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On finite H-spaces given by sphere extensions of classical groups

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§1. Introduction

An *H-space* is a path-connected (based) space admitting a continuous multiplication for which the base point is a homotopy unit. An H-space is called *finite* if it has the homotopy type of a finite CW-complex. Typical examples of finite H-spaces are the product spaces of Lie groups, the 7-sphere S^7 or the 7-projective space RP^7. The other examples are constructed by Hilton-Roitberg [6], Curtis-Mislin [4], A. Zabrodsky [17] and so on. These are given actually by sphere extensions of the classical groups $SO(n)$, $SU(n)$ or $Sp(n)$ which we shall discuss in this paper. To prove our main results we find a decomposition formula for cohomology operation in the BP-theory, which would be useful in the further study of H-spaces.

Let $d=1$, 2 or 4, and

(1.1) $$G(n, d) = SO(n),\ SU(n) \text{ or } Sp(n) \quad \text{according to} \quad d = 1, 2 \text{ or } 4.$$

Consider the commutative diagram

(1.2) $$\begin{array}{ccccc} G(n-1, d) & \xrightarrow{\iota} & G(n, d) & \xrightarrow{\pi} & G(n, d)/G(n-1, d) = S^{dn-1} \\ & & \uparrow{\scriptstyle \tilde{h}_\lambda} & & \uparrow{\scriptstyle h_\lambda} \\ G(n-1, d) & \xrightarrow{\iota_\lambda} & M(n, d, \lambda) & \xrightarrow{\pi_\lambda} & S^{dn-1} \end{array}$$

of the principal bundles for any integers $n \geqq 2$ and λ, where the lower bundle is induced from the upper one by the map h_λ of degree λ. The total space $M(n, d, \lambda)$ is called a *sphere extension* of $G(n-1, d)$. On the conditions for $M(n, d, \lambda)$ to be an H-space, the following are known:

(1.3) ([17; Cor.]) *When $G(n, 1)=SO(n)$ and n is even $\neq 2, 4, 8$, $M(n, 1, \lambda)$ is an H-space if and only if λ is odd.*

(1.4) ([4], [17; Cor.]) *When $G(n, 2)=SU(n)$, $M(n, 2, \lambda)$ is an H-space if and only if $n=2$, 4 or λ is odd.*

(1.5) ([17; Cor.], [18; Th. 3.10]) *When $G(n, 4)=Sp(n)$, $M(n, 4, \lambda)$ is an H-space if and only if λ is odd or $n=2$ and $\lambda \not\equiv 2$* mod 4.

The purpose of this paper is to complete (1.3) in case when $n=2, 4, 8$ or n is odd, and furthermore, to give the condition for the H-space $M(n, d, \lambda)$ to have the homotopy type of a loop space. Our main results are stated as follows:

THEOREM **A**. *$M(n, 1, \lambda)$ in (1.2) for $G(n, 1)=SO(n)$ is an H-space if and only if*

$$n = 2, 4, 8 \text{ or } \lambda \text{ is odd when } n \text{ is even}, \quad \text{and } \lambda = \pm 1 \text{ when } n \text{ is odd.}$$

Furthermore, in these cases, $M(n, 1, \lambda)$ has the homotopy type of a loop space, and in fact, it is homotopy equivalent to $SO(n)$.

THEOREM **B**. *$M(n, d, \lambda)$ in (1.2) for $G(n, 2)=SU(n)$ or $G(n, 4)=Sp(n)$ has the homotopy type of a loop space if and only if*

$$\lambda \not\equiv 0 \bmod p \qquad \text{for any prime } p \text{ with } 2p < dn;$$

and then $M(n, d, \lambda)$ is p-equivalent to $G(n, d)$ for any prime p.

We remark that $M(n, d, \lambda)$ in Theorem **B** is not homotopy equivalent to $G(n, d)$ if $\lambda \not\equiv \pm 1 \bmod (dn/2-1)!$ by A. Zabrodsky [19; Th. A].

Theorems **A** and **B** follow from Theorems A and B, respectively, which are presented in §2 by considering the conditions that $M(n, d, \lambda)$ is p-equivalent to an H-space or a loop space for a prime p. In addition to Theorems A and B, we state in Proposition 2.4 that ι_λ in (1.2) is a loop map up to homotopy type, which is proved in §4. Theorem A is proved in §3 assuming Proposition 3.2 which is proved in §5 by using the unstable secondary operations introduced by A. Zabrodsky. Theorem B is proved in §3 assuming Proposition 3.11 which is considered in a little more general situation than $M(n, d, \lambda)$. We prove (i) of Proposition 3.11 in §6 by studying the action of the Steenrod algebra, and (ii) in §8 after performing a decomposition formula (Proposition 7.7) for the Landweber-Novikov operations in the BP-theory in §7.

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§2. Restatement of results

Throughout this paper, we assume that all spaces, maps and homotopies are based, and all spaces are path-connected and have the homotopy type of CW-complexes.

Let p be a prime. Then, we say simply that a map $f: X \to Y$ is a *p-equivalence* if f is a mod p (co)homology equivalence, i.e., if

$$f_*: H_*(X; Z_p) \longrightarrow H_*(Y; Z_p) \quad (\text{or equivalently } f^*: H^*(Y; Z_p) \longrightarrow H^*(X; Z_p))$$

is an isomorphism. When such a map f exists, we say that X is *p-equivalent* to Y and denote by $X \simeq_p Y$. We notice that this relation $\simeq_p$ is an equivalence relation in the category of p-universal spaces and spaces treated in this paper are all in this category (see [13–15] for the definition and the properties of p-universal spaces). We say that X is a *mod p H-space* (resp. a *mod p loop space*) if it is p-equivalent to an H-space (resp. a loop space ΩY for some Y).

Now, for $G(n, d)$ and $M(\lambda)=M(n, d, \lambda)$ in (1.1–2), we consider the following conditions:

(pH) (resp. (pL), (pG)) $M(\lambda)$ is p-equivalent to an H-space (resp. a loop space, $G(n, d)$),

where p is a prime or ∞ and '∞-equivalent' means 'homotopy equivalent'. ((∞H) means that $M(\lambda)$ is an H-space.) Then, we can state Theorems A and B which are stronger versions of Theorems **A** and **B** in the introduction.

THEOREM A. *Let* $d=1$, $G(n, 1)=SO(n)$ *and* $M(\lambda)=M(n, 1, \lambda)$.

(I) *The case n is even*: (i) (pH), (pL) *and* (pG) *hold for any odd prime p.*

(ii) *The conditions* (2H), (2L), (2G), (∞H), (∞L), (∞G) *and the following* (2.1) *are equivalent to each other*:

(2.1) λ *is odd, or* $n = 2, 4, 8$.

(II) *The case n is odd*: *The conditions* (pH), (pL), (pG) *and the following* (2.2: p) *are equivalent to each other for any prime p or* $p=\infty$:

(2.2: p) $\lambda \not\equiv 0 \bmod p$ (*when p is a prime*); $\lambda = \pm 1$ (*when* $p = \infty$).

THEOREM B. *Let* $d=2$ *or* 4, $G(n, d)=SU(n)$ *or* $Sp(n)$ *and* $M(\lambda)=M(n, d, \lambda)$.

(i) *The conditions* (pL), (pG) *and the following* (2.3: p) *are equivalent to each other for any prime p*:

(2.3: p) $\lambda \not\equiv 0 \bmod p$, *or* $2p \geqq dn$.

(ii) *The condition* (∞L) *is equivalent to* (pG) *for all prime p and also to*

(2.3: ∞) $\lambda \not\equiv 0 \bmod p$ *for any prime p with* $2p < dn$.

In addition to these theorems, we have the following

PROPOSITION 2.4. *When* $M(n, d, \lambda)$ *is homotopy equivalent to a loop space, i.e., when* (2.1) *or* (2.2: ∞) *holds in Theorem A or when* (2.3: ∞) *holds in Theorem B, the map* ι_λ: $G(n-1, d)\to M(n, d, \lambda)$ *in* (1.2) *is homotopy equivalent to a loop map in the sense that we can choose a homotopy equivalence f of* $M(n, d, \lambda)$ *to* *a loop space so that the composition* $f \circ \iota_\lambda$ *is a loop map.*

We remark that $\tilde{h}_\lambda$: $M(n, d, \lambda) \to G(n, d)$ in (1.2) is not necessarily homotopy equivalent to a loop map unless $\lambda = \pm 1$, even if $M(n, d, \lambda)$ is homotopy equivalent to $G(n, d)$.

§3. Reduction of Theorems A and B to some propositions

In this section, assuming Propositions 3.2 and 3.11 stated below, we prove Theorems A and B by using mainly the results due to A. Zabrodsky [17] [20].

PROOF OF THEOREM A (I). The implications $(p\mathrm{G}) \Rightarrow (p\mathrm{L}) \Rightarrow (p\mathrm{H})$, $(\infty\mathrm{H}) \Rightarrow (p\mathrm{H})$, $(\infty\mathrm{L}) \Rightarrow (p\mathrm{L})$ and $(\infty\mathrm{G}) \Rightarrow (p\mathrm{G})$ are trivial for any $p \leqq \infty$. We notice that

(3.1) $$\pi_{n-1}(BSO(n-1)) = 0,\ Z_2 \text{ or } Z_2 \oplus Z_2 \qquad \text{for } n = 2k \geqq 4$$

(cf. [10; pp. 161–162]).

