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Quasi-artinian Lie algebras

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Introduction

The concept of 'quasi-artinian Lie algebras' was recently introduced by Aldosray in [1]. The class of all quasi-artinian Lie algebras contains $\mathrm{E}\mathfrak{A} \cup \text{Min-}\lhd$. We construct a quasi-artinian Lie algebra which does not belong to $\mathrm{E}\mathfrak{A} \cup \text{Min-}\lhd$ (Theorem 4.1). The existence of such a Lie algebra and a question left unanswered in [1] motivate us to rise the following problems:

1. Give a condition under which quasi-artinian Lie algebras are soluble.
2. Give a condition under which quasi-artinian Lie algebras satisfy the minimal condition for ideals.
3. Does every semisimple quasi-artinian Lie algebra satisfy the minimal condition for ideals?

The aim of this paper is to give answers to the above problems. In Section 2 we shall prove that every residually (ω)-central quasi-artinian Lie algebra is soluble (Theorem 2.3). This result is a generalization of Theorem 3.3 in [1]. The main result of Section 3 is that a quasi-artinian Lie algebra L satisfies the minimal condition for ideals if and only if L belongs to the largest Q-closed subclass of $\text{Min-}\lhd\,\mathfrak{A}$ (Theorem 3.3). In Section 4 we shall construct a Lie algebra by which we can give a negative answer to the third problem stated above (Corollary 4.2).

1.

Notations and terminology are based on Amayo and Stewart [3], and some of the notions used in this paper are found in [1] and [2]. But for the sake of convenience we list the terms that we use here.

Lie algebras will be of arbitrary dimension. For a Lie algebra L and an ordinal α, $L^{(\alpha)}$ and $\zeta_\alpha(L)$ denote the α-th terms of the (transfinite) derived and upper central series of L respectively. These are inductively defined by $L=L^{(0)}$, $L^{(\alpha+1)}=[L^{(\alpha)}, L^{(\alpha)}]$ and $L^{(\rho)}=\bigcap_{\alpha<\rho} L^{(\alpha)}$ for limit ordinals ρ; $\zeta_0(L)=0$, $\zeta_1(L)=$the center of L, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L)=\zeta_1(L/\zeta_\alpha(L))$ and $\zeta_\rho(L)=\bigcup_{\alpha<\rho}\zeta_\alpha(L)$ for limit ordinals ρ. The hypercenter $\zeta_*(L)$ of L is $\bigcup_\alpha \zeta_\alpha(L)$. For a subalgebra H of L the ideal closure series $(H_i)_{i\in N}$ is defined recursively by $H_0=L$, $H_{i+1}=\langle H^{H_i}\rangle$.

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A class $\mathfrak{X}$ is a collection of Lie algebras together with their isomorphic copies and the 0-dimensional Lie algebra. We will need the classes

$$\mathfrak{A},\ \mathrm{L}\mathfrak{N},\ \mathfrak{A}^d,\ \mathrm{E}\mathfrak{A},\ \mathfrak{Z},\ \text{Min-}\lhd,\ \text{Min-}\lhd\,\mathrm{E}\mathfrak{A},\ \text{Min-}\lhd\,\mathfrak{A},\ \text{Max-}\lhd,\ \text{Max-}\lhd^2,\ \mathrm{sE}\mathfrak{A}\text{-Fin},\ \mathrm{s}\mathfrak{A}\text{-Fin},\ \mathfrak{R}$$

($d \in N$) defined by: $L \in \mathfrak{A}$, $\mathrm{L}\mathfrak{N}$ if L is respectively abelian, locally nilpotent. $L \in \mathfrak{A}^d$ if $L^{(d)}=0$. $L \in \mathrm{E}\mathfrak{A}$ if $L \in \mathfrak{A}^d$ for some $d<\omega$. $L \in \mathfrak{Z}$ if $\zeta_*(L)=L$. $L \in \text{Min-}\lhd$, $\text{Min-}\lhd\,\mathrm{E}\mathfrak{A}$, $\text{Min-}\lhd\,\mathfrak{A}$ if L has the minimal condition on ideals, soluble ideals, abelian ideals respectively. Here we note that $\text{Min-}\lhd\,\mathrm{E}\mathfrak{A} \subsetneq \text{Min-}\lhd\,\mathfrak{A}$ ([4], [5]). $L \in \text{Max-}\lhd$, $\text{Max-}\lhd^2$ if L has the maximal condition on ideals, 2-step subideals respectively. $L \in \mathrm{sE}\mathfrak{A}\text{-Fin}$ (resp. $\mathrm{s}\mathfrak{A}\text{-Fin}$) if every soluble (resp. abelian) subideal of L is finite-dimensional. It is known that $\mathrm{sE}\mathfrak{A}\text{-Fin}=\mathrm{s}\mathfrak{A}\text{-Fin}$ ([3, Corollary 9.2.2]). $L \in \mathfrak{R}$ if $x \in L$ implies $x=0$ or $x \notin [x, L]^L$. We note here that $\mathrm{L}\mathfrak{N} \subseteq \mathfrak{R}$ ([2]).

