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Removability of polar sets for solutions of semi-linear equations on a harmonic space

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1. Introduction and preliminaries

In the classical potential theory, it is well known that polar sets (or the sets with null capacity) are removable for bounded, as well as Dirichlet-finite, harmonic functions (see., e.g., [1; §VII, Theorem 1]). The purpose of the present paper is to extend this result to solutions of semi-linear equations on harmonic spaces.

Let $(X, \mathscr{U})$ be a harmonic space in the sense of [2] and let $\mathscr{R}$ be the sheaf of functions which are locally expressible as differences of continuous superharmonic functions. We assume that X has a countable base and $1 \in \mathscr{R}(X)$. An open set in X possessing a positive potential is called a P-set (cf. [2]) and a relatively compact open set whose closure is contained in a P-set is called a PC-set (cf. [7], [8]). Let $\mathscr{M}$ be the sheaf of (signed) Randon measures on X and $\sigma: \mathscr{R} \to \mathscr{M}$ be a measure representation (see, [6], [7], [9]). Let $\mathscr{M}_\sigma$ be the image sheaf of σ and consider a sheaf morphism $F: \mathscr{R} \to \mathscr{M}_\sigma$ which satisfies the following two conditions (cf. [8]):

(F.1) (Monotonicity): For any open set U, if $f_1, f_2 \in \mathscr{R}(U)$ and $f_1 \leqq f_2$ on U, then $F(f_1) \leqq F(f_2)$ on U.

(F.2) (Local Lipschitz condition): For any PC-set U and for any $M>0$, there is a non-negative measure $\pi_{M,U}$ on U such that $\sigma(p_{M,U}) = \pi_{M,U}$ for some bounded continuous potential $p_{M,U}$ on U and

$$F(f_1) - F(f_2) \leqq (f_1 - f_2)\pi_{M,U} \quad \text{on} \quad U,$$

whenever $f_1, f_2 \in \mathscr{R}(U)$ and $-M \leqq f_2 \leqq f_1 \leqq M$ on U.

We are concerned with the semi-linear equation

$$\text{(1)} \qquad \sigma(u) + F(u) = 0.$$

For each open set U, let

$$\mathscr{H}_B^F(U) = \{u \in \mathscr{R}(U) \mid u \text{ satisfies (1) and is bounded on } U\},$$

$$\mathscr{H}_D^F(U) = \{u \in \mathscr{R}(U) \mid u \text{ satisfies (1) and } \delta_u(U) < +\infty\}.$$

Here δ_u is the gradient measure of u as defined in [6], [7]. The value $\delta_u(U)$ is regarded as the Dirichlet integral of u over U. Thus $\mathscr{H}_B^F(U)$ is the space of bounded solutions of (1) and $\mathscr{H}_D^F(U)$ that of Dirichlet-finite solutions of (1).

A set e in X is called polar if for any P-set U there is a potential p on U such that $p(x)=+\infty$ for all $x\in e\cap U$.

As to the $\mathscr{H}_B^F$-removability of polar sets, we obtain the following general theorem:

THEOREM 1. *Any closed polar set e in X is $\mathscr{H}_B^F$-removable, i.e., for any open set U and for any $u\in\mathscr{H}_B^F(U\setminus e)$, there is $\tilde{u}\in\mathscr{H}_B^F(U)$ such that $\tilde{u}|_{U\setminus e}=u$.*

This result is quite as expected in view of similar results for solutions of elliptic and parabolic equations on euclidean domains (see e.g., [3; Theorem 3.1], [10; IX, §8, Satz 21] for elliptic equations and [4] for parabolic equations).

In order to discuss $\mathscr{H}_D^F$-removability, we restrict ourselves to the self-adjoint case (cf. Remark 3 in section 6). By definition (cf. [6], [7]), a self-adjoint harmonic space is a Brelot space having a consistent system of symmetric Green functions. To such a system there corresponds a canonical measure representation σ (see [6], [7]). We shall prove

THEOREM 2. *Let X be a self-adjoint harmonic space and consider the equation (1) with respect to a canonical measure representation σ. Let e be a compact polar set contained in a P-set. Then, for any open set U containing e and for any $u\in\mathscr{H}_D^F(U\setminus e)$, there is $\tilde{u}\in\mathscr{H}_D^F(U)$ such that $\tilde{u}|_{U\setminus e}=u$.*

With respect to linear elliptic equations on euclidean domains, we may regard [10; IX, §8, Satz 20] as giving removability of polar sets for Dirichlet-finite solutions; but it seems that no results are known for non-linear equations.