(i) Let p be an odd prime. By (3.1) and the definition of $M(\lambda) = M(n, 1, \lambda)$ in (1.2), we see that $\tilde{h}_\lambda$: $M(\lambda) \simeq_p SO(n)$ for $n = 2k \geqq 4$. When $n = 2$, π: $SO(2) \to S^1$ is a homeomorphism and so is π_λ: $M(\lambda) \to S^1 = SO(2)$. Thus we see $(p\mathrm{G})$.

(ii) $(2\mathrm{H}) \Rightarrow (2.1)$: This is shown in [17; Cor.].

$(2.1) \Rightarrow (\infty\mathrm{G})$: If λ is odd, then $\tilde{h}_\lambda$: $M(\lambda) \simeq SO(n)$ for $n = 2k \geqq 4$ by (2.5). If $n = 2$, 4 or 8. then the upper principal bundle in (1.2) is trivial and so is the lower one. So, $M(\lambda)$ is homeomorphic to $SO(n)$. Q. E. D.

Theorem A (II) follows from the following proposition, which will be proved in §5:

PROPOSITION 3.2. *In Theorem* A (II), $(p\mathrm{H})$ *implies* $(2.2\colon p)$ *for any prime* p.

PROOF OF THEOREM A (II) FROM PROPOSITION 3.2. $(2.2\colon p)$ implies $(p\mathrm{G})$ for any $p \leqq \infty$ by definition, and $(2.2\colon \infty)$ means $(2.2\colon p)$ for all prime p. So, we see Theorem A (II) by the trivial implications and $(p\mathrm{H}) \Rightarrow (2.2\colon p)$ for any prime p. Q. E. D.

PROOF OF THEOREM B (i) FOR $p = 2$. $(2\mathrm{G}) \Rightarrow (2\mathrm{L})$ is trivial.

$(2.3\colon 2) \Rightarrow (2\mathrm{G})$: If λ is odd, then h_λ is a 2-equivalence and so is $\tilde{h}_\lambda$: $M(\lambda) \to G(n, d)$ in (1.2). If $4 \geqq dn$, then $d = n = 2$, $G(2, 2) = SU(2)$, and π_λ: $M(\lambda) \to S^3 = SU(2)$ is a homeomorphism, because so is π: $SU(2) \to S^3$.

$(2\mathrm{L}) \Rightarrow (2.3\colon 2)$: When $dn \neq 8$, this is shown in [17; Cor.]. Assume $dn = 8$. Then $G(n, d) = SU(4)$ or $Sp(2)$. We notice that

$$\pi_7(BSU(3)) = Z_6 \text{ and } \pi_7(BSp(1)) = Z_{12} \quad \text{(cf. [3; 26.10], [12; Th. 2.2])}$$

By (1.2), we see that $M(n, d, q\lambda)\simeq_p M(n, d, \lambda)$ if $q\not\equiv 0 \bmod p$ and p is a prime. So, $M(4, 2, \lambda)\simeq S^7\times SU(3)$ if $\lambda=0, 6$, $\simeq_2 M(4, 2, 6)$ if $\lambda=2$; and $M(2, 4, \lambda)\simeq S^7\times Sp(1)$ if $\lambda=0, 12$, $\simeq_p M(2, 4, 1)=Sp(2)$ if $\lambda=2$, $\simeq_2 M(2, 4, 12)$ if $\lambda=4$, $\simeq_p M(2, 4, 12)$ if $\lambda=6$, where p is any odd prime. Here, $S^7\times SU(3)$ and $S^7\times Sp(1)$ are not mod 2 loop spaces, because they admit no mod 2 homotopy associative H-structures by [5; Th. 2]. So, $M(4, 2, \lambda)$ (λ: even) and $M(2, 4, \lambda)$ ($\lambda=0, 4$) are not mod 2 loop spaces. Furthermore $M(2, 4, \lambda)$ ($\lambda=2, 6$) is a mod p H-space for any odd prime, and is not an H-space by (1.5). So, it is not a mod 2 H-space by [20; Prop. 4.5.3]. Thus $M(2, 4, \lambda)$ (λ: even) is not a mod 2 loop space.

Q. E. D.

Proof of (ii) from (i) in Theorem B. If $M(\lambda)$ satisfies (pG) for all prime p, then it has the same genus type as $G(n, d)$ and hence satisfies (∞L), according to [20; Cor. 4.7.4]. The implications (∞L)$\Rightarrow$(2.3: ∞)$\Rightarrow$(pG) follow from (i).

Q. E. D.

Now, let p be an odd prime in the rest of this section. Then, Theorem B (i) for p is proved in somewhat more general situation given as follows:

(3.3) Let G be a given simply connected finite mod p loop space such that $H^*(G; Z)$ has no p-torsion, i.e.,

(3.4) $$H^*(G; Z_p) = \Lambda(g_1,\dots, g_k),\quad \dim g_i = 2n_i-1,\ 2 \leqq n_1 \leqq\cdots\leqq n_k,$$

for some g_i of mod p universal transgressive. Furthermore, let

(3.5) $\pi\colon G\to S^m$ ($m=2n_k-1$) be a given fibering with $\pi^*\xi=g_k$ for a generator $\xi\in H^m(S^m; Z_p)$.

By replacing $G(n, d)$ by G in (1.2), we can define $M(G, \lambda)$ for any integer λ by the pullback diagram

(3.6) $$\begin{array}{ccc} M(G, \lambda) & \xrightarrow{\tilde{h}_\lambda} & G \\ \downarrow{\scriptstyle \pi_\lambda} & & \downarrow{\scriptstyle \pi} \\ S^m & \xrightarrow{h_\lambda} & S^m, \end{array}$$

where h_λ is the map of degree λ. Then we can prove the following theorem, where $\mathscr{A}$ denotes the mod p Steenrod algebra and $\tilde{\mathscr{A}}$ is its augmentation ideal:

Theorem 3.7. *Under the assumption that*

(3.8) $$g_k \notin \tilde{\mathscr{A}}(H^*(G; Z_p))\quad \textit{if}\quad n_k = p^a b\quad \textit{and}\quad 1 \leqq b < p,$$

$M(G, \lambda)$ in (3.6) is a mod p *loop space if and only if*

(3.9) $$\lambda \not\equiv 0 \bmod p,\quad \textit{or}\quad n_k - n_1 + 2 \leqq p;$$

and then $M(G, \lambda)$ is p-equivalent to G.

PROOF OF THEOREM B (i) FOR ODD PRIME p FROM THEOREM 3.7. We notice that (cf. [1; Prop. 9.1], [2; Cor. 11.4, Cor. 13.5])

$$H^*(G(n, d); Z_p) = \Lambda(e_3, e_{3+d}, \dots, e_{m-d}, e_m), \dim e_i = i,\ m = dn - 1,\ \pi^*(\xi) = e_m,$$

for $G(n, d)$ $(d=2, 4)$ and π in (1.1–2), where e_i is universal transgressive. Furthermore,

$$(*) \quad \mathscr{P}^i e_{2j-1} = \binom{j-1}{i} e_m \text{ where } 2j - 1 + 2i(p-1) = m, \text{ i.e., } j + i(p-1) = dn/2.$$

Assume that $dn/2 = p^a b$ and $1 \leqq b < p$. Let

$$i = c_0 p^t + c_1 p^{t+1} \ (t \geqq 0,\ 1 \leqq c_0 < p,\ c_1 \geqq 0) \quad \text{and} \quad j = p^a b - i(p-1) > 0.$$

Then, since $1 \leqq b < p$, we see that $t < a$, $(c_0 + c_1 p)(p-1) < p^{a-t} b$ and

$$j - 1 = c_0 p^t - 1 + c p^{t+1} \text{ where } c = p^{a-t-1} b - c_0 - c_1(p-1) \geqq 0.$$

So, the coefficients of p^t in the p-adic expansions of i and $j-1$ are c_0 and $c_0 - 1$, respectively, which implies $\binom{j-1}{i} \equiv 0 \bmod p$ as is well-known. Therefore $e_m \in \mathscr{A}(H^*(G(n, d); Z_p))$ by $(*)$; and the assumption (3.8) is satisfied for $G = G(n, d)$. Now Theorem B (i) for odd prime p is the special case of Theorem 3.7 for $G = G(n, d)$ with $n_2 = 2$ and $n_k = dn/2$. Q.E.D.

Theorem 3.7 follows immediately from the following propositions:

PROPOSITION 3.10. (3.9) *implies that $M(G, \lambda)$ is p-equivalent to G.*

PROPOSITION 3.11. *Assume that $M(G, \lambda)$ is a* mod p *loop space.*

(i) *If $b > p$ (where $n_k = p^a b$ and $b \not\equiv 0 \bmod p$), then $\lambda \not\equiv 0 \bmod p$.*

(ii) *If $1 \leqq b < p$ and* (3.8) *is valid, then $\lambda \not\equiv 0 \bmod p$ or $p \geqq n_k$.*

Proposition 3.11 will be proved in §§6–8.

PROOF OF PROPOSITION 3.10. If $\lambda \not\equiv 0 \bmod p$, then h_λ is a p-equivalence and so is $\tilde{h}_\lambda$ in (3.6).