For a class $\mathfrak{X}$ of Lie algebras the classes

$$\mathrm{s}\mathfrak{X},\quad \mathrm{I}\mathfrak{X},\quad \mathrm{E}\mathfrak{X},\quad \mathrm{R}\mathfrak{X},\quad \mathrm{Q}\mathfrak{X}$$

are defined as follows. $L \in \mathrm{s}\mathfrak{X}$ (resp. $\mathrm{I}\mathfrak{X}$) if L is a subalgebra (resp. subideal) of a Lie algebra in $\mathfrak{X}$. $L \in \mathrm{E}\mathfrak{X}$ if L has a finite series $0=L_0 \lhd \cdots \lhd L_n=L$ whose factors $L_{i+1}/L_i \in \mathfrak{X}$ for $0 \leq i \leq n-1$. $L \in \mathrm{R}\mathfrak{X}$ if L has a family $(I_\alpha)_{\alpha \in A}$ of ideals such that $L/I_\alpha \in \mathfrak{X}$ for all α and $\bigcap_{\alpha \in A} I_\alpha=0$. $L \in \mathrm{Q}\mathfrak{X}$ if L is a homomorphic image of a Lie algebra in $\mathfrak{X}$. We say that a class $\mathfrak{X}$ of Lie algebras is E-closed, I-closed, Q-closed, R-closed if $\mathrm{E}\mathfrak{X}=\mathfrak{X}$, $\mathrm{I}\mathfrak{X}=\mathfrak{X}$, $\mathrm{Q}\mathfrak{X}=\mathfrak{X}$, $\mathrm{R}\mathfrak{X}=\mathfrak{X}$ respectively.

A Lie algebra L is said to be semisimple if the sum of all soluble ideals of L is zero.

Let L be a Lie algebra over any field. L is said to be quasi-artinian if for every descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of L there exists a positive integer r such that $[L^{(r)}, I_r] \subseteq \bigcap_{i \geq 1} I_i$. We denote by qmin-$\lhd$ the class of all quasi-artinian Lie algebras. The following proposition is a characterization of the quasiartinian Lie algebras.

PROPOSITION 1.1. *Let L be a Lie algebra over any field. Then $L \in$ qmin-$\lhd$ if and only if there exists an ideal I of L such that $L/I \in \mathrm{E}\mathfrak{A}$ and the set $\{[K, I]: K \lhd L\}$ satisfies the minimal condition.*

PROOF. Let $L \in$ qmin-$\lhd$ and put $I=L^{(\omega)}$. Then $I=L^{(n)}$ for some $n<\omega$ and $L/I \in \mathrm{E}\mathfrak{A}$. Let $K_i \lhd L$ ($i \geq 1$) and suppose that $[K_1, I] \supseteq [K_2, I] \supseteq \cdots$. Since $L \in$ qmin-$\lhd$, there exists an integer $n \geq 1$ such that $[L^{(n)}, [K_n, I]] \subseteq \bigcap_{i \geq 1} [K_i, I]$. Observing $I^{(1)}=I$ we have $[K_n, I] \subseteq [[K_n, I], I]=[[K_n, I], L^{(n)}] \subseteq [K_{n+j}, I]$ for any $j \geq 0$.

Conversely take an ideal I of L such that $L/I \in \mathrm{E}\mathfrak{A}$ and the set $\{[K, I]: K \lhd L\}$

satisfies the minimal condition. Then $L^{(n)} \subseteq I$ for some $n < \omega$. Let $K_i \lhd L\,(i \geq 1)$ and suppose that $K_1 \supseteq K_2 \supseteq \cdots$. Then $\{[K_i, I]: i \geq 1\}$ has the minimal element $[K_m, I]$. Putting $r = \max\{m, n\}$ we have $[K_r, L^{(r)}] \subseteq [K_m, I] \subseteq \bigcap_{i \geq 1} [K_i, I] \subseteq \bigcap_{i \geq 1} K_i$.