2. Lemmas on polar sets

In this section, let $(X, \mathscr{U})$ be a general harmonic space. For an open set U let R^U denote the reducing operator on U, i.e.,

$$R^U f=\inf\{u \mid \text{hyperharmonic on } U,\ u\geqq f \text{ on } U\}.$$

LEMMA 1. *Let e be a compact polar set contained in a P-set U and let p be a potential on U such that $p(x)=+\infty$ for all $x\in e$. Then, for any $\varepsilon>0$ there is a continuous potential p_ε on U such that $p_\varepsilon\geqq 1$ on a neighborhood of e and $p_\varepsilon\leqq\varepsilon p$ on U.*

PROOF. Let $V_\varepsilon=\{x\in U \mid p(x)>1/\varepsilon\}$. Then V_ε is an open set containing e. Choose a continuous function φ_ε on U such that $\varphi_\varepsilon=1$ on a neighborhood of e, $0\leqq\varphi_\varepsilon\leqq 1$ on U and $\varphi_\varepsilon=0$ on $U\setminus V_\varepsilon$. Put $p_\varepsilon=R^U\varphi_\varepsilon$. Then, by [2; Proposition 2.3.1] (or [7; Propositions 2.6 and 2.7]), we see that p_ε is the required potential.

LEMMA 2. *Let e be a compact polar set contained in a P-set U. Then there exists a potential p on U such that $p(x)=+\infty$ if $x\in e$, $p(x)<+\infty$ if $x\in U\setminus e$ and p is harmonic outside a compact set in U.*

PROOF. By [2; Exercise 6.2.1], we can find a potential $\tilde{p}$ on U such that $\tilde{p}(x)=+\infty$ if $x\in e$ and $\tilde{p}(x)<+\infty$ if $x\in U\setminus e$. Then $p=R^U(\psi\tilde{p})$ serves our purpose, where ψ is a non-negative continuous function on U such that $\psi=1$ on e and has a compact support in U.

3. Proof of THEOREM 1.

Given $u\in\mathscr{H}_B^F(U\setminus e)$, let u^* and u_* be functions on U which are equal to u on $U\setminus e$ and

$$u^*(y) = \limsup_{x\to y, x\in U\setminus e} u(x), \quad u_*(y) = \liminf_{x\to y, x\in U\setminus e} u(x)$$

for $y\in e$. Let V be any PC-set such that $\bar{V}\subset U$. We know that V is resolutive, so that $H_{u^*}^V$ and $H_{u_*}^V$ are defined ([2; Theorem 2.4.2]). Since u^* and u_* differ only on a polar set, we see that $H_{u^*}^V=H_{u_*}^V$ on V (cf. [2; Corollary 6.2.4]). Then $H_u^{F,V}\in\mathscr{H}_B^F(V)$ is defined from $H_{u^*}^V=H_{u_*}^V$ as in [8; p. 476]. By [8; Theorem 4.2] and its proof, we see that $H_u^{F,V}$ is also given by

$$H_u^{F,V} = \inf \bar{\mathscr{F}}_{u^*}^{F,V} = \sup \underline{\mathscr{F}}_{u_*}^{F,V},$$

where

$$\bar{\mathscr{F}}_{u^*}^{F,V} = \{v\in\mathscr{R}(V)\,|\,\sigma(v) + F(v) \geqq 0 \text{ on } V, \liminf_{x\to\xi, x\in V} v(x) \geqq u^*(\xi) \text{ for all } \xi\in\partial V\},$$

$$\underline{\mathscr{F}}_{u_*}^{F,V} = \{w\in\mathscr{R}(V)\,|\,\sigma(w) + F(w) \leqq 0 \text{ on } V, \limsup_{x\to\xi, x\in V} w(x) \leqq u_*(\xi) \text{ for all } \xi\in\partial V\}.$$

We shall show that $u=H_u^{F,V}$ on $V\setminus e$.