Now suppose that $n_k - n_1 + 2 \leqq p$. Then, we have a homotopy equivalence

$$\varphi : S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_k-1}_{(p)} \longrightarrow G_{(p)}$$

by Kumpel [11], where $-_{(p)}$ denotes the localization at p (cf. [15] for the details on the localization). Here, we may assume that the composition $\pi_{(p)} \circ s$ of

$$s = \varphi \mid S^m_{(p)}: S^m_{(p)} \longrightarrow G_{(P)} \quad \text{and} \quad \pi_{(p)}: G_{(P)} \longrightarrow S^m_{(p)} \quad (m = 2n_k - 1)$$

is a homotopy equivalence, i.e., s is a homotopy section for $\pi_{(p)}$. Let $\iota: Y \to G_{(p)}$ be the homotopy fibre of $\pi_{(p)}$. Then, the composition

$$f = \mu \circ (s \times \iota): S^m_{(p)} \times Y \longrightarrow G_{(p)} \times G_{(p)} \longrightarrow G_{(p)}$$

is a homotopy equivalence, where μ is the multiplication. Now we define

$$g = \text{pr} \circ f^{-1}: G_{(p)} \longrightarrow S^m_{(p)} \times Y \longrightarrow Y \, (\text{pr denotes the projection}).$$

It is clear that $g \circ \iota \sim \text{id}: Y \to Y$. Let $\iota_\lambda: Y \to M(G, \lambda)_{(p)}$ be the homotopy fibre of $(\pi_\lambda)_{(p)}: M(G, \lambda)_{(p)} \to S^m_{(p)}$ so that $(\tilde{h}_\lambda)_{(p)} \circ \iota_\lambda \sim \iota$ for $(\tilde{h}_\lambda)_{(p)}: M(G, \lambda)_{(p)} \to G_{(p)}$. Then $g \circ (\tilde{h}_\lambda)_{(p)} \circ \iota_\lambda \sim \text{id}$, and

$$((\pi_\lambda)_{(p)}, g \circ (\tilde{h}_\lambda)_{(p)}): M(G, \lambda)_{(p)} \longrightarrow S^m_{(p)} \times Y$$

is a homotopy equivalence. Thus $M(G, \lambda)_{(p)} \simeq S^m_{(p)} \times Y \to G_{(p)}$ and $M(G, \lambda) \simeq_p G$ by [15; Cor. 5.4]. Q.E.D.

§4. Proof of Proposition 2.4

Proposition 2.4 is clear in the case (2.2: ∞) in Theorem A (II), and seen in the case (2.1) in Theorem A (I) (ii) by the proof of (2.1)$\Rightarrow$(∞G) in Theorem A (I) given in §3.

The case (2.3: ∞) in Theorem B (ii): If $dn \leqq 4$, then $d = n = 2$ and $G(1, 2) = SU(1) = *$. Thus $\iota_\lambda = *: * \to M(\lambda)$ is clearly a loop map.

Suppose $dn > 4$. Put $P_1 = \{p; \text{prime} \mid \lambda \equiv 0 \bmod p\}$ and $P_2 = \{p; \text{prime} \mid p \notin P_1\}$. Since P_1 is a finite set by definition, we write $P_1 = \{p_1, p_2, \dots, p_k\}$. We define integers μ_i $(0 \leqq i \leqq t)$ inductively so that $\lambda\mu_i \equiv 1 \bmod N = 2\{(dn/2-1)!\}$ and $\mu_i \not\equiv 0 \bmod p_j$ for any $j \leqq i$. Since $\lambda \not\equiv 0 \bmod N$ by (2.3: ∞) and $2 < dn/2$, there is an integer μ_0 such that $\lambda\mu_0 \equiv 1 \bmod N$. Suppose that we have μ_j for $j < i$ $(i \geqq 1)$ with the desired properties. If $\mu_{i-1} \not\equiv 0 \bmod p_i$, then $\mu_i = \mu_{i-1}$ satisfies the desired properties. If $\mu_{i-1} \equiv 0 \bmod p_i$, then $\mu_i = \mu_{i-1} + Np_1 \cdots p_{i-1}$ satisfies the desired properties since $N \not\equiv 0 \bmod p_i$ by the definition of P_1. Put $\mu = \mu_t$. Then

(4.1) $\qquad \lambda\mu \equiv 1 \bmod N = 2\{(dn/2-1)!\} \quad \text{and} \quad \mu \not\equiv 0 \bmod p \text{ for any } p \in P_1.$

Now we notice that

(4.2) $\qquad \pi_{dn-1}(BG(n-1, d)) = Z_{N/2} \text{ or } Z_N \quad$ (cf. [3; 26.10], [12; Th. 2.2]).

Then we have the following commutative diagram of the principal bundles:

$$(4.3)\qquad \begin{array}{ccccccc}
G(n-1) & = & G(n-1) & = & G(n-1) & = & G(n-1) \\
\downarrow \iota & & \downarrow \iota_{\lambda\mu} & & \downarrow \iota_{\lambda} & & \downarrow \iota \\
G(n) & \xrightarrow[\approx]{\varphi} & M(\lambda\mu) & \xrightarrow{\tilde{h}_{\mu}} & M(\lambda) & \xrightarrow{\tilde{h}_{\lambda}} & G(n) \\
\downarrow \pi & & \downarrow \pi_{\lambda\mu} & & \downarrow \pi_{\lambda} & & \downarrow \pi \\
S^{dn-1} & \longrightarrow & S^{dn-1} & \xrightarrow{h_{\mu}} & S^{dn-1} & \xrightarrow{h_{\lambda}} & S^{dn-1},
\end{array}$$

where $G(i)=G(i, d)$ and φ is a homeomorphism by (4.1) and (4.2).

Now we use the localization theory (cf. [15]). Let P be a set of primes or $P=\emptyset$. We denote $X_{(P)}$ (resp. $f_{(P)}$) for the localization of a space X (resp. a map f) at P. We also write $l(P)=l(X; P)$: $X\to X_{(P)}$ and $l(P, P')=l(X; P, P')$: $X_{(P)}\to X_{(P')}$ for the standard maps, where $P'\subset P$. Then (4.3) induces the homotopy commutative diagram

$$\begin{array}{ccccccc}
G(n-1) & & \xrightarrow{\qquad l(P_2) \qquad} & & G(n-1)_{(P_2)} & \xrightarrow{\iota_{(P_2)}} & G(n)_{(P_2)} \\
\| & & & & \downarrow \iota_{\lambda(P_2)} & & \| \\
G(n-1) & \xrightarrow{\iota_{\lambda}} & M & \xrightarrow{l_2} & M_{(P_2)} & \xrightarrow[\simeq]{\tilde{h}_{\lambda(P_2)}} & G(n)_{(P_2)} \\
\downarrow l(P_1) & & \downarrow l_1 & & \downarrow l'_2 & & \downarrow l(P_2, \emptyset) \\
G(n-1)_{(P_1)} & \xrightarrow{\iota_{\lambda(P_1)}} & M_{(P_1)} & \xrightarrow{l'_1} & M_{(\emptyset)} & \xrightarrow[\simeq]{\tilde{h}_{\lambda(\emptyset)}} & G(n)_{(\emptyset)} \\
\| & & \simeq\uparrow (\tilde{h}_{\mu}\circ\varphi)_{(P_1)} & & \simeq\uparrow (\tilde{h}_{\mu}\circ\varphi)_{(\emptyset)} & & \| \\
G(n-1)_{(P_1)} & \xrightarrow{\iota_{(P_1)}} & G(n)_{(P_1)} & \xrightarrow{l(P_1, \emptyset)} & G(n)_{(\emptyset)} & \xrightarrow{(\tilde{h}_{\lambda\mu}\circ\varphi)_{(\emptyset)}} & G(n)_{(\emptyset)},
\end{array}$$

where $M=M(\lambda)$, $l_i=l(P_i)$ and $l'_i=l(P_i, \emptyset)$ for $i=1, 2$. Since h_{λ} (resp. h_{μ}) is a P_2 (resp. P_1)-equivalence by the definition of P_i and (4.1), so is $\tilde{h}_{\lambda}$ (resp. $\tilde{h}_{\mu}$) Thus $\tilde{h}_{\lambda(P_2)}$, $\tilde{h}_{\lambda(\emptyset)}$, $(\tilde{h}_{\mu}\circ\varphi)_{(P_1)}$ and $(\tilde{h}_{\mu}\circ\varphi)_{(\emptyset)}$ are all homotopy equivalences. Now the middle square consisting of l_i and l'_i is homotopy equivalent to the weak pullback diagram by [15; Cor. 4.2]. Therefore M is homotopy equivalent to the weak pullback of $(\tilde{h}_{\lambda\mu}\circ\varphi)_{(\emptyset)}\circ l(P_1, \emptyset)$ and $l(P_2, \emptyset)$. Now $G(n)_{(\emptyset)}\simeq K(Q, 3)\times\cdots\times K(Q, dn-1)$ as loop spaces ([15; Lemma 7.4]) and $(\tilde{h}_{\lambda\mu}\circ\varphi)_{(\emptyset)}$ is represented by a diagonal matrix. Thus $(\tilde{h}_{\lambda\mu}\circ\varphi)_{(\emptyset)}$ is a loop map up to homotopy type. Furthermore $l(P_i, \emptyset)$ and $\iota_{(P_i)}\circ l(P_i)$ for $i=1, 2$ are all loop maps. Thus, up to homotopy type, M is a loop space and threre is a loop map f: $G(n-1)\to M$ so that $l_i\circ f\sim\iota_{\lambda(P_i)}\circ l(P_i)\sim l_i\circ\iota_{\lambda}$ for $i=1, 2$. But according to Hilton-Mislin-Roitberg [7; Th. 1], two maps g_i: $G(n-1)\to M$ $(i=1, 2)$ are mutually homotopic if and only if $l_i\circ g_1\sim l_i\circ g_2$ for $i=1, 2$. Thus $\iota_{\lambda}\sim f$ and the proposition is proved. Q. E. D.