For an integer $n \geq 1$ we put $\boldsymbol{Q}(n) = \{p/q \in \boldsymbol{Q}: q > n,\ p$ and q are relatively prime$\} + \boldsymbol{Z}$. Let V be a vector space over a field F of characteristic zero with basis $\{v(a): a \in \boldsymbol{Q}\}$. Considered as an abelian Lie algebra V has derivations $x: v(a) \mapsto v(a+1)$, $y: v(a) \mapsto a(a-1)v(a-1)$, $z: v(a) \mapsto 2av(a)$. Let L be the split extension $V \dotplus \langle x, y, z \rangle$.

Let $V_n = \sum_{a \in \boldsymbol{Q}(n)} Fv(a)$. Then $V_1 \supsetneqq V_2 \supsetneqq \cdots$. Since $\boldsymbol{Q}(n) + \boldsymbol{Z} = \boldsymbol{Q}(n)$, we have $[V_n, L] \subseteq V_n$. On the other hand, take an element a in $\boldsymbol{Q}(n)$. Then $v(a) = [v(a-1), x] \in [V_n, L]$. Hence we have $[V_n, L] = V_n$. Since $L^{(\omega)} = L$, for any integer $m \geq 1$ a descending chain $[V_1, L^{(m)}] \supsetneqq [V_2, L^{(m)}] \supsetneqq \cdots$ does not terminate finitely. Therefore by [1, Theorem 3.1], $L \notin$ qmin-$\lhd$.

Observing that V, $L/V \in$ qmin-$\lhd$, we have the following

Proposition 1.2. *Over any field of characteristic zero* qmin-$\lhd$ *is not* E*-closed.*

2.

In this section we shall give classes $\mathfrak{X}$ of Lie algebras such that qmin-$\lhd \cap \mathfrak{X} =$ E$\mathfrak{A}$.

To begin with, we generalize the notion of residually central Lie algebras. We say that a Lie algebra L is *residually* (ω)*-central* if $x \in L$ implies $x = 0$ or $x \notin [x, L^{(\omega)}]^L$, and denote by $\mathfrak{R}_{(\infty)}$ the class of all residually (ω)-central Lie algebras.

Lemma 2.1. (1) s$\mathfrak{R}_{(\infty)} =$ R$\mathfrak{R}_{(\infty)} = \mathfrak{R}_{(\infty)}$.

(2) RE$\mathfrak{A} \cup \mathfrak{R} \subseteq \mathfrak{R}_{(\infty)}$.

Proof. (1) Clearly s$\mathfrak{R}_{(\infty)} = \mathfrak{R}_{(\infty)}$. Let $x \in L \in$ R$\mathfrak{R}_{(\infty)}$ with $x \neq 0$ and take an ideal I of L such that $x \notin I$ and $L/I \in \mathfrak{R}_{(\infty)}$. Then $x + I \notin [x+I, (L/I)^{(\omega)}]^{L/I}$, so $x \notin [x, L^{(\omega)}]^L$. Therefore $L \in \mathfrak{R}_{(\infty)}$.

(2) If $L \in$ RE$\mathfrak{A}$, then $L^{(\omega)} = 0$ and so $L \in \mathfrak{R}_{(\infty)}$. It is obvious that $\mathfrak{R} \subseteq \mathfrak{R}_{(\infty)}$.

Lemma 2.2. *Let* $I \lhd L \in \mathfrak{R}_{(\infty)}$, $x \in L$. *Then for any ordinal* α, $x \notin \zeta_\alpha(I)$ *if and only if* $x \notin [x, I^{(\omega)}]^L + \zeta_\alpha(I)$.