By (F.2), there is a bounded continuous potential g on V such that $\sigma(g)=F(M)^- + M\sigma(1)^-$, where $M=\sup_{V\setminus e}|u|$. Put $f=M+g$ and $M'=2M+\sup_V g$. Let U' be a P-set such that $\bar{V}\subset U'\subset U$. By Lemmas 1 and 2, we can find a non-increasing sequence $\{p_n\}$ of continuous potentials on U' such that $p_n\geqq 1$ on a neighborhood W_n of $e\cap\bar{V}$ and $p_n\downarrow 0$ $(n\to\infty)$ on $U'\setminus(e\cap\bar{V})$. Set

$$v_n = \begin{cases} \min(u+M'p_n, f) & \text{on } V\setminus e \\ f & \text{on } V\cap e. \end{cases}$$

Since $u+M'p_n\geqq u+M'\geqq f$ on $W_n\cap V\setminus e$, we see that $v_n\in\mathscr{R}(V)$. Furthermore, since

$$\sigma(f) = M\sigma(1) + \sigma(g) \geqq -M\sigma(1)^- + F(M)^- + M\sigma(1)^-$$
$$\geqq -F(M) \geqq -F(f) \quad \text{on} \quad V$$

and

$$\sigma(u+M'p_n) \geqq \sigma(u) = -F(u) \geqq -F(u+M'p_n) \quad \text{on} \quad V \setminus e,$$

[8; Corollary to Theorem 3.3] implies that $\sigma(v_n)+F(v_n)\geqq 0$ on V. It is easy to see that

$$\liminf_{x\to\xi, x\in V} v_n(\xi) \geqq u^*(\xi) \qquad \text{for all} \quad \xi \in \partial V.$$

Thus, $v_n \in \bar{\mathscr{F}}_{u^*}^{F,V}$, so that $v_n \geqq H_u^{F,V}$. Letting $n\to\infty$, we conclude that $u\geqq H_u^{F,V}$ on $V\setminus e$.

Similarly, letting $\tilde{g}$ be the bounded continous potential on V such that $\sigma(\tilde{g})=F(-M)^+ + M\sigma(1)^-$, $\tilde{f}=-M-\tilde{g}$, $\tilde{M}=2M+\sup_V \tilde{g}$ and considering

$$\tilde{v}_n = \begin{cases} \max(u-\tilde{M}p_n, \tilde{f}) & \text{on} \quad V\setminus e \\ \tilde{f} & \text{on} \quad V\cap e, \end{cases}$$

we can prove that $u\leqq H_u^{F,V}$ on $V\setminus e$. Thus $u=H_u^{F,V}$ on $V\setminus e$. Since $H_u^{F,V}$ is continuous on V, it follows that $u^*=u_*=H_u^{F,V}$ on V. Since PC-sets V with $\bar{V}\subset U$ cover U, $u^*=u_*$ on U and it belongs to $\mathscr{H}_B^F(U)$.

4. Auxiliary properties of gradient measures

As preparations for the proof of Theorem 2, we give in this section some properties of gradient measures. Thus, in what follows, we assume that $(X, \mathscr{H})$ is a self-adjoint harmonic space, $\{G_U\}_{U:\text{P-set}}$ is a consistent system of symmetric Green functions, σ is the associated canonical measure representation and δ_f, $f\in \mathscr{R}(U)$, are considered with respect to this σ. For a P-set U and a signed measure μ on U such that $x\mapsto \int_U G_U(x, y)d|\mu|(y)$ is continuous, let $G_U\mu(x)=\int_U G_U(x, y)d\mu(y)$. Then $\sigma(G_U\mu)=\mu$ by definition.