§5. Zabrodsky's secondary operations and the proof of Proposition 3.2

In this section, let $d=1$, $G(n, 1)=SO(n)$, $n=2k+1$, $M(\lambda)=M(n, 1, \lambda)$ in

(1.2) and p be a prime.

LEMMA 5.1. *If* $\lambda\equiv 0 \bmod p$, *then we have the following isomorphism of algebras over the* mod p *Steenrod algebra* $\mathscr{A}$:

$$H^*(M(\lambda); Z_p) \cong H^*(S^{2k}; Z_p) \otimes H^*(SO(2k); Z_p).$$

PROOF. The case p is odd: Consider the bundle $SO(2k)\xrightarrow{\iota}SO(2k+1)\xrightarrow{\pi}S^{2k}$ in (1.2). We notice that (cf. [1; Prop. 10.2])

$$H^*(SO(2k); Z_p) = \Lambda(x_3, x_7, \dots, x_{4k-5}, e_{2k-1}),$$
$$H^*(SO(2k+1); Z_p) = \Lambda(y_3, y_7, \dots, y_{4k-5}, y_{4k-1}),$$

and $\iota^* y_i = x_i$ $(i \leqq 4k-5)$, $=0$ $(i=4k-1)$. Furthermore, in the Serre spectral sequence $\{E_r^{**}, d_r\}$ of mod p cohomology for the above bundle with $E_2^{**} = H^*(S^{2k}; Z_p)\otimes H^*(SO(2k); Z_p)$, d_r: $E_r^{s,t} \to E_r^{s+r,t-r+1}$ vanishes except for

$$d_{2k}(1\otimes e_{2k-1}a) = \xi \otimes a \quad (\xi \in H^{2k}(S^{2k}; Z_p), \text{ a generator; } a \in H^*(SO(2k); Z_p)).$$

Now, let $\{\tilde{E}_r^{**}, \tilde{d}_r\}$ be the spectral sequence for $SO(2k)\xrightarrow{\iota_\lambda}M(\lambda)\xrightarrow{\pi_\lambda}S^{2k}$ in (1.2) and h^*: $E_r^{**}\to\tilde{E}_r^{**}$ be the map induced by h_λ and $\tilde{h}_\lambda$ in (1.2). Then, $\tilde{E}_2^{**} = E_2^{**}$ and

$$h^*(1\otimes x_i) = 1 \otimes x_i, \quad h^*(1\otimes e_{2k-1}) = 1 \otimes e_{2k-1}, \quad h^*(\xi\otimes 1) = 0$$

because $\lambda\equiv 0 \bmod p$. So, $\tilde{d}_{2k}(1\otimes e_{2k-1}a) = h^* d_{2k}(1\otimes e_{2k-1}a) = h^*(\xi\otimes a) = 0$. Thus $\{E_r^{**}\}$ collapses, and we have the lemma.

The case $p=2$: Then (cf. [1; Prop. 10.3, (10.6)])

(5.2) $$H^*(SO(2k); Z_2) = Z_2[x_1, x_3, \dots, x_{2k-1}]/(x_i^{s(i)}: i = 1, 3, \dots, 2k-1),$$
$$H^*(SO(2k+1); Z_2) = Z_2[y_1, y_3, \dots, y_{2k-1}]/(y_i^{t(i)}: i = 1, 3, \dots, 2k-1)$$

and $\iota^* y_i = x_i$, where $s(i)$ (resp. $t(i)$) is the least power of 2 not less than $2k/i$ (resp. $(2k+1)/i$). Furthermore, if $2k = u(2v-1)$ and u is a power of 2, then $\pi^*(\xi) = (y_{2v-1})^u$ for a generator $\xi \in H^{2k}(S^{2k}; Z_2)$.

Now, put $z_i = \tilde{h}_\lambda^* y_i \in H^*(M(\lambda); Z_2)$. Then $\iota_\lambda^* z_i = x_i$. If $i \neq 2v-1$, then $s(i) = t(i)$ and $z_i^{s(i)} = \tilde{h}_\lambda^* y^{t(i)} = 0$. If $i = 2v-1$, then $s(i) = u$ and $z_i^{s(i)} = \tilde{h}_\lambda^*(y_{2v-1})^u = \tilde{h}_\lambda^* \pi^* \xi = \pi_\lambda^*(\lambda\xi) = 0$ since $\lambda \equiv 0 \bmod 2$. So, we can define an $\mathscr{A}$-algebra homomorphism

$$\varphi: H^*(SO(2k); Z_2) \longrightarrow H^*(M(\lambda); Z_2) \quad \text{by} \quad \varphi x_i = z_i \quad (i = 1, 3, \dots, 2k-1),$$

which satisfies $\iota_\lambda^* \circ \varphi = \mathrm{id}$. Hence, by the theorem of Leray-Hirsch, we have an $\mathscr{A}$-algebra isomorphism

$$\psi: H^*(S^{2k}; Z_2) \otimes H^*(SO(2k); Z_2) \cong H^*(M(\lambda); Z_2) \quad \text{by} \quad \psi(b\otimes a) = (\pi_\lambda^* b)(\phi a).$$

Q.E.D.

PROOF OF PROPOSITION 3.2 FOR ODD p. Suppose $\lambda\equiv 0 \mod p$. Then, $H^*(M(\lambda); Z_p)\cong H^*(S^{2k}; Z_p)\otimes H^*(SO(2k); Z_p)$ as algebras by Lemma 5.1, which admits no Hopf algebra structures by Borel's structure theorem. So, $M(\lambda)$ is not a mod p H-space. Q.E.D.

Now we consider the case $p=2$.

(5.3) ([16; Prop. 1.5]) *Let*

$$\tilde{X}_0 \xrightarrow{\iota} X_0 \xrightarrow{f} X, \quad E \longrightarrow \prod K(Z_2, m_s) \xrightarrow{h} \prod K(Z_2, l_t)$$

be fibrations such that X_0 and X are H-spaces, f is an H-map, the products are finite products and h is a loop map so that $\tilde{X}_0$ and E have the H-structures induced by H-maps f and h, respectively. Assume that

$$\text{(5.4)} \qquad \operatorname{Im}(f^*: H^*(X; Z_2) \longrightarrow H^*(X_0; Z_2)) \supset \textstyle\sum_{i<n} H^i(X_0; Z_2)$$

for some n with $n\geqq m_s$ and $2n\geqq l_t$. Then, for any map $g: X_0\to E$, the composition $g\circ\iota: \tilde{X}_0\to E$ is an H-map.

LEMMA 5.5. (i) *Let $f: X_0\to X$ be an H-map between H-spaces satisfying* (5.4) *for some n. Then, for any map $f': X_0\to K=K(Z_2, n)$, $X\times K$ has an H-structure so that $(f, f'): X_0\to X\times K$ is an H-map.*

(ii) *Let X_0 be an H-space and $X=\prod_{r=1}^t K(Z_2, n_r)$. If a map $f: X_0\to X$ satisfies* (5.4) *for some n with $n\geqq n_r$, then X has an H-structure so that f is an H-map.*

PROOF. (i) Let μ_0 and μ be H-structures of X_0 and X, respectively. Consider

$$D: X_0 \times X_0 \longrightarrow K \text{ given by } D(y, y') = f'(y')^{-1}f'(y)^{-1}f'(\mu_0(y, y')),$$

where $K=K(Z_2, n)$ is regarded to be a group. Then, $D|X_0\vee X_0\sim *$ and

$$D \sim \hat{D}\circ\pi: X_0 \times X_0 \xrightarrow{\pi} X_0 \wedge X_0 \xrightarrow{\hat{D}} K \text{ (π denotes the projection)}$$

for some map $\hat{D}$. By the assumption (5.4), $\hat{D}\in H^n(X_0\wedge X_0; Z_2)$ is contained in the image of $(f\wedge f)^*: H^n(X\wedge X; Z_2)\to H^n(X_0\wedge X_0; Z_2)$. So, we get a map

$$d: X \wedge X \longrightarrow K = K(Z_2, n) \quad \text{with} \quad d\circ(f\wedge f) \sim \hat{D}.$$

Thus, we see by definition that $X\times K$ has an H-structure

$$\mu': X \times K \times X \times K \longrightarrow X \times K \text{ given by } \mu'(x, k, x', k') = (\mu(x, x'), kk'd(x, x')),$$

and that $(f, f'): X_0\to X\times K$ is an H-map with respect to μ_0 and μ'.