Proof. We use transfinite induction on α. It is trivial for $\alpha = 0$. Let $\alpha > 0$ and assume that the result is true for any ordinal $\beta < \alpha$. Let $x \notin \zeta_\alpha(I)$ and assume that $x \in [x, I^{(\omega)}]^L + \zeta_\alpha(I)$. If α is limit, then there exists an ordinal $\beta < \alpha$ such that

$x\in[x, I^{(\omega)}]^L+\zeta_\beta(I)$. By induction hypothesis we have $x\in\zeta_\beta(I)\subseteq\zeta_\alpha(I)$, which is a contradiction. In the case that α is non-limit, take elements $y\in[x, I^{(\omega)}]^L$ and $z\in\zeta_\alpha(I)$ such that $x=y+z$. Then $[z, I^{(\omega)}]^L\subseteq\zeta_{\alpha-1}(I)$. Hence we have $y\in[y, I^{(\omega)}]^L+[z, I^{(\omega)}]^L\subseteq[y, I^{(\omega)}]^L+\zeta_{\alpha-1}(I)$. By induction hypothesis $y\in\zeta_{\alpha-1}(I)\subseteq\zeta_\alpha(I)$ and $x=y+z\in\zeta_\alpha(I)$, which is a contradiction.

We now set about showing the main theorem in this section.

THEOREM 2.3. qmin-$\lhd\cap\mathfrak{X}=\mathrm{E}\mathfrak{A}$ *for any class* $\mathfrak{X}$ *of Lie algebras such that* $\mathrm{E}\mathfrak{A}\subseteq\mathfrak{X}\subseteq\mathfrak{N}_{(\infty)}$.

PROOF. By Lemma 2.1(2) we have $\mathrm{E}\mathfrak{A}\subseteq$ qmin-$\lhd\cap\mathfrak{N}_{(\infty)}$. Assume that $\mathrm{E}\mathfrak{A}\subsetneq$ qmin-$\lhd\cap\mathfrak{N}_{(\infty)}$ and take a Lie algebra $L\in$ qmin-$\lhd\cap\mathfrak{N}_{(\infty)}$ with $L\notin\mathrm{E}\mathfrak{A}$. Put $I=L^{(\omega)}$. Since $L\in$ qmin-$\lhd$, $I=L^{(d)}$ for some $d<\omega$ and $I^{(\beta)}=I\neq 0$ for any ordinal β. Then by [3, Lemma 8.1.1], we have $I\notin\mathfrak{Z}$ and $\zeta_*(I)\subsetneq I$. Take an element $x_1\in I$ with $x_1\notin\zeta_*(I)$. Since $I^{(\omega)}=I$, by Lemma 2.2 we have $x_1\notin[x_1, I]^L+\zeta_*(I)$. Clearly $\zeta_*(I)\subsetneq[x_1, I]^L+\zeta_*(I)$. Next we take an element $x_2\in[x_1, I]^L+\zeta_*(I)$ with $x_2\notin\zeta_*(I)$. Then by using Lemma 2.2 again we have $x_2\notin[x_2, I]^L+\zeta_*(I)$. By continuing this procedure, we can find a sequence $(x_i)_{i=1}^\infty$ of elements of $I\backslash\zeta_*(I)$ such that for any integer $i\geq 1$

$$x_i\notin[x_i, I]^L+\zeta_*(I) \quad\text{and}\quad x_{i+1}\in[x_i, I]^L+\zeta_*(I).$$

Set $I_i=[x_i, I]^L+\zeta_*(I)$. Then $I_i\lhd L$ $(i\geq 1)$ and $I_1\supsetneq I_2\supsetneq\cdots$. Since $L\in$ qmin-$\lhd$, there exists an integer $n\geq 1$ such that $[I_n, L^{(d)}]=[I_n, I]\subseteq I_{n+1}$. Then we have $[x_n, I]^L\subseteq[[x_n, I], I]^L\subseteq[I_n, I]^L\subseteq I_{n+1}$. Therefore $I_n=[x_n, I]^L+\zeta_*(I)\subseteq I_{n+1}\subsetneq I_n$, which is a contradiction.

COROLLARY 2.4. (1) qmin-$\lhd\cap\mathfrak{N}\subseteq\mathrm{E}\mathfrak{A}$.

(2) Min-$\lhd\cap\mathfrak{N}_{(\infty)}\subseteq\mathrm{E}\mathfrak{A}$.

Recalling the fact that $\mathrm{L}\mathfrak{N}\subseteq\mathfrak{N}$, we have qmin-$\lhd\cap\mathrm{L}\mathfrak{N}\subseteq\mathrm{E}\mathfrak{A}$ as an immediate consequence of (1) in the above corollary. This is just Theorem 3.3 in [1]. Another immediate consequence of this corollary is that Min-$\lhd\cap\mathfrak{N}\subseteq\mathrm{E}\mathfrak{A}$. This is a part of Theorem 3.5 in [2].