First we prove

LEMMA 3. *Let U be a PC-set and e be a compact polar set in U. Then there exists a sequence $\{f_n\}$ of functions in $\mathscr{R}(U)$ satisfying the following conditions*:

(a) *$f_n=1$ on a neighborhood of e for each n,*
(b) *$0\leqq f_n\leqq 1$ on U for each n,*
(c) *$f_n(x)\to 0$ $(n\to\infty)$ if $x\in U\setminus e$,*
(d) *$\delta_{f_n}(U)\to 0$ $(n\to\infty)$.*

PROOF. For the given U and e, choose a potential p on U as in Lemma 2.

By Lemma 1, we can find continuous potentials p_n, $n=1, 2,\dots$, on U such that $p_n \geqq 1$ on a neighborhood of e and $p_n \leqq \min(p/n, p_1)$ on U for each n. Put $f_n = \min(1, p_n)$. Then $f_n \in \mathscr{R}(U)$ and satisfies (a), (b) and (c). Furthermore, by [7; Corollary 4.7],

$$\delta_{f_n}(U) \leqq \delta_{p_n}(U) \leqq \beta_U \int_U p_n \, d\sigma(p_n)$$
$$\leqq \frac{\beta_U}{n} \int_U p d\sigma(p_n) = \frac{\beta_U}{n} \int p_n \, d\sigma(p) \leqq \frac{\beta_U}{n} \int p_1 d\sigma(p)$$

with a constant $\beta_U \geqq 1$ (see [7; p. 72]). Since $\sigma(p)$ has a compact support in U, $\int p_1 d\sigma(p) < +\infty$ and (d) is satisfied.

Given an open set U and a function $h \in \mathscr{R}(U)$ which is positive everywhere on U, the harmonic space $(U, \mathscr{H}_{U,h})$ given by

$$\mathscr{H}_{U,h} = \{\mathscr{H}^{(h)}(V)\}_{V:\mathrm{open} \subset U}, \quad \mathscr{H}^{(h)}(V) = \{u/h \mid u \in \mathscr{H}(V)\}$$

is a self-adjoint harmonic space with a canonical measure representation $\sigma^{(h)}$: $\sigma^{(h)}(f) = h\sigma(fh)$ for $f \in \mathscr{R}(V)$, and the corresponding gradient measure $\delta_f^{(h)} = h^2 \delta_f$ for $f \in \mathscr{R}(V)$.

The rest of this section is devoted to the proof of the next proposition, which will be used to reduce the proof of Theorem 2 to the case $1 \in \mathscr{H}(X)$.

Proposition 1. *Let U' be a P-set, U a PC-set such that $\overline{U} \subset U'$ and e a compact polar set in U. Let $h \in \mathscr{R}(U')$ be positive everywhere on U'. Then for any $u \in \mathscr{R}(U' \setminus e)$ such that $\delta_u(U \setminus e) < +\infty$, we have $\delta_{u/h}^{(h)}(U \setminus e) < +\infty$.*

For the proof of this proposition, we need a few more lemmas. The first one is valid on a general harmonic space:

Lemma 4. *For any $f, g \in \mathscr{R}(U)$,*

$$\delta_{fg} \leqq 2(f^2 \delta_g + g^2 \delta_f) \qquad \text{on} \quad U.$$

Proof. By [7; Theorem 3.2], it is enough to show

$$2fg\delta_{[f,g]} \leqq f^2 \delta_g + g^2 \delta_f,$$

which can be easily proved by using [7; Proposition 3.3] and the continuity of f, g.

Lemma 5. *Let U be a P-set and suppose there exists $h \in \mathscr{H}(U)$ such that $m \equiv \inf_U h > 0$ and $M \equiv \sup_U h < +\infty$. Let μ be a non-negative measure such that $G_U \mu$ is bounded continuous on U. Then, for any*

$$f \in \mathscr{Q}_{IC}(U) = \{G_U\nu \mid G_U|\nu| \text{ is continuous and } \int_U G_U|\nu| d|\nu| < +\infty\},$$

$$\int_U f^2 d\mu \leqq \left(\frac{M}{m}\right)^2 (\sup_U G_U\mu)\, \delta_f(U).$$

PROOF. Since $\sigma^{(h)}(1)=0$, by [7; Lemma 4.12 and Theorem 4.3], we have

$$\int_U f^2 d\mu \leqq (\sup_U G_U^{(h)}\mu)\, \delta_f^{(h)}(U),$$

where $G_U^{(h)}(x, y)=h(x)^{-1}h(y)^{-1}G_U(x, y)$. Since $G_U^{(h)}\mu \leqq m^{-2}G_U\mu$ and $\delta_f^{(h)}(U)=\int_U h^2 d\delta_f \leqq M^2\delta_f(U)$, we obtain the required inequality.

