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Quadratic extensions of quasi-pythagorean fields

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Let F be a field of characteristic different from 2 and K be a quadratic extension of F. We let $N: K \to F$ be the norm map and $R(F)$ (resp. $R(K)$) be Kaplansky's radical of F (resp. K). Formerly we proposed the following conjecture: Is $N^{-1}(R(F))$ equal to $\dot{F} \cdot R(K)$? In [3], we gave a necessary and sufficient condition under which both F and K are quasi-pythagorean (see §1) and showed that the conjecture is true in this case.

The purpose of this paper is to show that the conjecture is true, whenever F is quasi-pythagorean and satisfies the finiteness condition for the space of orderings (see Theorem 6.1).

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§1. Quasi-pythagorean fields

Throughout this paper, F shall be a field of characteristic not equal to 2. First we recall a few basic notation. For a field F, WF shall denote the Witt ring of F consisting of the Witt classes of all quadratic forms over F, and IF shall denote the fundamental ideal in WF consisting of the Witt classes of all even-dimensional forms. The notation $\langle a_1, \dots, a_n \rangle$ shall mean the diagonal form $a_1 x_1^2 + \cdots + a_n x_n^2$, where $a_i \in \dot{F} := F \setminus \{0\}$. The nth power of the fundamental ideal shall be denoted by $I^n F$; it is additively generated by the n-fold Pfister forms $\langle\langle a_1, \dots, a_n \rangle\rangle := \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$. For a form $f = \langle a_1, \dots, a_n \rangle$, we define $D_F(f)$ to be the set $\{\sum a_i x_i^2 \neq 0; x_i \in F\}$. We note that if $n \geqq 2$, then $D_F\langle a_1, \dots, a_n \rangle = D_F\langle r_1 a_1, \dots, r_n a_n \rangle$ for $r_i \in R(F)$, where $R(F)$ is Kaplansky's radical of F. We also note that, for a Pfister form ρ and $x \in \dot{F}$, $x \in D_F(\rho)$ if and only if $\rho \otimes \langle\langle -x \rangle\rangle$ is isotropic.

As in [4], a field F is called quasi-pythagorean if $R(F) = D_F(2)$. It can be shown that F is quasi-pythagorean if and only if $I^2 F$ is torsion free. In [3], the subgroup H_a of F is defined by $H_a = \{x \in \dot{F}; D_F\langle 1, -x \rangle D_F\langle 1, -ax \rangle = \dot{F}\}$ and, in case F is quasi-pythagorean, it is shown that H_a is a subgroup of $D_F\langle 1, a \rangle$.

PROPOSITION 1.1. *Let F be a quasi-pythagorean field and $K = F(\sqrt{a})$ be a quadratic extension of F. Then the following statements are equivalent*:

(1) $N^{-1}(R(F)) = \dot{F} \cdot R(K)$, *where N is the norm map $N: \to F$.*

(2) $D_F\langle 1, a\rangle R(K)=D_K(2)$.

PROOF. We first note that $R(K)\cap \dot{F}=H_a$ by [3], Proposition 1.4 and Proposition 1.9. We also have $D_K(2)\cap \dot{F}=D_F\langle 1, a\rangle$ by [1], Lemma 3.5 and $N^{-1}(R(F))=\dot{F}\cdot D_K(2)$ by the norm principle ([2], 2.13). We assume that $N^{-1}(R(F))=\dot{F}\cdot D_K(2)=\dot{F}\cdot R(K)$. Then $D_K(2)\subseteq \dot{F}\cdot R(K)$. Thus for any $\alpha\in D_K(2)$, there exist $f\in\dot{F}$ and $\beta\in R(K)$ such that $\alpha=f\beta$; and so $f=\alpha\beta^{-1}\in D_K(2)\cap\dot{F}=D_F\langle 1, a\rangle$, which implies $\alpha=f\beta\in D_F\langle 1, a\rangle R(K)$. The reverse inclusion is clear and we have $D_F\langle 1, a\rangle R(K)=D_K(2)$. Conversely we assume $D_F\langle 1, a\rangle R(K)=D_K(2)$. Then $\dot{F}\cdot R(K)\supseteq D_K(2)$ and we have $\dot{F}\cdot R(K)=\dot{F}\cdot D_K(2)=N^{-1}(R(F))$.

Q. E. D.

PROPOSITION 1.2. *Let F be a quasi-pythagorean field and $K=F(\sqrt{a})$ be a quadratic extension of F. Further we assume that $R(F)$ is of finite index. Then the following statements hold*:

(1) *If* $\dim D_K(2)/R(K)\leqq\dim D_F\langle 1, a\rangle/H_a$, *then* $N^{-1}(R(F))=\dot{F}\cdot R(K)$.

(2) *For* $d_1,\dots, d_m\in\dot{F}$, *if F is generated by the set* $\{d_1,\dots, d_m, -1\}\cup R(F)$, *then* $R(K)=D_K(2)\cap(\bigcap_{i=1,\dots,m}D_K\langle 1, -d_i\rangle)$.

PROOF. Since $D_K(2)\cap\dot{F}=D_F\langle 1, a\rangle$ and $R(K)\cap\dot{F}=H_a$, we have a canoncial injection $\varphi: D_F\langle 1, a\rangle/H_a\to D_K(2)/R(K)$; hence $\dim D_K(2)\geqq\dim D_F\langle 1, a\rangle/H_a$. Therefore if $\dim D_K(2)/R(K)\leqq\dim D_F\langle 1, a\rangle/H_a$, then φ is bijective and this implies $D_F\langle 1, a\rangle R(K)=D_K(2)$. Thus the assertion (1) follows from Proposition 1.1.