3.

In this section we shall present classes $\mathfrak{X}$ of Lie algebras such that qmin-$\lhd$ $\cap\,\mathfrak{X}=$ Min-$\lhd$.

For any class $\mathfrak{X}$ of Lie algebras, let

$$\mathfrak{X}^{\mathrm{Q}}$$

denote the largest Q-closed subclass of $\mathfrak{X}$. It is easy to see that for a Lie algebra L, $L \in \mathfrak{X}^{\mathrm{Q}}$ if and only if $I \lhd L$ implies $L/I \in \mathfrak{X}$.

It is obvious that

$$(*) \qquad \text{Min-}\lhd \subseteq (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}} \subseteq (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}.$$

We first consider the first inclusion. Let S be a non-abelian simple Lie algebra over any field. For any integer $i \geq 1$, let S_i be an isomorphic copy of S. Consider a Lie algebra $L = \oplus_{i=1}^{\infty} S_i$. Then $L \notin \text{Min-}\lhd$. Let $I \lhd L$. By [3, Lemma 13.4.3] there exists a subset M of N such that $L/I \simeq \oplus_{i \in M} S_i$. This shows that $I \lhd L$implies $L/I \in \text{Min-}\lhd\ \mathrm{E}\mathfrak{A}$, and that $L \in (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}}$. Therefore we have the following

PROPOSITION 3.1. $\text{Min-}\lhd \subsetneq (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}}$ *over any field.*

For the second inclusion of $(*)$ we have the following

LEMMA 3.2. $(\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}} = (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}$.

PROOF. Let $L \in (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}$. We show that for any $1 \leq d < \omega$, L satisfies the minimal condition for soluble ideals of derived length $\leq d$. As a consequence we will have $L \in \text{Min-}\lhd\ \mathrm{E}\mathfrak{A}$ and $(\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}} \subseteq (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}}$ The assertion is trivial for $d=1$. Assume that the assertion is true for $d-1$. Let $K_1 \supseteq K_2 \supseteq \cdots$ be a descending chain of ideals of L with $K_i \in \mathfrak{A}^d$ for any i. Since $K_i^{(d-1)} \in \mathfrak{A}$ $(i \geq 1)$ and $L \in \text{Min-}\lhd\ \mathfrak{A}$, there exists an integer $n \geq 1$ such that $K_n^{(d-1)} = K_{n+j}^{(d-1)}$ for any $j \geq 0$. Put $K = K_n^{(d-1)}$. Then $L/K \in (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}$ and $K_{n+j}/K \in \mathfrak{A}^{d-1}$ for any $j \geq 0$. By using induction hypothesis the chain $K_n \supseteq K_{n+1} \supseteq \cdots$ terminates finitely.

We now give an answer to the second problem stated in the introduction.

THEOREM 3.3. $\text{qmin-}\lhd \cap \mathfrak{X} = \text{Min-}\lhd$ *for any class* $\mathfrak{X}$ *of Lie algebras such that* $\text{Min-}\lhd \subseteq \mathfrak{X} \subseteq (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}$.

PROOF. By Lemma 3.2 it is enough to show that $\text{qmin-}\lhd \cap (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}} \subseteq \text{Min-}\lhd$. Let $L \in \text{qmin-}\lhd \cap (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}}$ and put $I = L^{(\omega)}$. Then $I = L^{(d)}$ for some $d < \omega$. Let $K_i \lhd L (i \geq 1)$ and suppose that $K_1 \supseteq K_2 \supseteq \cdots$. Since $L \in \text{qmin-}\lhd$, there exists an integer $n \geq 1$ such that $[K_n \cap I, I] \subseteq \bigcap_{i \geq 1} (K_i \cap I)$. Put $K = \bigcap_{i \geq 1} (K_i \cap I)$. Then $(K_{n+j} \cap I)^{(1)} \subseteq [K_n \cap I, I] \subseteq K$ $(j \geq 0)$. Since $K \lhd L \in (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}}$, $L/K \in \text{Min-}\lhd\ \mathrm{E}\mathfrak{A}$. Therefore a descending chain $(K_{n+1} \cap I)/K \supseteq (K_{n+2} \cap I)/K \supseteq \cdots$ of abelian ideals of L/K terminates finitely. On the other hand since $L/I \in \mathrm{E}\mathfrak{A} \cap (\text{Min-}\lhd\ \mathrm{E}\mathfrak{A})^{\mathrm{Q}}$, we have $L/I \in \text{Min-}\lhd$. Hence a descending chain $(K_{n+1}+I)/I \supseteq (K_{n+2}+I)/I \supseteq \cdots$ terminates finitely, and so does the descending chain $K_1 \supseteq K_2 \supseteq \cdots$. This shows that $L \in \text{Min-}\lhd$.