LEMMA 6. *Let U be a PC-set and e be a compact polar set in U. Suppose $u \in \mathscr{R}(U \setminus e)$ and $\delta_u(U \setminus e) < +\infty$. Then, for any compact set K in U and for any non-negative measure μ on U such that $G_U\mu$ is bounded continuous, we have*

$$\int_{K \setminus e} u^2 d\mu < +\infty.$$

PROOF. Choose $\varphi \in \mathscr{R}(U)$ such that $\varphi=1$ on a neighborhood W of $K \cup e$, has a compact support in U and $0 \leqq \varphi \leqq 1$ on U (cf. [7; Proposition 2.17]). Let $\{f_n\}$ be a sequence as is given in Lemma 3. For each $l>0$, we consider the function

$$u_l = \max(-l, \min(u, l)) \qquad \text{on} \quad U \setminus e.$$

Then, $\varphi(1-f_n)u_l \in \mathscr{R}(U)$ if it is extended by 0 on e. Since $\varphi(1-f_n)u_l$ has a compact support in U, it follows that $\varphi(1-f_n)u_l \in \mathscr{Q}_{IC}(U)$ ([7; Lemma 6.4]). Since U is a PC-set, there is $h \in \mathscr{H}(U)$ such that $m \equiv \inf_U h > 0$ and $M \equiv \sup_U h < +\infty$. Hence, by the previous lemma,

$$(2) \qquad \int_{K \setminus e} (1-f_n)^2 u_l^2 d\mu \leqq \int_U [\varphi(1-f_n)u_l]^2 d\mu \leqq \beta\left(\frac{M}{m}\right)^2 \delta_{\varphi(1-f_n)u_l}(U),$$

where $\beta = \sup_U G_U\mu$. By Lemma 4,

$$\begin{aligned} \delta_{\varphi(1-f_n)u_l} &\leqq 2[(1-f_n)^2\delta_{\varphi u_l} + (\varphi u_l)^2\delta_{f_n}] \\ &\leqq 2[\delta_{\varphi u_l} + l^2\delta_{f_n}] \qquad \text{on} \quad U \setminus e. \end{aligned}$$

Since $\delta_{f_n}(U) \to 0$ $(n \to \infty)$, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \delta_{\varphi(1-f_n)u_l}(U) &= \limsup_{n\to\infty} \delta_{\varphi(1-f_n)u_l}(\mathrm{Supp}\,\varphi \setminus e) \\ &\leqq 2\delta_{\varphi u_l}(\mathrm{Supp}\,\varphi \setminus e). \end{aligned}$$

Hence, by (2), we obtain

$$(3) \qquad \int_{K \setminus e} u_l^2 d\mu \leqq 2\beta\left(\frac{M}{m}\right)^2 \delta_{\varphi u_l}(\operatorname{Supp} \varphi \setminus e).$$

Let W' be a relatively compact open set such that

$$\operatorname{Supp} \varphi \setminus W \subset W' \subset \overline{W'} \subset U \setminus e.$$

Since u is bounded on $\overline{W'}$,

$$\delta_{\varphi u_l}(\operatorname{Supp} \varphi \setminus W) = \delta_{\varphi u}(\operatorname{Supp} \varphi \setminus W)$$

for large l. On $W \setminus e$, $\varphi u_l = u_l$, so that $\delta_{\varphi u_l} = \delta_{u_l} \leqq \delta_u$. Hence

$$\delta_{\varphi u_l}(W \setminus e) \leqq \delta_u(W \setminus e).$$

Thus, letting $l \to \infty$ in (3), we obtain

$$\int_{K \setminus e} u^2 d\mu \leqq 2\beta\left(\frac{M}{m}\right)^2 \{\delta_{\varphi u}(\operatorname{Supp} \varphi \setminus W) + \delta_u(W \setminus e)\} < +\infty.$$