Next we proceed to the assertion (2). It is easy to show that $D_K\langle 1, -x\rangle\cap D_K\langle 1, -y\rangle\subseteq D_K\langle 1, -xy\rangle$ for any $x, y\in\dot{F}$. We also have $R(K)=\bigcap_{x\in\dot{F}}D_K\langle 1, -x\rangle$ by [3], Proposition 1.10. The assertion (2) follows from these facts.

Q. E. D.

By a preordering of a field F, we mean a subgroup $P\subsetneqq\dot{F}$ such that $P+P\subseteq P$ and $\dot{F}^2\subseteq P$. A preordering P is called an ordering if P is of index 2 in $\dot{F}$. Let P be a preordering of F. We denote by $X(F)$ the space of orderings of F and by $X(F/P)$ the subspace of all orderings σ with $P(\sigma)\supseteq P$, where $P(\sigma)$ is the positive cone of σ.

We say that two forms f, g are P-isometric or isometric over P (in symbols $f\cong_P g$) if f, g have the same dimension and the same signature with respect to any $\sigma\in X(F/P)$. For a form $f=\langle a_1,\dots, a_n\rangle$ and $b\in F$ (which may be zero), if there exist $p_1,\dots, p_n\in P\cup\{0\}$ such that $a_1p_1+\cdots+a_np_n=b$ and $(p_1,\dots, p_n)\neq(0,\dots, 0)$, then we say that the form f represents b over P. We put $D_P(f)=\{b\in\dot{F}; f$ represents b over $P\}$. We say that f is P-isotropic or f is isotropic over P if f represents 0 over P and P-anisotropic or anisotropic over P otherwise. For two forms f, g, we say that f contains a subform g over P if f is

P-isometric to $g \perp h$ for some form h. In this case such a form h is uniquely determined and we write $h=f-g$.

For a form f, it is well known that f is P-isotropic if and only if $\pm x \in D_P(f)$ for some $x \in \dot{F}$. Let F be a formally real, quasi-pythagorean field. Then $R(F)$ is a preordering of F, and we can see that for a form f with $\dim f \geqq 2$, $D_F(f) = D_{R(F)}(f)$.

PROPOSITION 1.3. *Let F be a quasi-pythagorean field and $K=F(\sqrt{a})$ be a quadratic extention of F. Then the following statements hold*:

(1) *Let ρ be an n-fold Pfister form $(n \geqq 1)$ over F. If $a \in D_F(\rho)$, then we have $D_K(\rho) \cap \dot{F} = D_F(\rho)$.*

(2) *For $x, y \in \dot{F}$, if $a \in D_F\langle\langle x, y\rangle\rangle$ and $D_F\langle 1, x\rangle \subseteq D_F\langle 1, y\rangle$, then we have $D_K\langle 1, x\rangle \subseteq D_K\langle 1, y\rangle$.*

PROOF. As for (1), we note that if F is not formally real, then $R(F)=\dot{F}$ and the assertion (1) is clear. So we may assume that F is formally real. It is clear that $D_F(\rho) \subseteq D_K(\rho) \cap \dot{F}$, and we must show the reverse inclusion $D_F(\rho) \supseteq D_K(\rho) \cap \dot{F}$. We take an element $x \in D_K(\rho) \cap \dot{F}$. It is sufficient to show that the $(n+1)$-fold Pfister form $\rho \otimes \langle\langle -x\rangle\rangle$ is isotropic. Since $(\rho \otimes \langle\langle -x\rangle\rangle) \otimes K$ is isotropic, the form $\rho \otimes \langle\langle -x\rangle\rangle$ contains a subform $b\langle 1, -a\rangle$ for some $b \in \dot{F}$([6], p. 200, Lemma 3.1). Now b is an element of $D_F(\rho \otimes \langle\langle -x\rangle\rangle) = G_F(\rho \otimes \langle\langle -x\rangle\rangle)$ and so $\rho \otimes \langle\langle -x\rangle\rangle$ contains a subform $\langle 1, -a\rangle$; in particular, $\rho \otimes \langle\langle -x\rangle\rangle$ represents $-a$. Since $\pm a \in D_{R(F)}(\rho \otimes \langle\langle -x\rangle\rangle)$, $\rho \otimes \langle\langle -x\rangle\rangle$ is $R(F)$-isotropic and we have $x \in D_{R(F)}(\rho) = D_F(\rho)$. This implies that $D_K(\rho) \cap \dot{F} = D_F(\rho)$.

Next the assumption $D_F\langle 1, x\rangle \subseteq D_F\langle 1, y\rangle$ implies that $\dot{F} \cdot D_K\langle 1, x\rangle \subseteq F \cdot D_K\langle 1, y\rangle$ by the norm principle ([2], 2.13). So for $\alpha \in D_K\langle 1, x\rangle$, there exists $f \in \dot{F}$ such that $f\alpha \in D_K\langle 1, y\rangle$. We see that $f \in \alpha D_K\langle 1, y\rangle \subseteq D_K\langle 1, x\rangle D_K\langle 1, y\rangle \subseteq D_K\langle\langle x, y\rangle\rangle$ and it follows from the assertion (1) that $f \in D_K\langle\langle x, y\rangle\rangle \cap \dot{F} = D_F\langle\langle x, y\rangle\rangle$. The fact $D_F\langle 1, x\rangle \subseteq D_F\langle 1, y\rangle$ implies $x \in D_F\langle 1, y\rangle = G_F\langle 1, y\rangle$ and so $\langle\langle x, y\rangle\rangle \cong \langle 1, y\rangle \perp x\langle 1, y\rangle \cong \langle 1, y\rangle \perp \langle 1, y\rangle \cong \langle\langle 1, y\rangle\rangle$. Hence we can easily show that $D_F\langle\langle x, y\rangle\rangle = D_F\langle\langle 1, y\rangle\rangle = D_F\langle 1, y\rangle$, since F is quasi-pythagorean. Therefore we have $f \in D_F\langle 1, y\rangle \subseteq D_K\langle 1, y\rangle$. This implies $\alpha \in f D_K\langle 1, y\rangle = D_K\langle 1, y\rangle$ and the assertion (2) is proved. Q.E.D.

