=y(|x|)$, $w(x)=z(|x|)$. We then have the following one-dimensional initial value problems for $y(t)$ and $z(t)$:

(8) $$\begin{cases} y'' + \dfrac{1}{t}y' = \lambda\phi_*(t)y^{1/2}, & t > 0, \\ y(0) = \eta, \quad y'(0) = 0, \end{cases}$$

(9) $$\begin{cases} z'' + \dfrac{1}{t}z' = \lambda\phi^*(t)e^z, & t > 0, \\ z(0) = \zeta, \quad z'(0) = 0, \end{cases}$$

where $'=d/dt$, and η and ζ are positive constants.

In order to solve (9) we transform it into the equivalent integral equation

$$z(t) = \zeta + \lambda \int_0^t s \log (t/s) \cdot \phi^*(s) e^{z(s)} ds, \quad t \geq 0. \tag{10}$$

Define the functions k, ℓ: $[0, \infty) \to (0, \infty)$ by

$$k(t) = 1 \text{ for } 0 \leq t \leq 1, \quad k(t) = t \text{ for } t \geq 1,$$

$$\ell(t) = 1 \text{ for } 0 \leq t \leq e, \quad \ell(t) = \log t \text{ for } t \geq e.$$

Choose $\zeta \in (0, c/2]$, define the set Z by

$$Z = \{z \in C[0, \infty);\ \zeta \leq z(t) \leq 2\zeta \ell(t) \text{ for } t \geq 0\},$$

and consider the mapping F: $Z \to C[0, \infty)$ defined by

$$Fz(t) = \zeta + \lambda \int_0^t s \log (t/s) \cdot \phi^*(s) e^{z(s)} ds, \qquad t \geq 0.$$

Finally let λ be small enough so that

$$\lambda \int_0^\infty k(t) \phi^*(t) e^{2\zeta \ell(t)} dt \leq \zeta/2.$$

Then proceeding as in the proof of Theorem 1 of [2], it is shown that F is continuous and maps Z into a compact subset of Z, so that the Schauder-Tychonoff fixed point theorem implies that F has a fixed point z in Z. This fixed point z is a solution of (10) [hence of (9)], and so the function $w(x) = z(|x|)$ satisfies (7) in R^2. It is clear that $w(x)$ has logarithmic growth as $|x| \to \infty$.

We now turn to equation (8) with λ chosen as above. Since condition (2) implies that $\int_1^\infty t(\log t)^{1/2} \phi^*(t) dt < \infty$, from the proof of Theorem 1 of [2] we see that (8) has a positive solution $y(t)$ with logarithmic growth provided η is sufficiently large. The function $v(x) = y(|x|)$ then gives a solution of (6) in R^2. We require additionally that η be so large that

$$\eta > \lambda \eta^{1/2} \int_0^e t \phi_*(t) dt > 2\zeta.$$

Then, it follows that with this choice of λ, η and ζ the functions v and w satisfy $w \leq v$ in R^2 (see the proof of Theorem 1 of [2] again), and so the functions v and w have all the required properties. This completes the proof.

We note that Theorem 2 allows a slight extension as follows.

THEOREM 3. *Consider the equation*

$$\Delta u = \lambda \phi(x) e^u + \mu \psi(x), \qquad x \in R^2, \tag{11}$$

where ϕ, ψ: $R^2 \to (0, \infty)$ *are locally Hölder continuous (with exponent* $\theta \in (0, 1)$*)*

and λ, μ are positive constants. Suppose that (2) *holds for some $c>0$ and*

$$\int_0^\infty t\psi^*(t)dt < \infty,$$

where $\psi^*(t)=\max_{|x|=t}\psi(x)$. Then equation (11) *has a positive entire solution with logarithmic growth as $|x|\to\infty$ provided λ and μ are sufficiently small.*

PROOF OF THEOREM 1. Choose a constant $\lambda>0$ so that equation (4) has a positive entire solution $\tilde{u}$ satisfying (3) for some k_1, k_2 and R. For this $\lambda>0$ there exist positive constants C_1 and C_2 large enough so that $\lambda e^{-C_1}\leq 1$ and $\lambda e^{C_2}\geq 1$. Define the functions V, $W\in C^{2+\theta}_{loc}(R^2)$ by

$$V(x) = \tilde{u}(x) + C_1, \qquad W(x) = \tilde{u}(x) - C_2.$$

Then we have

$$\Delta V = \lambda e^{-C_1}\phi(x)e^V \leq \phi(x)e^V,$$
$$\Delta W = \lambda e^{C_2}\phi(x)e^W \geq \phi(x)e^W$$

in R^2. Since $W\leq V$ in R^2, from Theorem 2.10 of [3] we conclude that there exists an entire solution u of (1) squeezed between W and V. It is obvious that this solution has the required asymptotic property. This completes the proof.

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Department of Mathematics,
Faculty of Science,
Hiroshima University