Proof of Proposition 1. By Lemma 4 and [7; Theorem 3.3],

$$\delta_{u/h}^{(h)} = h^2 \delta_{u/h} \leqq 2\left(\delta_u + \frac{u^2}{h^2}\delta_h\right) \qquad \text{on} \quad U \setminus e.$$

Let V be a relatively compact open set such that $e \subset V \subset \overline{V} \subset U$. Since u/h is bounded on $U \setminus \overline{V}$ and $\delta_h(U \setminus \overline{V}) < +\infty$, we have

$$\int_{U \setminus \overline{V}} \frac{u^2}{h^2} d\delta_h < +\infty.$$

On the other hand, $\mu = h^{-2}\delta_h|_V$ is a non-negative measure such that $G_U\mu$ is bounded continuous. Hence, by Lemma 6,

$$\int_{V \setminus e} \frac{u^2}{h^2} d\delta_h < +\infty.$$

Therefore, $\int_{U \setminus e} (u^2/h^2) d\delta_h < +\infty$, and hence $\delta_{u/h}^{(h)}(U \setminus e) < +\infty$.

5. Proof of THEOREM 2

By assumption, given an open set U containing e, we can find a P-set V' such that $e \subset V' \subset U$. Therefore we may assume from the beginning that U is a PC-set containing e and $\overline{U}$ is contained in a P-set U' on which there exists $h \in \mathscr{H}(U')$ such that $h > 0$ everywhere on U'.

First, we reduce our problem to the case $\sigma(1) = 0$ on U'. Consider the harmonic space $(U', \mathscr{H}_{U',h})$ and the sheaf morphism $F_1 : \mathscr{R}|_{U'} \to \mathscr{M}|_{U'}$ defined by

$$F_1(f) = hF(hf) \qquad \text{on} \quad W$$

for $f \in \mathscr{R}(W)$, $W \subset U'$. Then F_1 satisfies conditions (F.1) and (F.2) for the harmonic space $(U', \mathscr{H}_{U',h})$. If $u \in \mathscr{H}_D^F(U \setminus e)$, then

$$\sigma^{(h)}(u/h) = h\sigma(u) = -hF(u) = -F_1(u/h) \qquad \text{on} \quad U \setminus e$$

and $\delta_{u/h}^{(h)}(U \setminus e) < +\infty$ by Proposition 1. Hence $u/h \in \mathscr{H}_D^{(h)F_1}(U \setminus e)$. Obviously, e is also polar for the structure $\mathscr{H}_{U',h}$. Thus, if the theorem is true for the harmonic space $(U', \mathscr{H}_{U',h})$, then there is $\tilde{v} \in \mathscr{H}_D^{(h)F_1}(U)$ such that $\tilde{v}|_{U \setminus e} = u/h$. Then, we see that $\tilde{u} \equiv h\tilde{v} \in \mathscr{H}_D^F(U)$ and $\tilde{u}|_{U \setminus e} = u$, and hence the theorem is proved. Since $\sigma^{(h)}(1) = 0$ on U', this means that it is enough to prove the theorem in case $\sigma(1) = 0$ on U'.

Thus, assume $\sigma(1) = 0$ on U'. By considering each component of U, we may further assume that U is connected. Then there exists a regular domain V such that $e \subset V \subset \bar{V} \subset U$ (cf. [5; Corollary 4.2]). By [8; Theorem 2.1], $v = \mathscr{H}_u^{F,V}$ is defined. Put $w = u - v$ on $V \setminus e$. We shall show that $w = 0$; then it suffices to let $\tilde{u} = v$ on e.