COROLLARY 1.4. *Let F be a quasi-pythagorean field and $K=F(\sqrt{a})$ be a quadratic extension of F. Then for $x \in \dot{F}$, the following statements hold*:

(1) *If $a \in D_F\langle 1, x\rangle$, then we have $D_K(2) \subseteq D_K\langle 1, x\rangle$.*

(2) *If $x \in D_F\langle 1, a\rangle$, then we have $D_K\langle 1, x\rangle \subseteq D_K(2)$.*

§2. The group $H_a(P)$

Throughout this section, a field always means a formally real field. For a subset Y of $X(F)$, we denote by $Y^{\perp}$ the preordering $\cap P(\sigma)$, $\sigma \in Y$. We have $P=X(F/P)^{\perp}$ and in particular, $X(F)^{\perp}=D_F(\infty)=\Sigma \dot{F}^2$. The topological structure of $X(F)$ is determined by Harrison sets $H(x)=\{\sigma \in X(F);\ x \in P(\sigma)\}$ as its subbasis, where x ranges over F. For a preordering P of F, we write $H_P(x)=H(x) \cap X(F/P)$.

LEMMA 2.1. *Let P be a preordering of a field F. Then for x, y and $a \in \dot{F}$, the following conditions are equivalent*:

(1) $x \in D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle$.

(2) $\langle 1, -x, -y, axy\rangle$ *is P-isotropic.*

(3) $y \in D_P\langle 1, -x\rangle D_P\langle 1, -ax\rangle$.

PROOF. (1)$\Rightarrow$(2): We write $x=\alpha\beta$ for some $\alpha \in D_P\langle 1, -y\rangle$ and $\beta \in D_P\langle 1, -ay\rangle$. Since $\alpha\beta^2-x\beta=0$, $\alpha\beta^2 \in D_P\langle 1, -y\rangle$ and $-x\beta \in -xD_P\langle 1, -ay\rangle = D_P\langle -x, axy\rangle$, we see that $\langle 1, -y, -x, axy\rangle$ is P-isotropic.

(2)$\Rightarrow$(1): From the assumption, there exists a non-trivial relation $p_1-yp_2-xp_3+axyp_4=0$ with $p_i \in P \cup \{0\}$, $i=1, 2, 3, 4$. If $p_1-yp_2=x(p_3-ayp_4)=0$, then at least one of the forms $\langle 1, -y\rangle$ and $\langle 1, -ay\rangle$ is P-isotropic and we have $D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle=\dot{F}$. If $p_1-yp_2=x(p_3-ayp_4)\neq 0$, then $x(p_3-ayp_4)^2=(p_1-yp_2)(p_3-ayp_4) \in D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle$. Therefore in any case we have $x \in D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle$.

The equivalence of the conditions (2) and (3) is quite similar to the above one. Q. E. D.

For $a \in \dot{F}$, the subgroup H_a of $\dot{F}$ is defined by $H_a=\{x \in \dot{F};\ D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle=\dot{F}\}$ in [3], §1. Generally, we put $H_a(P)=\{x \in \dot{F};\ D_P\langle 1, -x\rangle D_P\langle 1, -ax\rangle=\dot{F}\}$, where P is a preordering of F. We note that $H_{-1}(P)$ is the group $H(P)$ defined in [5], §2. By [5], Remark 2.3, (1), we have $H(P)=\dot{F}$ if and only if the space $X(F/P)$ satisfies S. A. P..

PROPOSITION 2.2. *Let P be a preordering of a field F. Then for $a \in \dot{F}$, we have $H_a(P)=\bigcap_{y \in \dot{F}} D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle$. In particular, $H_a(P)$ is a subgroup of $\dot{F}$ and $P \subseteq H_a(P) \subseteq D_P\langle 1, a\rangle$.*

PROOF. Since $H_a(P)=\{x \in \dot{F};\ D_P\langle 1, -x\rangle D_P\langle 1, -ax\rangle=\dot{F}\}$, it follows from Lemma 2.1 that $H_a(P)=\bigcap_{y \in \dot{F}} D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle$. Thus $H_a(P)$ is a multiplicative subgroup of $\dot{F}$ which contains P. For $y=-1$, $D_P\langle 1, -y\rangle D_P\langle 1, -ay\rangle = D_P\langle 1, 1\rangle D_P\langle 1, a\rangle = D_P\langle 1, a\rangle$, which implies $H_a(P) \subseteq D_P\langle 1, a\rangle$. Q. E. D.