COROLLARY 3.4. $\text{qmin-}\lhd \cap \text{Max-}\lhd^2 \subsetneq \text{Min-}\lhd$.

PROOF. Let $L \in \text{Max-}\lhd^2$. Then every abelian ideal of L is finite-dimensional. Hence $L \in \text{Min-}\lhd\ \mathfrak{A}$. Since $\text{Max-}\lhd^2$ is Q-closed, we have $\text{Max-}\lhd^2 \subseteq (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}$. The result immediately follows from the above theorem.

REMARK. We can not replace $\text{Max-}\lhd^2$ by $\text{Max-}\lhd$ in the above corollary. In fact, let V be a vector space over any field with basis $\{x_i\colon i \geq 1\}$. Considered as an abelian Lie algebra V has a derivation $z\colon x_i \mapsto x_{i+1}$ $(i \geq 1)$. Let L be the split extension $V \dotplus \langle z \rangle$. Then it is easy to see that every non-zero ideal of L has a finite codimension. Therefore $L \in \text{Max-}\lhd$. Since $L \in \mathrm{E}\mathfrak{A} \subseteq \text{qmin-}\lhd$, we have $L \in \text{qmin-}\lhd \cap \text{Max-}\lhd$. But $L \notin \text{Min-}\lhd$.

We state a necessary condition for an I-closed class $\mathfrak{X}$ of Lie algebras to satisfy $\text{qmin-}\lhd \cap \mathfrak{X} \subseteq \text{Min-}\lhd$.

PROPOSITION 3.5. *Let $\mathfrak{X}$ be an* I-*closed class of Lie algebras. If* $\text{qmin-}\lhd \cap \mathfrak{X} \subseteq \text{Min-}\lhd$, *then* $\mathfrak{X} \subseteq \mathrm{sE}\mathfrak{A}\text{-Fin}$.

PROOF. Suppose that $\text{qmin-}\lhd \cap \mathfrak{X} \subseteq \text{Min-}\lhd$. Let $L \in \mathfrak{X}$ and A be an abelian subideal of L. Since $\mathfrak{X}$ is I-closed, $A \in \mathfrak{X} \cap \mathfrak{A} \subseteq \text{Min-}\lhd$, and hence A is finite-dimensional. This shows that $L \in \mathrm{s}\mathfrak{A}\text{-Fin}$. The statement follows from the fact that $\mathrm{sE}\mathfrak{A}\text{-Fin} = \mathrm{s}\mathfrak{A}\text{-Fin}$ ([3, Corollary 9.2.2]).

The converse of this proposition fails. In fact, let L be the Lie algebra constructed in Section 4. Then $L \in \text{qmin-}\lhd \cap \mathrm{sE}\mathfrak{A}\text{-Fin}$ but $L \notin \text{Min-}\lhd$. However we have the following

COROLLARY 3.6. *Let $\mathfrak{X}$ be a class of Lie algebras. If $\mathfrak{X}$ is* I-*closed and* Q-*closed, then the following conditions are equivalent*:

(1) $\text{qmin-}\lhd \cap \mathfrak{X} \subseteq \text{Min-}\lhd$.

(2) $\mathfrak{X} \subseteq \mathrm{sE}\mathfrak{A}\text{-Fin}$.

(3) $\mathfrak{X} \subseteq \text{Min-}\lhd\ \mathfrak{A}$.

PROOF. (1)$\Rightarrow$(2) is the assertion of Proposition 3.5 and (2)$\Rightarrow$(3) is obvious. Assume that the condition (3) holds. Since $\mathrm{Q}\mathfrak{X} = \mathfrak{X} \subseteq \text{Min-}\lhd\ \mathfrak{A}$, $\mathfrak{X} \subseteq (\text{Min-}\lhd\ \mathfrak{A})^{\mathrm{Q}}$. Therefore by Theorem 3.3 the condition (1) holds.