Suppose $w \neq 0$ on $V \setminus e$. Since $V \setminus e$ is connected (cf. [2; Proposition 6.2.5]), [7; Theorem 5.4] implies that $\delta_w \neq 0$ on $V \setminus e$. Then there is $\alpha > 0$ such that

$$\delta_w(\{x \in V \setminus e \mid \alpha < |w(x)| < \alpha + 1\}) > 0. \tag{4}$$

Choose a continuous function χ on $\boldsymbol{R}$ such that $\chi(t) > 0$ if $\alpha < |t| < \alpha + 1$ and $\chi(t) = 0$ otherwise. Put $\psi(t) = \int_0^t (t-s)\chi(s)ds$. Then $\psi \in \mathscr{C}^2(\boldsymbol{R})$, $\psi \equiv 0$ on $[-\alpha, \alpha]$, $\psi \geqq 0$ everywhere, ψ' is bounded on $\boldsymbol{R}$, $\psi'(t)\,\mathrm{sgn}\, t \geqq 0$ for all $t \in \boldsymbol{R}$ and $\psi'' = \chi$. Since $w(x) \to 0$ as $x \to \xi$ for all $\xi \in \partial V$ (cf. [8; Proposition 3.3]), there is a compact set K in V containing e such that $|w(x)| < \alpha$ for all $x \in V \setminus K$. Choose $\varphi_0 \in \mathscr{R}(V)$ such that $\varphi_0 = 1$ on K, $0 \leqq \varphi_0 \leqq 1$ on V and φ_0 has a compact support in V; let $\{f_n\}$ be a sequence as is given in Lemma 3 for V and e. For each n, $\varphi_0(1 - f_n) \in \mathscr{R}(V)$ and has a compact support in $V \setminus e$. Since $\psi \circ w \in \mathscr{R}(V \setminus e)$ by [7; Theorem 3.3], by Green's formula [7; Theorem 5.3], we have

$$\delta_{[\psi \circ w, \varphi_0(1-f_n)]}(V \setminus e) = \int_{V \setminus e} \varphi_0(1 - f_n) d\sigma(\psi \circ w). \tag{5}$$

By [7; Theorem 3.3],

$$\sigma(\psi \circ w) = -(\psi'' \circ w)\delta_w + (\psi' \circ w)\sigma(w) \qquad \text{on} \quad V \setminus e.$$

Since u and v satisfy (1) on $V \setminus e$,

$$\sigma(w) = \sigma(u) - \sigma(v) = -F(u) + F(v) = F(u - w) - F(u).$$

By (F.1), we see that $(\psi' \circ w)\{F(u - w) - F(u)\} \leqq 0$. Hence

$$\sigma(\psi \circ w) \leqq -(\psi'' \circ w)\delta_w = -(\chi \circ w)\delta_w.$$

Thus, by (5)

(6) $$\delta_{[\psi\circ w,\varphi_0(1-f_n)]}(V\setminus e) \leqq -\int_{V\setminus e}\varphi_0(1-f_n)(\chi\circ w)d\delta_w.$$

Since $\chi\circ w=0$ on $V\setminus K$ and $\varphi_0=1$ on K, the right hand side of (6) is equal to $-\int_{K\setminus e}(1-f_n)(\chi\circ w)d\delta_w$, which tends to $-\int_{K\setminus e}(\chi\circ w)d\delta_w$ as $n\to\infty$. Note that $\delta_w(K\setminus e)<+\infty$ and $\chi\circ w$ is bounded on $K\setminus e$. On the other hand,

$$\begin{aligned}|\delta_{[\psi\circ w,\varphi_0(1-f_n)]}(V\setminus e)| &= |\delta_{[\psi\circ w,1-f_n]}(K\setminus e)| \\ &\leqq \delta_{\psi\circ w}(K\setminus e)^{1/2}\cdot\delta_{f_n}(K\setminus e)^{1/2} \\ &\to 0 \quad (n\to\infty),\end{aligned}$$

where we used the fact that $\delta_{\psi\circ w}(K\setminus e)=\int_{K\setminus e}(\psi'\circ w)^2 d\delta_w$ (cf. [7; Theorem 3.3]) which is finite since ψ' is bounded and $\delta_w(K\setminus e)<+\infty$. Hence (6) implies that $\int_{K\setminus e}(\chi\circ w)d\delta_w\leqq 0$. Since $\chi\geqq 0$ and $\delta_w\geqq 0$, it follows that $\int_{K\setminus e}(\chi\circ w)d\delta_w=0$. This is impossible in view of (4) and the choice of χ. Hence $w=0$ and the proof is completed.