In the rest of this section, we fix an element $a \in \dot{F}$ and a preordering P of F. We denote the sets $D_P\langle 1, a\rangle$, $D_P\langle 1, -a\rangle$ by T, T' respectively. If $\pm a \notin P$, then T and T' are preorderings of F and we have $H_P(a)=X(F/T)$, $H_P(-a)=X(F/T')$. The following three lemmas are generalizations of [3], Lemma 2.6, Lemma 2.8 and Lemma 2.9 respectively, and the proofs are omitted.

LEMMA 2.3. *Assume that* $\pm a \notin P$. *Then for* $x \in T$, *the following statements hold*:

(1) $D_P\langle 1, x\rangle = D_{T'}\langle 1, x\rangle \cap T$.

(2) $D_P\langle 1, -x\rangle = D_{T'}\langle 1, -x\rangle$.

(3) *If we further assume* $a \in H(P)$, *then we have* $D_P\langle 1, x\rangle D_P\langle 1, -x\rangle = D_{T'}\langle 1, x\rangle D_{T'}\langle 1. -x\rangle$.

LEMMA 2.4. *Assume that* $\pm a \notin P$. *Then we have* $H(T') \cap T = H_a(P)$.

LEMMA 2.5. *Assume that* $X(F/P)$ *is finite and connected. Then we have* $H_a(P) = P \cup aP$.

We shall say that two orderings $\sigma, \tau \in X(F/P)$ are connected in $X(F/P)$ if $\sigma = \tau$ or there exists a fan of index 8 which contains σ and τ. We denote this relation by $\sigma \sim \tau$. Marshall ([7], Theorem 4.7) showed that the relation $\sim$ is an equivalence relation in $X(F/P)$. An equivalence class of this relation is called a conncected component of $X(F/P)$ and a union of some connected components is called full. If P is of finite index, then $X(F/P)$ is a finite space; let $X_1, \dots, X_n$ be the connected components of $X(F/P)$. We write $P_i = X_i^{\perp}$ $(i = 1, \dots, n)$. By [5], Corollary 2.7, the canonical map $\varphi: \dot{F}/P \rightarrow \Pi \dot{F}/P_i$ $(i = 1, \dots, n)$ is isomorphic. It is clear by the definition that $H_a(P) \subseteq H_a(P_i)$ for any $i = 1, \dots, n$. Therefore the map $\psi: H_a(P)/P \rightarrow \Pi H_a(P_i)/P_i$ $(i = 1, \dots, n)$ is well-defined, where ψ is the restriction of φ to $H_a(P)/P$.

PROPOSITION 2.6. *Let* P *be a preordering of* F *of finite index and* $X_1, \dots, X_n$ *be the connected components of* $X(F/P)$. *We write* $P_i = X_i^{\perp}$ $(i = 1, \dots, n)$. *Then the canonical map* ψ: $H_a(P)/P \rightarrow \Pi H_a(P_i)/P_i$ $(i = 1, \dots, n)$ *is isomorphic.*

PROOF. Clearly ψ is injective. We shall show that ψ is surjective. Let X_i be any connected component of $X(F/P)$. Since X_i is a full subspace of $X(F/P)$, $X_i = H_P(b)$ for some $b \in H(P)$ by [5], Proposition 2.4. Then we have $X(F/P) \setminus X_i = H_P(-b)$, $D_P\langle 1, -b\rangle = \cap_{j \neq i} P_j$ and $D_P\langle 1, b\rangle = P_i$. We can take an element $x \in D_P\langle 1, -b\rangle \cap aP_i$, since $D_P\langle 1, -b\rangle D_P\langle 1, b\rangle = \dot{F}$. The fact $x \in D_P\langle 1, -b\rangle = \cap_{j \neq i} P_j$ shows that $H_P(-x) \subseteq X_i$ and this implies $D_P\langle 1, -x\rangle = (H_P(-x))^{\perp} \supseteq X_i^{\perp} = D_P\langle 1, b\rangle$. Since $ax \in P_i$, we have $H_P(-ax) \subseteq H_P(-b)$ and so $H_P\langle 1, -ax\rangle \supseteq H_P\langle 1, -b\rangle$. Thus $D_P\langle 1, -x\rangle D_P\langle 1, -ax\rangle \supseteq D_P\langle 1, b\rangle D_P\langle 1, -b\rangle = \dot{F}$, which implies $x \in H_a(P)$. Hence $\varphi(H_a(P)/P) \supseteq \varphi(xP \cup P/P)$. By Lemma 2.5, we have $H_a(P_i) = P_i \cup aP_i$; since $x \in aP_i \cap (\cap_{j \neq i} P_j)$, $\varphi(xP \cup P/P) = H_a(P_i)/P_i \times \Pi_{j \neq i}(P_j/P_j)$. From these facts, it is easily shown that ψ is surjective. Q.E.D.

Let S be a subgroup of $\dot{F}$ which contains P. Then S/P has the structure of $\boldsymbol{Z}_2$-vector space, and we denote its dimension by $\dim S/P$. For a connected component X_i of $X(F/P)$, we can readily see that $a \notin P_i$ if and only if $X_i \cap$

$H_P(-a)\neq\phi$. Now we have the following corollary to Proposition 2.6.

COROLLARY 2.7. *Let P be a preordering of F of finite index and $X_1,\dots, X_n$ be the connected components of $X(F/P)$. Then* $\dim H_a(P)/P=|I|$, *where* $I=\{i;\ a\notin P_i\}$.