(ii) We may assume that $n_r\leqq n_s$ if $r<s$. Put $K_s=\prod_{r=1}^s K(Z_2, n_r)$ and let

$f_s\colon X_0 \to K_s$ be the composition of f and the projection $X = K_t \to K_s$. Then, since $n_r \leqq n_s \leqq n$ for $r < s$, (5.4) shows that

$$\operatorname{Im}(f_{s-1}^*\colon H^*(K_{s-1}; Z_2) \longrightarrow H^*(X_0; Z_2)) \supset \sum_{i<n_s} H^i(X_0; Z_2).$$

So, we see that $K_s = K_{s-1} \times K(Z_2, n_s)$ has an H-structure so that $f_s\colon X_0 \to K_s$ is an H-map, by induction starting from $K_0 = *$ and by using (i). Thus, (ii) holds for $X = K_t$ and $f_t = f$. Q.E.D.

Now, let

$$K_0 = K(Z_2, 2k) \xrightarrow{h_0} K_1 = K(Z_2, 4k) \xrightarrow{h_1} K_2 = K(Z_2, 4k+1)$$

be the maps such that

$$h_0^* \iota_{4k} = (\iota_{2k})^2,\ h_1^* \iota_{4k+1} = Sq^1 \iota_{4k}\ (\iota_t \in H^t(K(Z_2, t); Z_2) \text{ is the fundamental class}).$$

Then, $h_0^* h_1^* \iota_{4k+1} = Sq^1(\iota_{2k})^2 = 0$ and so $h_1 \circ h_0$ is homotopic to $*$. Thus, we have the following 2-stage Postnikov system

$$\begin{array}{ccc} \Omega K_1 = K(Z_2, 4k-1) \xrightarrow{j} & E & \xrightarrow{\hat{h}_1} \Omega K_2 = K(Z_2, 4k) \\ & \downarrow r & \\ & K_0 \xrightarrow{h_0} K_1 \xrightarrow{h_1} K_2, & \end{array} \tag{5.6}$$

where $r\colon E \to K_0$ is the homotopy fibre of h_0, j is the natural map and $\hat{h}_1$ is the map induced from a homotopy of $h_1 \circ h_0$ to $*$ so that $\hat{h}_1 \circ j \sim \Omega h_1$. Then, A. Zabrodsky proved the following

(5.7) ([20; Lemma 3.4.1]) $\mu^* v = v \otimes 1 + 1 \otimes v + u \otimes u$

for $v = \hat{h}_1^* \iota_{4k}$ and $u = r^* \iota_{2k}$, where $\mu\colon E \times E \to E$ is the loop multiplication.

Proof of Proposition 3.2 for $p = 2$. Contrary to Proposition 3.2 for $p = 2$, suppose that $M(\lambda) = M(n, 1, \lambda)$ is a mod 2 H-space for even λ, where $n = 2k+1$. Then, we have an H-space X and a 2-equivalence $\varphi\colon X \to M(\lambda)$. According to Lemma 5.1 and (5.2), the algebra $H^*(X, Z_2)$ over $\mathscr{A}$ is given by

$$(*) \quad H^*(X; Z_2) = \Lambda(\zeta) \otimes Z_2[z_1, z_3, \dots, z_{2k-1}]/(z_i^{s(i)}\colon i = 1, 3, \dots, 2k-1),$$
$$\zeta = \varphi^* \pi_\lambda^* \xi, \quad z_i = \varphi^* \tilde{h}_\lambda^* y_i, \quad \pi_\lambda^* \xi \notin \operatorname{Im} \tilde{h}_\lambda^*,$$

where π_λ, ι_λ and $\tilde{h}_\lambda$ are the maps in (1.2), $\xi \in H^{2k}(S^{2k}; Z_2)$ is a generator and x_i, y_i and $s(i)$ are given in (5.2). Consider the map

$$f\colon X \longrightarrow K_0 = K(Z_2, 2k) \quad \text{with} \quad f^* \iota_{2k} = \zeta.$$

Then, $f^* h_0^* \iota_{4k} = f^*(\iota_{2k})^2 = \zeta^2 = 0$ and we have a lift

$$\tilde{f}\colon X \longrightarrow E \quad \text{with} \quad r\circ\tilde{f} \sim f \qquad \text{for} \quad r\colon E \longrightarrow K_0 \quad \text{in} \quad (5.6).$$

Furthermore, consider the fibering

$$\tilde{X} \xrightarrow{\ \iota\ } X \xrightarrow{\ g\ } \prod_{i=1}^{k} K(Z_2.\, 2i-1) \quad \text{with} \quad g^*\iota_{2i-1} = z_{2i-1}.$$

Then, by Lemma 5.5 (ii) and (5.3), $\tilde{X}$ is an H-space with multiplication $\tilde{\mu}$ so that $\tilde{f}\circ\iota\colon \tilde{X}\to E$ is an H-map. So, (5.7) shows that

$$(**) \qquad \tilde{\mu}^*(\tilde{f}\circ\iota)^*v = (\tilde{f}\circ\iota)^*v\otimes 1 + 1\otimes(\tilde{f}\circ\iota)^*v + (\tilde{f}\circ\iota)^*u\otimes(\tilde{f}\circ\iota)^*u,$$

where $(\tilde{f}\circ\iota)^*u = \iota^*\tilde{f}^*r^*\iota_{2k} = \iota^*\xi$. Now, by using $(*)$, we see that

$$\operatorname{Im} g^* = \operatorname{Im}(\varphi^*\circ\tilde{h}_\lambda^*) \not\ni \zeta, \quad \operatorname{Im}\iota^* \cong H^*(X; Z_2)//\operatorname{Im} g^* = \Lambda(\zeta) \text{ and } \iota^*\zeta \neq 0.$$

Furthermore, $\tilde{f}^*v \in H^{4k}(X; Z_2) = (\operatorname{Im} g^*)\cdot\tilde{H}^*(X; Z_2)$ and $(\tilde{f}\circ\iota)^*v = 0$. Thus, the left and the right hand sides of $(**)$ are zero and non-zero, respectively, which is a contradiction. So, Proposition 3.2 for $p=2$ is proved. Q.E.D.

§6. Proof of Proposition 3.11 (i)

The rest of this paper is devoted to prove Proposition 3.11.

Let p be an odd prime and $\mathscr{A}$ denotes the mod p Steenrod algebra.

LEMMA 6.1. *Let $m = p^s t$ with $t \not\equiv 0 \bmod p$. Then,*

$$\mathscr{P}^m = \sum_{i=0}^{s} \mathscr{P}^{p^i}\alpha_i \qquad \textit{for some } \alpha_i \in \mathscr{A}.$$

PROOF. When $t=1$, the equality is trivial.

Assume that $t \geqq 2$. If $s=0$, then $m = t \not\equiv 0 \bmod p$ and the Adem relation shows that

$$\mathscr{P}^1\mathscr{P}^{m-1} = m\mathscr{P}^m \quad \text{and} \quad \mathscr{P}^m = \mathscr{P}^1\alpha_0 \qquad \text{for} \quad \alpha_0 = m^{-1}\mathscr{P}^{m-1}.$$

Now, assume inductively that the equality is true for $s \leqq l-1$, and consider the case $m = qt$ with $q = p^l$, $l \geqq 1$. Consider the Adem relation

$$\mathscr{P}^q\mathscr{P}^{q(t-1)} = \sum_{i=0}^{q/p}(-1)^{i+q/p}a_i\mathscr{P}^{qs-i}\mathscr{P}^i, \quad a_i = \binom{(q(t-1)-i)(p-1)-1}{q-pi}.$$

Then, $a_0 \not\equiv 0 \bmod p$ because $q(t-1)(p-1)-1 = aq+q-1$, $a = (t-1)(p-1)-1 \equiv -t \not\equiv 0 \bmod p$ and $q = p^l$. Also, $qt - i \not\equiv 0 \bmod p^l$ for $0 < i \leqq q/p = p^{l-1}$. Thus, we see the equality by induction. Q.E.D.

In the rest of this section, the coefficient Z_p in cohomology is omitted to simplify the notation.

PROOF OF PROPOSITION 3.11 (i). By the assumption that $M(G, \lambda)$ is a mod p loop space, let $f: \Sigma M(G, \lambda) \to Y$ be the adjoint of a p-equivalence $M(G, \lambda) \to \Omega Y$. Contrary to (i), suppose that $\lambda \equiv 0 \bmod p$. Then, by (3.4–6),

(6.2) $$H^*(M(G, \lambda)) = \Lambda(e_1, \dots, e_k) \ (\dim e_i = 2n_i - 1), \quad e_k = \pi_\lambda^* \xi,$$
$$\tilde{h}_\lambda^* x_i \equiv e_i \ (\text{if } i < k), \ \equiv 0 \ (\text{if } i = k) \bmod D_M,$$

where $D_M = DH^*(M(G, \lambda))$ is the decomposable module. Furthermore,

(6.3) $$H^*(Y) = Z_p[y_1, \dots, y_k] \ (\dim y_i = 2n_i), \quad f^* y_i \equiv e_i \bmod D_M,$$

where $f^*: H^*(Y) \to H^*(\Sigma M(G, \lambda)) \cong H^{*-1}(M(G, \lambda))$. So, for any $t < 2n_k$,

$$f^* \tilde{\mathscr{A}}(H^t(Y)) = \tilde{\mathscr{A}}(f^* H^t(Y)) \subset \tilde{\mathscr{A}}(H^{t-1}(M(G, \lambda)))$$
$$= \tilde{\mathscr{A}}(\tilde{h}_\lambda^* H^{t-1}(G)) \subset \tilde{h}_\lambda^* \tilde{\mathscr{A}}(H^*(G)),$$

where $\tilde{\mathscr{A}}$ is the augmentation ideal of $\mathscr{A}$, since $\tilde{h}_\lambda^*: H^{t-1}(G) \cong H^{t-1}(M(G, \lambda))$. Furthermore $e_k \notin \operatorname{Im} \tilde{h}_\lambda^* + D_M$. Thus $e_k \notin f^* \tilde{\mathscr{A}}(H^*(Y)) + D_M$ and

$$y_k \notin \tilde{\mathscr{A}}(H^*(Y)) + D_Y \ (D_Y = DH^*(Y)).$$

Now, by changing generators except for y_k, we may assume that

(6.4) $$\tilde{\mathscr{A}}(H^*(Y)) \subset Z_p\{1, y_1, \dots, y_{k-1}\} + D_Y.$$