6. Remarks

REMARK 1. Theorem 1 remains valid without the monotonicity condition (F.1) for F; more precisely, if F satisfies only the condition (F.2) in which (iii) is replaced by

(iii)′ $|F(f_1)-F(f_2)|\leqq(f_1-f_2)\pi_{M,U}$ on U, whenever $f_1, f_2\in\mathscr{R}(U)$ and $-M\leqq f_2\leqq f_1\leqq M$ on U.

This is seen as follows. Let U be any PC-set and $u\in\mathscr{H}_B^F(U\setminus e)$ be given. For $M=\sup_{U\setminus e}|u|$, we consider a linear perturbation of the original harmonic structure on U so that the perturbed space $(U, \tilde{\mathscr{U}})$ has a measure representation $\tilde{\sigma}$ such that $\tilde{\sigma}(f)=\sigma(f)-f\pi_{M,U}$ (cf., e.g., [11]). Then e is also polar for $\tilde{\mathscr{U}}$. Consider the sheaf morphism $\tilde{F}$: $\mathscr{R}|_U\to\mathscr{M}|_U$ defined by

$$\tilde{F}(f) = F(\max(-M, \min(f, M))) + f\pi_{M,U}.$$

Then $\tilde{F}$ satisfies (F.1) and (F.2) for the space $(U, \tilde{\mathscr{U}})$. Since $u\in\tilde{\mathscr{H}}_B^{\tilde{F}}(U\setminus e)=\{v \mid v$ is bounded and $\tilde{\sigma}(v)+\tilde{F}(v)=0$ on $U\setminus e\}$, Theorem 1 implies that u has an extension $\tilde{u}\in\tilde{\mathscr{H}}_B^{\tilde{F}}(U)$. Since $|\tilde{u}|\leqq M$, it follows that $\tilde{u}\in\mathscr{H}_B^F(U)$. Then it is easy to see that the assertion of Theorem 1 holds for any open set U.

REMARK 2. The following simple example shows that the monotonicity condition (F.1) cannot be suppressed for the validity of Theorem 2.

Let X be the unit ball in $\boldsymbol{R}^n$ $(n\geqq 3)$ with center at 0 and consider the classical harmonic structure on X, so that $\sigma(f)=-\Delta f$ (in the distribution sense). Let $e=\{0\}$, which is a polar set. For $\alpha>(n+2)/(n-2)$, let $F(f)=-|f|^\alpha m$, where m is the Lebesgue measure on X. Then F satisfies (i), (ii) of (F.2) and (iii)′ in the above Remark with $\pi_{M,U}=\alpha M^{\alpha-1}m$. Let

$$u(x)=\frac{(\alpha-1)^2}{2(\alpha n-n-2\alpha)}|x|^{2/(1-\alpha)}.$$

Then $\Delta u(x)+|u(x)|^\alpha=0$ for $x\in X\setminus\{0\}$, so that $\sigma(u)+F(u)=0$ on $X\setminus\{0\}$. Furthermore, by direct computation, we see

$$\int_{X\setminus\{0\}}|\nabla u|^2dx<+\infty,$$

i.e., $u\in\mathscr{H}_D^F(X\setminus\{0\})$. Since $u(x)\to+\infty$ $(x\to 0)$, u has no extension to a function in $\mathscr{H}_D^F(X)$.

REMARK 3. The self-adjointness condition in Theorem 2 may appear to be too stringent. In fact, [10; IX, §8, Satz 20] suggests that Theorem 2 would remain valid for non self-adjoint elliptic harmonic space. In non-elliptic case, e.g., for parabolic equations on euclidean domains (even for heat equations), there seems to be no known result on the removability of polar sets for Dirichlet-finite solutions.

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