LEMMA 2.8. *Assume that P is of finite index and let $X(F/T)$ be a full subspace of $X(F/P)$. Then for any $x\in\dot{F}$, we have $D_P\langle 1, x\rangle T=D_T\langle 1, x\rangle$.*

PROOF. We put $T^c=(X(F/P)\backslash X(F/T))^{\perp}$. We note that $D_T\langle 1, x\rangle=(H_P(x)\cap X(F/T))^{\perp}$ and $D_{T^c}\langle 1, x\rangle=(H_P(x)\cap X(F/T^c))^{\perp}$ for any $x\in\dot{F}$, hence $D_P\langle 1, x\rangle= D_T\langle 1, x\rangle\cap D_{T^c}\langle 1, x\rangle$. Therefore $D_P\langle 1, x\rangle T=D_T\langle 1, x\rangle\cap D_{T^c}\langle 1, x\rangle T$ and, since $TT^c=\dot{F}$, the assertion follows. Q.E.D.

PROPOSITION 2.9. *Assume that P is of finite index and let $X(F/P_1),\dots, X(F/P_n)$ be the connected components of $X(F/P)$. Then the following statements are equivalent*:

(1) $H_a(P)=D_P\langle 1, a\rangle$.

(2) *a is P_i-rigid for any $i=1,\dots, n$.*

PROOF. (1)$\Rightarrow$(2): By the assumption, $D_P\langle 1, a\rangle P_i=H_a(P)P_i$ and also $H_a(P)P_i=H_a(P_i)$ by Proposition 2.6. Since $H_a(P_i)=P_i\cup aP_i$ by Lemma 2.5, $D_P\langle 1, a\rangle P_i=P_i\cup aP_i$. Thus the assertion follows immediately from Lemma 2.8.

(2)$\Rightarrow$(1): Assume that a is P_i-rigid for any i. Then $D_P\langle 1, a\rangle P_i=D_{P_i}\langle 1, a\rangle =P_i\cup aP_i$ by Lemma 2.8, and so for any i, $D_P\langle 1, a\rangle P_i=H_a(P_i)$ by Lemma 2.5. Thus, by Proposition 2.6, we see that $D_P\langle 1, a\rangle\subseteq H_a(P)$. On the other hand the reverse inclusion holds always by Proposition 2.2 and so the assertion (1) is proved. Q.E.D.

As a corollary of Proposition 2.9, we have the following assertion, which generalizes [3], Corollary 2.10.

COROLLARY 2.10. *Suppose that $X(F)$ is a finite space. Let $K=F(\sqrt{a})$ be a quadratic extension of F and $X(F/P_1),\dots, X(F/P_n)$ be the connected components of $X(F)$. Then the following statements are equivalent*:

(1) *K is quasi-pythagorean.*

(2) *F is quasi-pythagorean and a is P_i-rigid for any $i=1,\dots, n$.*

LEMMA 2.11. *Let F be a formally real, quasi-pythagorean field. We assume that $X(F)$ is a finite space and denote by $X(F/P_1),\dots, X(F/P_n)$ the connected components of $X(F)$. Then the following statements hold*:

(1) *If $H(-x)\subseteq X(F/P_i)$, then $D_F\langle 1, x\rangle D_F\langle 1, -x\rangle=D_{P_i}\langle 1, x\rangle D_{P_i}\langle 1, -x\rangle$.*

(2) *If $a\notin -R(F)$ and $H(-x)\subseteq X(F/P_i)\cap H(-a)$, then we have*

$$D_F\langle 1, x\rangle D_F\langle 1, ax\rangle = D_{P_i}\langle 1, x\rangle D_{P_i}\langle 1, -x\rangle \cap T,$$

where T denotes the preordering $D_F\langle 1, a\rangle$ of F.

PROOF. If $X(F)$ is connected, then the assertion (1) is clear. Therefore, to prove (1), we may suppose $n \geqq 2$. Then there exists $b \in H(R(F))$ such that $X(F/P_i) = H(-b)$ by [5], Proposition 2.4. Since $H(x) \supseteq X(F) \backslash X(F/P_i) = H(b)$, we have $x \in D_F\langle 1, x\rangle \subseteq D_F\langle 1, b\rangle$. Hence by Lemma 2.3. (3), we see $D_F\langle 1, x\rangle D_F\langle 1, -x\rangle = D_{P_i}\langle 1, x\rangle D_{P_i}\langle 1, -x\rangle$.

Since $H(-x) \subseteq H(-a)$, $H(x) \supseteq H(a)$ and $x \in D_F\langle 1, a\rangle = T$; hence it is clear that $D_F\langle 1, x\rangle D_F\langle 1, ax\rangle \subseteq T$. Also the inclusion $H(x) \supseteq H(a)$. implies $H(ax) = H(-x) \cup H(a)$ and so $D_F\langle 1, ax\rangle = H(-x)^\perp \cap H(a)^\perp = D_F\langle 1, -x\rangle \cap T$. Therefore $D_F\langle 1, x\rangle D_F\langle 1, ax\rangle$ is contained in $D_F\langle 1, x\rangle D_F\langle 1, -x\rangle$ and we have

$$D_F\langle 1, x\rangle D_F\langle 1, ax\rangle \subseteq D_{P_i}\langle 1, x\rangle D_{P_i}\langle 1, -x\rangle \cap T.$$

For the reverse inclusion, we take $z \in D_{P_i}\langle 1, x\rangle D_{P_i}\langle 1, -x\rangle \cap T$. From the assertion (1), there exist $\alpha \in D_F\langle 1, x\rangle$ and $\beta \in D_F\langle 1, -x\rangle$ such that $z = \alpha\beta$. The fact $\alpha \in D_F\langle 1, x\rangle \subseteq T$ implies implies $\beta \in T$ and so $\beta \in T \cap D_F\langle 1, -x\rangle = D_F\langle 1, ax\rangle$. Thus we have $z = \alpha\beta \in D_F\langle 1, x\rangle D_F\langle 1, ax\rangle$ and the conclusion follows. Q. E. D.