Since $n_k = p^a b$ and $b \not\equiv 0 \bmod p$, Lemma 6.1 implies that

(6.5) $$y_k^p = \mathscr{P}^{n_k} y_k = \sum_{i=0}^a \mathscr{P}^{p^i} \alpha_i y_k \qquad \text{for some} \quad \alpha_i \in \mathscr{A}.$$

On the other hand, we see the following

(6.6) *For any* $u \in H^*(Y)$ *with* $\dim u > 2n_k(p-1)$, $\mathscr{P}^j u$ $(j > 0)$ *is a polynomial in* $y_1, \dots, y_k$ *without including the term* y_k^p.

In fact, $\dim u > 2n_k(p-1)$ implies that $u \in D_Y^{(p)}$ where

(6.7) $D_Y^{(t)} = D^{(t)} H^*(Y)$ is given by $D_Y^{(2)} = D_Y$ and $D_Y^{(t+1)} = D_Y^{(t)} \cdot \tilde{H}^*(Y)$ $(t \geqq 2)$.

So, $u \equiv \sum c y_{i_1} \cdots y_{i_p}$ $(c \in Z_p)$ mod $D_Y^{(p+1)}$ and

$$\mathscr{P}^j u \equiv \sum_{l_1 + \cdots + l_p = j} c \mathscr{P}^{l_1} y_{i_1} \cdots \mathscr{P}^{l_p} y_{i_p} \bmod D_Y^{(p+1)}.$$

Hence $\mathscr{P}^j u$ does not contain y_k^p by (6.4).

Now, $\dim \alpha_i y_k = 2n_k p - 2p^i(p-1) > 2n_k(p-1)$ since $b > p$ in (6.5). So, y_k^p does not appear in the right hand side of (6.5), which is a contradiction. Therefore, (i) is proved. Q. E. D.

§7. BP-theory and the Landweber-Novikov operation

In this section, we summarize the known facts on the BP-theory and prove Proposition 7.7, which are used to prove Proposition 3.11 (ii) in §8. The main references are [8] and [9].

Let p be an odd prime and $Z_{(p)}$ be the integers localized at p. Then,

$$BP^* = Z_{(p)}[v_1, v_2,\dots], \quad |v_i| = \dim v_i = -2(p^i-1).$$

(7.1) (cf. e.g. [8]) *Let Y have the homotopy type of a CW-complex of finite type. Then, the BP-cohomology BP^*Y at p of Y is a module over BP^*. Furthermore, the Thom map*

$$T\colon BP^*Y \longrightarrow H^*(Y; Z_p) \quad (\textit{which is a ring homomorphism})$$

is epimorphic and $\ker T=(p, v_1, v_2,\dots)$ *(the ideal generated by* $\{p, v_1, v_2,\dots\}$*), if $H^*(Y; Z)$ has no p-torsion.*

Let $E=(e_1, e_2,\dots)$ be an exponential sequence, i.e., a sequence of integers $e_i \geqq 0$ being 0 except for a finite number of i. Then, we have the Landweber-Novikov operation

$$r_E \in BP^*BP \quad \text{with} \quad \deg r_E = |r_E| = |E| = 2\sum e_i(p^i-1).$$

(7.2) ([8; (1.1)]) *r_E acts on BP^*Y so that the diagram*

$$\begin{array}{ccc} BP^*Y & \xrightarrow{\ r_E\ } & BP^*Y \\ \downarrow T & & \downarrow T \\ H^*(Y; Z_p) & \xrightarrow{\ \chi(\mathscr{P}^E)\ } & H^*(Y; Z_p) \end{array}$$

is commutative, where $\chi\colon \mathscr{A}\to\mathscr{A}$ is the canonical anti-automorphism on the mod p *Steenrod algebra $\mathscr{A}$.*

(7.3) ([9; (2.1)]) *Put $v_0=p$ and $t\Delta_i=(0,\dots, 0, t, 0,\dots)$ where t is in the i-th position. Then,*

$$r_E v_n \equiv v_{n-i} \ \textit{if}\ E = p^{n-i}\Delta_i, \ \equiv 0 \ \textit{otherwise}, \ \bmod\,(p, v_1, v_2,\dots)^2;$$

$$r_E((p, v_1, v_2,\dots)^n) \subset (p^n, v_1, v_2,\dots) \ (\text{cf. } [9; (2.3)]).$$

In the rest of this section, we concern mainly with the following composition law:

(7.4) ([8; (1. 2)]) $r_E r_F \equiv \sum_{R(X)=F,S(X)=E} b(X) r_{T(X)} \bmod (v_1, v_2,\dots),$

where X ranges over all matrices (x_{ij}) $(i, j=0, 1,\dots)$ being omitted the term x_{00}

and consisting of integers x_{ij} which are 0 except for a finite number of (i, j). Furthermore, for such a matrix $X=(x_{ij})$, the exponential sequences

$$R(X)=(r_1, r_2,\dots), \quad S(X)=(s_1, s_2,\dots), \quad T(X)=(t_1, t_2,\dots)$$

and $b(X)\in Z$ are defined as follows:

$$r_i = \sum_j p^j x_{ij}, \quad s_j = \sum_i x_{ij}, \quad t_n = \sum_{i+j=n} x_{ij}, \quad b(X) = \prod(t_n!)/\prod(x_{ij}!).$$

For exponential sequences $E=(e_1, e_2,\dots)$ and $F=(f_1, f_2,\dots)$, put $E+E=(e_1+f_1, e_2+f_2,\dots)$. Also, put $E-F=(e_1-f_1, e_2-f_2,\dots)$ if $e_i\geqq f_i$ for any i. Furthermore, a linear ordering $E<F$ is defined in [8] as follows:

(7.5) $E<F$ if and only if (1) $|E|<|F|$, or
(2) $|E|=|F|$ and $e_i=f_i$ if $i>s$ while $e_s>f_s$, for some s.

Lemma 7.6. *$|T(X)|=|R(X)+S(X)|$ and $T(X)\leqq R(X)+S(X)$ in* (7.4).

Proof. If $x_{ij}=0$ for $ij\neq 0$, then $r_i=x_{i0}$, $s_i=x_{0i}$ and $t_i=x_{i0}+x_{0i}=r_i+s_i$. So $T(X)=R(X)+S(X)$. Assume that $x_{ij}\neq 0$ for some $ij\neq 0$, and let $x_{ab}\neq 0$ $(ab\neq 0)$ and $x_{ij}=0$ for $i+j>a+b$ and $ij\neq 0$. Then, for $i\geqq a+b$, we have $r_i=x_{i0}$, $s_i=x_{0i}$ and

$$t_i = x_{i0} + x_{oi} = r_i + s_i \quad \text{if} \quad i > a + b, \; t_i = x_{i0} + x_{0i} > r_i + s_i \quad \text{if} \quad i = a + b.$$

So $T(X)<R(X)+S(X)$. $|T(X)|=|R(X)+S(X)|$ is clear by definition. Q.E.D.

Now, we prove the following decomposition formula of pr_{p^m} by using (7.4), where $r_t=r_{t\Delta_1}$:

Proposition 7.7. *Let $m\geqq 1$. Then,*

$$pr_{p^m} \equiv \sum r_{E_s}\theta_s \mod (p^2, v_1, v_2,\dots)$$

for some $\theta_s\in BP^*BP$ and some exponential sequences E_s with

(1) *$|E_s|<4p$ if $m=1$, $|E_s|<2p^m$ if $m\geqq 2$; and*
(2) *$E_s\neq\Delta_i$ for all $i\geqq 1$.*

To prove this proposition, we notice the following

Lemma 7.8. *Let $m\geqq 2$ and E be an exponential sequence with $|E|=2p^m(p-1)$ and $E\neq p^m\Delta_1$.* Then

$$r_E \equiv r_{E_1}\theta_1 + \sum a_F r_F \equiv \sum r_{E_s}\theta_s \mod (p^2, v_1, v_2,\dots),$$

where $\theta_s\in BP^*BP$, E_s satisfies (1) *for $m\geqq 2$ and* (2) *in Proposition* 7.7, *$a_F\in Z_{(p)}$, $|F|=|E|$ and $F<E$.*

PROOF OF PROPOSITION 7.7 FROM LEMMA 7.8. The case $m=1$: By (7.4), we have $r_2 r_{p-2} \equiv \binom{p}{2} r_p \bmod (v_1, v_2, \ldots)$ and hence

$$pr_p \equiv -2r_2 r_{p-2} \bmod (p^2, v_1, v_2, \ldots).$$

Since $|2\Delta_1| = 4(p-1) < 4p$ and $2\Delta_1 \neq \Delta_i$, this is the desired formula.