4.

The purpose of this section is to construct a Lie algebra which gives a negative answer to the third problem stated in the introduction. This problem was asked as an open question in [1] and has been left unanswered.

Let K be any field, F the quotient field of polynomial algebra $K[x]$ and $R = \sum_{i \in \mathbf{Z}} Kx^i$. Let W be a Lie algebra over F with basis $\{w(r)\colon r \in R\}$ and multiplication

$$[w(g), w(h)] = (g-h)w(g+h), \quad g, h \in R.$$

Then it is proved in [3, Theorem 10.3.1] that if K has characteristic $\neq 2$ then W is a non-abelian simple Lie algebra and that if K has characteristic 2 then $W^{(1)}$ is a non-abelian simple Lie algebra. For the sake of convenience we set $S=W$ in the case that K has characteristic $\neq 2$, and $S=W^{(1)}$ in the case that K has characteristic 2.

For any integer n we define a homomorphism λ_n of an abelian group R into an abelian group F by $\lambda_n(\sum a_i x^i)=a_n$, and define a derivation δ_n of S by

$$w(r)\delta_n = \lambda_n(r)w(r), \quad w(r) \in S.$$

Let $D=\sum_{n\in \mathbf{Z}} F\delta_n$. Then D is an infinite-dimensional abelian subalgebra of a Lie algebra $\mathrm{Der}_F(S)$. We now let L be the split extension $S \dotplus D$.

We first claim that every non-zero subideal of L contains S. Let H be a non-zero subideal of L and H_i be the i-th ideal closure of H in L. By induction on i we show that $S \subseteq H_i$ $(i \geq 0)$. As a consequence we will have $S \subseteq H$, since $H=H_n$ for some $n<\omega$. It is trivial for $i=0$. Let $i \geq 0$ and assume that $S \subseteq H_i$. Suppose that $[S, H_{i+1}]=0$ and take any element $z=\sum_{i=1}^m a_i w(r_i)+\sum_{j=p}^q b_j\delta_j$ $(a_i, b_j \in F)$ in H_{i+1}. We may assume that $\{w(r_1),\dots, w(r_m)\}$ is contained in the subspace spanned by $\{w(r)\colon r \in \sum_{i<s} Kx^i\}$. Let $t=\max\{s, q+1\}$. Then

$$0 = [S, H_{i+1}] \ni [z, w(x^t)] = \textstyle\sum_{i=1}^m a_i(r_i - x^t)w(r_i+x^t).$$

Hence $a_1=\cdots=a_m=0$. Further $0=[z, w(x^j)] = -b_j w(x^j)\,(p \leq j \leq q)$, and we have $z=0$. This is a contradiction. Hence we have $[S, H_{i+1}]\neq 0$. Since $S \lhd H_i$ and $H_{i+1} \lhd H_i$, $[S, H_{i+1}] \lhd H_i \cap S = S$. By the simplicity of S we have $S=[S, H_{i+1}] \subseteq [H_i, H_{i+1}] \subseteq H_{i+1}$.

We next prove that every soluble subideal of L is zero. Let H be a non-zero soluble subideal of L. Then $S \subseteq H$ by the above result, which contradicts the simplicity of S.

We thirdly prove that $L \in \mathrm{qmin}\text{-}\lhd$. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of non-zero ideals of L. Since S is simple, L/S is abelian and $S \subseteq I_n$ for any $n \geq 1$, we have

$$[L^{(1)}, I_1] \subseteq [S, L] = S \subseteq I_n \quad \text{for any } n \geq 1.$$

Therefore L is quasi-artinian.

Finally $L \notin \mathrm{Min}\text{-}\lhd$ since L/S is infinite-dimensional and abelian.

Summing up these facts we have

THEOREM 4.1. *Over any field there is a Lie algebra L satisfying the following conditions*:

(1) $L \in \mathrm{qmin}\text{-}\lhd$.

(2) *L has no non-zero soluble subideals.*

(3) $L \notin \mathrm{E}\mathfrak{A} \cup \mathrm{Min}\text{-}\lhd$.

COROLLARY 4.2. *Over any field there is a semisimple quasi-artinian Lie algebra which does not satisfy the minimal condition for ideals.*

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