§3. Connected spaces of orderings

Let F be a formally real field and P be a preordering of F. We denote by $\mathrm{gr}(X(F/P))$ the translation group of $X(F/P)$ in the terminology of [7], namely $\mathrm{gr}(X(F/P)) = \{\alpha \in \chi(\dot{F}/P);\ \alpha X(F/P) = X(F/P)\}$, where $\chi(\dot{F}/P) = \mathrm{Hom}(\dot{F}/P, \{\pm 1\})$ is the character group of $\dot{F}/P$. For a preordering P of finite index, $X(F/P)$ is connected if and only if $X(F/P) = 1$ or $|X(F/P)| > 3$ and $\mathrm{gr}(X(F/P)) \neq 1$. In what follows we assume that $X(F/P)$ is connected and $|X(F/P)| > 3$. For $x \in \dot{F}$, we define the subgroup $J_P(x)$ of $\dot{F}$ by $J_P(x) = D_P\langle 1, x\rangle D_P\langle 1, -x\rangle$ as in [5]. Since $X(F/P)$ is connected and $|X(F/P)| > 3$, there exists $\alpha \in \mathrm{gr}(X(F/P))$, $\alpha \neq 1$ and we fix it in this section. Then we can write $\alpha = \sigma_1\sigma_2$ with $\sigma_1, \sigma_2 \in X(F/P)$ and there exist orderings $\sigma_3, \ldots, \sigma_n \in X(F/P)$ such that $\{\sigma_1, \ldots, \sigma_n\}$ is a basis of $X(F/P)$, namely $\{\sigma_1, \ldots, \sigma_n\}$ is a basis of the subgroup of $\chi(\dot{F}/P)$ generated by the set $X(F/P)$. For a subspace Y of $X(F/P)$, we denote by $\dim Y$ the dimension of the subgroup of $\chi(\dot{F}/P)$ generated by Y; it is well known that $\dim Y$ is equal to the index of $Y^\perp$ in $\dot{F}$. Let $\{a_1, \ldots, a_n\}$, $a_i \in \dot{F}$, be the dual basis of $\{\sigma_1, \ldots, \sigma_n\}$ and we put $c_j = a_j$ $(j \neq 2)$and $c_2 = a_1a_2$. The subgroups of $\dot{F}$ generated by $\{c_2, \ldots, c_n\} \cup P$ and $\{c_2, \ldots, \check{c}_j, \ldots, c_n\} \cup P$ are denoted by L and L_j $(2 \leqq j \leqq n)$ respectively. It is clear that $\dot{F} = L \cup c_1L$, $-1 \in L$ and $-1 \notin L_j$, since $c_2 \cdots c_n = a_1 \cdots a_n \in -P$. Also it is easily shown that $\alpha(c_1) = -1$, $\alpha(c_j) = 1$ $(j = 2, \ldots, n)$. We note that $H_P(c_1) \ni \sigma_j$ $(j = 2, \ldots, n)$, $H_P(c_2) \ni \alpha\sigma_3$, σ_j $(j = 3, \ldots, n)$ and, for $i \geqq 3$, $H_P(c_i) \ni \sigma_j$ $(j \neq i)$. We can readily see c_i is P-rigid for any $i = 1, \ldots, n$, since an element x of $\dot{F}$ is P-rigid if and only if $\dim H_P(x) \geqq \dim X(F/P) - 1$. For $\alpha \in \chi(\dot{F}/P)$, the same symbol

α will often stand for the composite map $\alpha\circ p$, p being the canonical projection $\dot{F}\to\dot{F}/P$, as far as there is no fear of confusion.

PROPOSITION 3.1. *The following statements hold*:

(1) $\operatorname{Ker}\alpha=L$.

(2) *Any element of* c_1L *is* P-*rigid*.

(3) *For* $x\in L$, *if* $x\notin -P$, *then* $D_P\langle 1, x\rangle\subseteq L$.

PROOF. Since $\sigma_i(a_i)=-1$ and $\sigma_i(a_j)=1$ $(i\neq j)$, we have $\alpha(c_1)=\sigma_1\sigma_2(a_1)=-1$, $\alpha(c_2)=\sigma_1\sigma_2(a_1a_2)=1$ and $\alpha(c_i)=\sigma_1\sigma_2(a_i)=1$ for $i\geqq 3$. The assertion (1) follows immediately from these observations.

Let x be any element of c_1L. Then $\alpha(x)=-1$ by (1). Now for any $\sigma\in X(F/P)$, if $\sigma(x)=-1$, then $\alpha\sigma(x)=\alpha(x)\sigma(x)=1$. Thus for each $\sigma\in X(F/P)$, there exists $e(\sigma)\in\{0, 1\}$ such that $\alpha^{e(\sigma)}\sigma(x)=1$. This implies that x is an element of the preordering $\cap_{\sigma\in X(F/P)}\operatorname{Ker}(\alpha^{e(\sigma)}\sigma)$, and hence, $D_P\langle 1, x\rangle\subseteq\cap_{\sigma\in X(F/P)}\operatorname{Ker}(\alpha^{e(\sigma)}\sigma)$. Note however that

$$\cap_{\sigma\in X(F/P)}\operatorname{Ker}(\alpha^{e(\sigma)}\sigma)\cap\operatorname{Ker}(\alpha)=\cap_{\sigma\in X(F/P)}\operatorname{Ker}(\sigma)=P.$$

Since $\operatorname{Ker}(\alpha)$ is of index 2 in $\dot{F}$, this implies that $\cap_{\sigma\in X(F/P)}\operatorname{Ker}(\alpha^{e(\sigma)}\sigma)$ has order at most 2 in $\dot{F}/P$. So we have $D_P\langle 1, x\rangle=P\cup xP$, which settles the assertion (2).