The case $m \geqq 2$: Put $q = p^{m-1}$. Then, (7.4) shows that

$$(*) \qquad r_q r_{pq-q} \equiv \sum_{t=0}^{q-q/p} a_t r_{(pq-tp-t,t)}, \ a_t = \binom{pq-tp-t}{q-t}, \bmod (v_1, v_2, \ldots).$$

Here, the term for $t=0$ is

$$a_0 r_{pq} = p\binom{pq-1}{q-1} r_{pq} \equiv pr_{pq} \bmod p^2.$$

Also, in the left hand side of $(*)$, $q\Delta_1$ satisfies (1) and (2), i.e., $|q\Delta_1| = 2q(p-1) < 2pq$ and $q\Delta_1 \neq \Delta_i$ for all i. Furthermore, if $t \geqq 1$, then $E = (pq-tp-t, t)$ in the right hand side of $(*)$ satisfies $|E| = 2pq(p-1)$ and $E \neq pq\Delta_1$, and hence $r_{(pq-tp-t,t)}$ is decomposed into the form given in Lemma 7.8. Therefore, $(*)$ implies the desired formula. Q.E.D.

PROOF OF LEMMA 7.8. We prove the first congruence. Then, it can be applied also for r_F there, since $F < E$ with $|F| = |E|$ also satisfies the assumption of the lemma. Also for E, the number of F's with $F < E$ and $|F| = |E|$ is finite. Therefore, we see the second congruence using the first one finite times.

Let $E = (e_1, e_2, \ldots)$ satisfy $|E| = 2p^m(p-1)$ and $E \neq p^m\Delta_1$. If $e_t \neq 0$, then $2(p^t-1) \leqq |E| = 2p^m(p-1)$ and so $t \leqq m$. Suppose $e_t \leqq 1$ for all t. Then $2p^m(p-1) = |E| \leqq 2\sum_{t=1}^{m}(p^t-1) < 4p^m$, which contradicts $p \geqq 3$. Therefore $e_t \geqq 2$ for some t. Let $e_t = \sum u_i p^i$ be the p-adic expansion. Then, $u_i \neq 0$ for some $i \geqq 1$ or $u_0 \geqq 2$.

Assume $u_i \neq 0$ for some $i \geqq 1$. Then, $2p^i(p^t-1) \leqq |E| = 2p^m(p-1)$ and so $i+t \leqq m$ or $(i, t) = (m, 1)$. If $(i, t) = (m, 1)$, then $E = p^m\Delta_1$ which contradicts the assumption. Thus $i+t \leqq m$. Now, (7.4) shows that

$$r_{p^i\Delta_t} r_{E-p^i\Delta_t} \equiv \binom{e_t}{p^i} r_E + \sum a_F r_F \bmod (v_1, v_2, \ldots),$$

where $a_F \in Z$, $|F| = |E|$ and $F < E$ by Lemma 7.6. Here, $\binom{e_t}{p^i} \not\equiv 0 \bmod p$ since $u_i \neq 0$. Furthermore, $|p^i\Delta_t| = 2p^i(p^t-1) < 2p^m$ since $i+t \leqq m$. So, we see the desired congruence.

Assume $u_0 \geqq 2$. Then, (7.4) and Lemma 7.6 show that

$$r_{2\Delta_1} r_{E-2\Delta_1} \equiv \binom{e_t}{2} r_E + \sum a_F r_F \bmod (v_1, v_2, \ldots),$$

where $a_F \in Z$, $|F|=|E|$ and $F<E$. Since $\binom{e_t}{2} \not\equiv 0 \bmod p$ by $u_0 \geqq 2$, this shows the desired congruence. Q.E.D.

§ 8. Proof of Proposition 3.11 (ii)

In this section, we assume that G is a simply connected finite mod p loop space and $H^*(G; Z)$ has no p-torsion (p: odd prime), and that

$$\lambda \equiv 0 \bmod p \quad \text{and} \quad M = M(G, \lambda) \simeq_p \Omega Y \qquad \text{for some} \quad Y.$$

We continue to use the notations given in (3.4–6) and (6.2–3), and the coefficient Z_p in cohomology is omitted.

LEMMA 8.1.
$$BP^*G = \Lambda_{BP^*}(\bar{g}_1, \dots, \bar{g}_k) \qquad (\dim \bar{g}_i = 2n_i - 1),$$
$$BP^*M = \Lambda_{BP^*}(\bar{e}_1, \dots, \bar{e}_k) \qquad (\dim \bar{e}_i = 2n_i - 1),$$
$$BP^*Y = BP^*[[\bar{y}_1, \dots, \bar{y}_k]] \qquad (\dim \bar{y}_i = 2n_i)$$

($BP^*[[\ \ ]]$ denotes the ring of power series), and the generators $\bar{g}_i$, $\bar{e}_i$ and $\bar{y}_i$ can be taken to satisfy $T\bar{g}_i = g_i$, $T\bar{e}_i = e_i$, $T\bar{y}_i = y_i$,

$$\tilde{h}_\lambda^* \bar{g}_i \equiv \begin{cases} \bar{e}_i & (\text{if } i<k), \\ \lambda \bar{e}_k & (\text{if } i=k), \end{cases} \qquad f^* \bar{y}_i \equiv \bar{e}_i \bmod \bar{D}_M = DBP^*M,$$

and $\pi_\lambda^* \bar{\xi} = \bar{e}_k$, where T denotes the Thom map and $BP^*S^m = \Lambda_{BP^*}(\bar{\xi})$, $T\bar{\xi} = \xi$ ($m = 2n_{k-1}$).

PROOF. We notice that (7.1) is valid for G, M and Y.

Take $\bar{g}_i \in BP^*G$ with $T\bar{g}_i = g_i \in H^*(G)$ for $i<k$, and put $\bar{g}_k = \pi^* \bar{\xi} \in BP^*G$. Then $T\bar{g}_k = \pi^* \xi = g_k$ by (3.5). We have $\bar{g}_i^2 = 0$ since $\dim \bar{g}_i$ is odd. So, we see the equality for G by (7.1).

In the second place, we define $\bar{e}_i \in BP^*M$ inductively. Put $\bar{e}_1 = \tilde{h}_\lambda^* \bar{g}_1$. Then $T\bar{e}_1 = \tilde{h}_\lambda^* g_1 = e_1$ by (6.2), since $DH^{2n_1-1}(M) = 0$. Let $j>1$ and assume that $\bar{e}_i$ is defined for any $i<j$ so that $T\bar{e}_i = e_i$ and $\tilde{h}_\lambda^* \bar{g}_i \equiv \bar{e}_i \bmod \bar{D}_M$. If $j<k$, then $T\tilde{h}_\lambda^* \bar{g}_j = \tilde{h}_\lambda^* g_j = e_j + d_j$ for some $d_j \in D_M$, and d_j is a polynomial of e_i ($i<j$) by (6.2). So, we can take $\bar{d}_j \in \bar{D}_M$ such that $T\bar{d}_j = d_j$ by the inductive assumption. Put $\bar{e}_j = \tilde{h}_\lambda^* \bar{g}_j - \bar{d}_j$. Then, $T\bar{e}_j = e_j$ and $\tilde{h}_\lambda^* \bar{g}_j \equiv \bar{e}_j \bmod \bar{D}_M$ as desired. When $j=k$, put $\bar{e}_k = \pi_\lambda^* \bar{\xi}$. Then, $\tilde{h}_\lambda^* \bar{g}_k = \pi_\lambda^* h_\lambda^* \bar{\xi} = \lambda \pi_\lambda^* \bar{\xi} = \lambda \bar{e}_k$ and $T\bar{e}_k = \pi_\lambda^* \xi = e_k$. Thus, we have defined $\bar{e}_i$ and the equality for M holds by the same reason as that for G.

Finally, we define $\bar{y}_i \in BP^*Y$ inductively. Take $\tilde{y}_i \in BP^*Y$ with $T\tilde{y}_i = y_i \in H^*(Y)$ for any i. Then $Tf^*\tilde{y}_i = f^* y_i \equiv e_i \bmod D_M$. Let $0 = l(0) < l(1) < \cdots < l(t) < l(t+1) = k$ be the sequence of integers such that $n_i = n_{l(s)}$ for $l(s-1) < i \leqq l(s)$. By the equality for M,

$$BP^m M/DBP^m M \cong Z_{(p)}\{\bar{e}_i \mid l(t) < i \leqq k\} \quad (m = 2n_k - 1)$$

because $|v_i| < 0$ for $i > 0$. Therefore, for any i with $l(t) < i \leqq k$,

$$f^*\tilde{y}_i \equiv \sum a_{ij}\bar{e}_j \bmod \bar{D}_M \quad (l(t) < j \leqq k),$$

where $a_{ij} \in Z_{(p)}$ and $a_{ij} \equiv \delta_{ij}$ (the Kronecker delta) mod p. Consider the matrix $A = (a_{ij})$. Then $\det A \equiv 1 \bmod p$ and we have the inverse matrix $A^{-1} = (b_{ij})$. Since A is the identity matrix mod p, so is A^{-1} and $b_{ij} \equiv \delta_{ij}$ mod p. Now, put

$$\bar{y}_i = \sum_j b_{ij}\tilde{y}_j \qquad \text{for} \quad l(t) < i \leqq k.$$

Then, we see that $f^*\bar{y}_i \equiv \bar{e}_i \bmod \bar{D}_M$ and $T\bar{y}_i = T\tilde{y}_i = y_i$.