Since $x\notin -P$, there exists $\sigma\in X(F/P)$ such that $\sigma(x)=1$. Let y be an element of $D_P\langle 1, x\rangle$. We suppose $y\notin L$; then $-y\notin L$ because $-1\in L$. Thus $-y\in c_1L$, so $-y$ is P-rigid. But $y\in D_P\langle 1, x\rangle$ and this is equivalent to $-x\in D_P\langle 1, -y\rangle$. So we have $-x\in P$ or $-x\in -yP$. The former case is impossible since $\sigma(x)=1$ and the latter case is also impossible since $x\in L$, $Y\notin L$. This proves the assertion (3). Q. E. D.

LEMMA 3.2. *Let T be a preordering of a field F and f, g be forms over F. If any element of* $D_T(f)D_T(g)=\{\alpha\beta;\ \alpha\in D_T(f),\ \beta\in D_T(g)\}$ *is T-rigid, then* $D_T(f\perp g)=D_T(f)\cup D_T(g)$.

PROOF. It suffices to show that $D_T(f\perp g)\subseteq D_T(f)\cup D_T(g)$. Let y be an element of $D_T(f\perp g)$. Then we can write $y=\alpha+\beta$, where $\alpha\in D_T(f)\cup\{0\}$ and $\beta\in D_T(g)\cup\{0\}$. If $\alpha=0$ or $\beta=0$, then the assertion follows immediately. Thus we may assume $\alpha\neq 0$ and $\beta\neq 0$. We note that $\alpha^{-1}\beta=(\alpha^{-1})^2\alpha\beta\in D_T(f)D_T(g)$. So by the assumption, we have $1+\alpha^{-1}\beta\in T$ or $1+\alpha^{-1}\beta\in\alpha^{-1}\beta T$. In the former case, $y=\alpha(1+\alpha^{-1}\beta)\in\alpha T\subseteq D_T(f)$ and in the latter case, $y\in\alpha(\alpha^{-1}\beta)T=\beta T\subseteq D_T(g)$. Therefore we have $y\in D_T(f)\cup D_T(g)$, and so $D_T(f\perp g)\subseteq D_T(f)\cup D_T(g)$.

Q. E. D.

We say that two forms f, g are P-similar if $f\cong_P ag$ for some $a\in\dot{F}$.

LEMMA 3.3. *Let x_i, y_i, z_i $(i=1, 2)$ be elements of the group L such that x_1x_2, $y_1y_2 \notin z_1z_2P$ and x_1x_2, $y_1y_2 \notin -P$. Then the form $\varphi=\langle x_1, x_2\rangle \perp c_1\langle y_1, y_2\rangle$ does not contain a subform which is P-similar to $\langle z_1, z_2\rangle$.*

PROOF. First we note that $D_P\langle x_1, x_2\rangle = x_1D_P\langle 1, x_1x_2\rangle \subseteq L$ and $D_P\langle y_1, y_2\rangle = y_1D_P\langle 1, y_1y_2\rangle \subseteq L$ by Proposition 3.1.
Since $D_P\langle x_1, x_2\rangle D_P(c_1\langle y_1, y_2\rangle) \subseteq c_1L$ and any element of c_1L is P-rigid, $D_P(\varphi) = D_P\langle x_1, x_2\rangle \cup c_1D_P\langle y_1, y_2\rangle$ by Lemma 3.2. We now suppose on the contrary that φ contains a subform $b\langle z_1, z_2\rangle$ over P for some $b \in \dot{F}$. We consider the following two cases.

Case 1. $bz_1 \in D_P\langle x_1, x_2\rangle$. In this case, we have $\langle x_1, x_2\rangle \cong_P \langle bz_1, bz_1x_1x_2\rangle$ which implies that the form $\varphi - \langle bz_1\rangle = \langle bz_1x_1x_2\rangle \perp c_1\langle y_1, y_2\rangle$ represents bz_2 over P. We note that $bz_1 \in L$ and so $D_P\langle bz_1x_1x_2\rangle D_P(c_1\langle y_1, y_2\rangle) \subseteq c_1L$. By Lemma 3.2, we have $D_P(\langle bz_1x_1x_2\rangle \perp c_1\langle y_1, y_2\rangle) = bz_1x_1x_2P \cup D_P(c_1\langle y_1, y_2\rangle)$, hence $bz_2 \in bz_1x_1x_2P$. This contradicts the assumption $x_1x_2 \notin z_1z_2P$.

Case 2. $bz_1 \notin c_1D_P\langle y_1, y_2\rangle$. Similarly to the case 1, we can show that $y_1y_2 \notin z_1z_2P$, a contradiction. Q.E.D.