Suppose inductively that $\bar{y}_i \in BP^*Y$ is defined for any $i > l(s)$ $(s \leqq t)$ so that $T\bar{y}_i = y_i$ and $f^*\bar{y}_i \equiv \bar{e}_i \bmod \bar{D}_M$. By the equality for M, $BP^{m'}M/DBP^{m'}M$ $(m' = 2n_{l(s)} - 1)$ is isomorphic to

$$Z_{(p)}\{\bar{e}_j,\, u_i\bar{e}_i \mid l(s-1) < j \leqq l(s) < i,\, u_i \in \widetilde{BP}^*,\, |u_i| + 2n_i - 1 = m'\}.$$

So, for any i with $l(s-1) < j \leqq l(s)$,

$$f^*\tilde{y}_j \equiv \sum_{j'} a_{jj'}\bar{e}_{j'} + \sum_i c_{ji}u_i\bar{e}_i \bmod \bar{D}_M \quad (l(s-1) < j' \leqq l(s) < i),$$

where $a_{jj'}$, $c_{ji} \in Z_{(p)}$ and $a_{jj'} \equiv \delta_{jj'}$ mod p. Hence

$$f^*(\tilde{y}_j - \sum_i c_{ji}u_i\bar{y}_i) \equiv \sum_{j'} a_{jj'}\bar{e}_{j'} \bmod \bar{D}_M$$

since $f^*y_i \equiv \bar{e}_i \bmod \bar{D}_M$ for $i > l(s)$. Therefore, by the same argument as above, we can obtain $\bar{y}_j$ $(l(s-1) < j \leqq l(s))$ from $\tilde{y}_j$ so that $f^*\bar{y}_j \equiv \bar{e}_j \bmod \bar{D}_M$ and $T\bar{y}_j = y_j$.

Thus, we have defined $\bar{y}_i$ and the equality $BP^*Y = BP^*[[\bar{y}_1, \ldots, \bar{y}_k]]$ is seen by (7.1). Q.E.D.

Now we assume that

(8.2) $\qquad n_k = p^a b,\ 1 \leqq b < p \quad \text{and} \quad g_k \notin \tilde{\mathscr{A}}(H^*(G)) = \tilde{\mathscr{A}}(H^*(G; Z_p)),$

which is the assumption in Proposition 3.11 (ii). We may also assume that

(8.3) $\qquad \tilde{\mathscr{A}}(H^*(G)) \subset Z_p\{1, g_1, \ldots, g_{k-1}\} + D_G \quad (D_G = DH^*(G))$

by changing generators g_i except for g_k.

LEMMA 8.4. *$r_E\bar{y}_i \in BP^*\{1, \bar{y}_1, \ldots, \bar{y}_{k-1}\} + \bar{D}_Y + (p^2, v_1, v_2, \ldots)$ for any $i < k$, where $\bar{D}_Y = DBP^*Y$.*

PROOF. Since $i < k$, $Tr_E\bar{g}_i = \chi(\mathscr{P}^E)g_i \in Z_p\{1, g_1, \ldots, g_{k-1}\} + D_G$ by (7.2), Lemma 8.1 and (8.3). Hence, by (7.1),

$$r_E\bar{g}_i \equiv c\bar{g}_k \bmod BP^*\{1, \bar{g}_1, \ldots, \bar{g}_{k-1}\} + \bar{D}_G + (p^2, v_1, v_2, \ldots) \quad (\bar{D}_G = DBP^*G),$$

where $c\equiv 0 \bmod p$. So, $f^*r_E\bar{y}_i\equiv r_E\bar{e}_i\equiv \tilde{h}_\lambda^* r_E\bar{g}_i\equiv c\lambda\bar{g}_k\equiv 0 \bmod BP^*\{1, \bar{e}_1,\dots, \bar{e}_{k-1}\}+\bar{D}_M+(p^2, v_1, v_2,\dots)$, since $\lambda\equiv 0 \bmod p$. This shows the lemma since $\operatorname{Ker} f^*=\bar{D}_Y$ by Lemma 8.1. Q.E.D.

PROOF OF PROPOSITION 3.11 (ii). In addition to the assumptions stated in the beginning of this section and in (8.2), we assume that $p<n_k$. Then, we arrive at a contradiction as is seen below; and so we see Proposition 3.11 (ii).

We notice that $a\geqq 1$ by (8.2) and $p<n_k$. Now, in the right hand side of (6.5), $\dim \alpha_i y_k>2n_k(p-1)$ if $i<a$. So, by (6.6) and (6.5), $\mathscr{P}^{p^a}\alpha_a y_k$ includes y_k^p. On the other hand,

$$H^n(Y) = D^{(p)}H^n(Y) \equiv N + Z_p\{y_k^p\} \bmod D^{(p+1)}H^n(Y) \qquad \text{for} \quad n = 2n_k p,$$

where $N=Z_p\{y_{i_1}\cdots y_{i_p} \mid l<i_1\leqq\cdots\leqq i_p\leqq k \text{ and } i_1<k\}$ for l with $n_l<n_{l+1}=n_k$. So,

$$y_k^p \equiv \mathscr{P}^{p^a}\alpha_a y_k \bmod N + D^{(p+1)}H^*(Y).$$

Here, $\mathscr{P}^{p^a}=-\chi(\mathscr{P}^{p^a})+\sum_{j=1}^{p^a-1}\mathscr{P}^j\chi(\mathscr{P}^{p^a-j})$ and we see that $\mathscr{P}^j\chi(\mathscr{P}^{p^a-j})\alpha_a y_k$ does not include y_k^p for $0<j<p^a$ by Lemma 6.1 and (6.6). Therefore, $y_k^p\equiv -\chi(\mathscr{P}^{p^a})\alpha_a y_k \bmod N+D^{(p+1)}H^*(Y)$. This implies that

$$\bar{y}_k^p \equiv r_{p^a}\bar{z} \bmod \bar{N} + \bar{D}_Y^{(p+1)} + (p, v_1, v_2,\dots) \quad (\bar{D}_Y^{(t)}=D^{(t)}BP^*Y)$$

by (7.2) and (7.1), where $\bar{N}=BP^*\{\bar{y}_{i_1}\cdots\bar{y}_{i_p} \mid l<i_1\leqq\cdots\leqq i_p\leqq k \text{ and } i_1<k\}$. Applying Proposition 7.7 to this equality, we have

$$(8.5) \qquad p\bar{y}_k^p \equiv pr_{p^a}\bar{z} \equiv \sum r_{E_s}\theta_s\bar{z} \mod \bar{N} + \bar{D}_Y^{(p+1)} + (p^2, v_1, v_2,\dots),$$

where $|E_s|<4p$ if $a=1$, $|E_s|<2p^a$ if $a\geqq 2$ and $E_s\neq \Delta_i$ for all $i\geqq 1$. We remark that $(a, b)\neq(1, 1)$ since $p^a b=n_k>p>b$ by assumption. Now, in (8.5),

$$\dim\theta_s\bar{z} = \dim\bar{y}_k^p - |E_s| = 2n_k p - |E_s| > 2n_k(p-1),$$

since $2n_k=2p^a b\geqq 4p$ if $a=1$ and $b>1$. Thus $\theta_s\bar{z}\in\bar{D}_Y^{(p)}$ by the dimensional reason and $|v_i|<0$ for $i>0$. Therefore, we may write as follows:

$$\theta_s\bar{z} \equiv \bar{w} + p\bar{w}_0 + \sum v_i\bar{w}_i \mod (p, v_1, v_2,\dots)^2,$$

where $\bar{w}, \bar{w}_0, \bar{w}_i\in D^{(p)}Z_{(p)}[\bar{y}_1,\dots, \bar{y}_k]$. Thus, we see that

$$r_{E_s}\theta_s\bar{z} \equiv r_{E_s}\bar{w} + pr_{E_s}\bar{w}_0 + p\sum_{e_i>0} r_{E_s-\Delta_i}\bar{w}_i \mod (p^2, v_1, v_2,\dots)$$

for $E_s=(e_1, e_2,\dots)$, by (7.3) and the Cartan formula $r_F(\bar{u}_1\bar{u}_2)=\sum_{F_1+F_2=F}(r_{F_1}\bar{u}_1)(r_{F_2}\bar{u}_2)$ for the Landweber-Novikov operation (cf. e.g. [8]). Here, $|E_s-\Delta_i|\neq 0$ for any i with $e_i>0$ since $E_s\neq\Delta_i$. Therefore, we have

$$r_{E_s}\theta_s\bar{z}\in\bar{N} + D_Y^{(p+1)} + (p^2, v_1, v_2,\dots)$$

by Lemma 8.4 and $\bar{w}, \bar{w}_0, \bar{w}_i \in D^{(p)}Z_{(p)}[\bar{y}_1, \dots, \bar{y}_k]$. This contradicts (8.5); and Proposition 3.11 (ii) is proved completely. Q. E. D.

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