LEMMA 3.4. *Let a, x, y be elements of the group L and z an element of c_1L. If $y \notin D_P\langle 1, -x\rangle D_P\langle 1, -ax\rangle$, then the the form $\varphi = \langle 1, -x, -y, z\rangle$ does not contain a subform which is P-similar to $\langle 1, -a\rangle$. In particular, if $y \notin D_P\langle 1, -x\rangle D_P\langle 1, x\rangle$, then the form $\varphi = \langle 1, x, y, z\rangle$ does not contain a subform which is P-similar to $\langle 1, 1\rangle$.*

PROOF. First we shall show that the form $\langle 1, -x, -y\rangle$ does not contain a subform which is P-similar to $\langle 1, -a\rangle$. Assume on the contrary that the form $\langle 1, -x, -y\rangle$ contains a subform $b\langle 1, -a\rangle$ over P for some $b \in \dot{F}$. Then we have $\langle 1, -x, -y\rangle \cong_P \langle b, -ab, -axy\rangle$, which implies $-axy \in D_P\langle 1, -x, -y\rangle$. Thus the form $\langle 1, -x, -y, acy\rangle$ is P-isotropic and hence $y \in D_P\langle 1, -x\rangle D_P \langle 1, -ax\rangle$ by Lemma 2.1, a contradiction.

Next we assume that φ contains subform $b\langle 1, -a\rangle$ over P for some $b \in \dot{F}$. From the assumption $y \notin D_P\langle 1, -x\rangle \cdot D_P\langle 1, -ax\rangle$, it follows that $xy \notin -P$ and so $D_P\langle -x, -y\rangle = -xD_P\langle 1, xy\rangle \subseteq L$ by Proposition 3.1. Let z be any element of $D_P\langle -x, -y\rangle$. Then $z \notin -P$, because the form $\langle 1, -x, -y, axy\rangle$ is P-anisotropic by Lemma 2.1. From these observation, it is easy to show that $D_P\langle 1, -x, -y\rangle \subseteq L$ by Proposition 3.1. Therefore, since $z \in c_1L$, $D_P(\varphi) = D_P\langle 1, -x, -y\rangle \cup zP$ by Lemma 3.2. We now treat the following two cases separately.

Case 1. $b \in zP$. In this case, since the form $\varphi - \langle b\rangle$ represents $-ab$, we have $-ab \in D_P\langle 1, -x, -y\rangle \subseteq L$. This contradicts the fact that $-ab \in -azP \subseteq c_1L$.

Case 2. $b \in D_P\langle 1, -x, -y\rangle$. In this case, the form $\langle 1, -x, -y\rangle$ is P-isometric to $\langle b, b_1, b_2\rangle$ for some $b_1, b_2 \in \dot{F}$. So we have $-ab \in D_P\langle b_1, b_2, z\rangle$. By Lemma 3.2, $D_P\langle b_1, b_2, z\rangle = D_P\langle b_1, b_2\rangle \cup zP$, and this shows that $-ab \in D_P\langle b_1, b_2\rangle$ because $-ab \in L$ and $zP \subseteq c_1 L$. Thus the form $\langle 1, -x, -y\rangle$ contains a subform $b\langle 1, -a\rangle$ over P. This contradicts the first step of our proof.

We now assume $y \notin D_P\langle 1, -x\rangle D_P\langle 1, x\rangle$. We put $y' = -y$, $x' = -x$ and $a = -1$. Then $y' \notin D_P\langle 1, -x'\rangle D_P\langle 1, -ax'\rangle$, so by the first assertion, $\langle 1, -x', -y', z\rangle = \langle 1, x, y, z\rangle$ does not contain a subform which is P-similar to $\langle 1, 1\rangle$. Q.E.D.

§4. Quadratic extensions of quasi-pythagorean fields

In this section, we assume that F is a formally real, quasi-pythagorean field and $R(F)$ is of finite index in $\dot{F}$. Let $X(F)$ be the space of orderings of F and $X(F/P_1), \dots, X(F/P_n)$ be the connected components of $X(F)$. We write $n(i) = \dim X(F/P_i)$ and $P_i^c = \cap_{j \neq i} P_j$ for $i = 1, \dots, n$. It is easily shown that for any $x \in P_i^c$, $H(x) \supseteq X(F) \backslash X(F/P_i)$, and $H(-x) \subseteq X(F/P_i)$. Since $X(F/P_i)$ is a full subspace of $X(F)$, there exists $b \in H(R(F))$ such that $X(F/P_i) = H(b)$ by [5], Proposition 2.4. We have $P_i = D_F\langle 1, b\rangle$ and $P_i^c = D_F\langle 1, -b\rangle$, and this implies $P_i \cdot P_i^c = D_F\langle 1, b\rangle D_F\langle 1, -b\rangle = \dot{F}$. Therefore we can take a basis of $\dot{F}/P_i$, consisting of elements in P_i^c.

In what follows, whenever we say that a subset B of $\dot{F}$ is a basis of $\dot{F}/P_i$, we understand that B consists of elements in P_i^c. We fix a quadratic extension $K = F(\sqrt{a})$.

PROPOSITION 4.1. *If $a \in P_i$, then we have $D_K\langle 1, -x\rangle \supseteq D_K(2)$ for any $x \in P_i^c$. In particular if $\{c_i\}$ $i = 1, \dots, n(i)$ is a basis of $\dot{F}/P_i$, then we have $D_K\langle 1, -c_i\rangle \supseteq D_K(2)$ for any $i = 1, \dots, n(i)$.*

PROOF. Since $x \in P_i^c$, we have $H(-x) \subseteq X(F/P_i)$, and so $D_F\langle 1, -x\rangle = H(-x)^{\perp} \supseteq P_i$. Thus we have $a \in D_F\langle 1, -x\rangle$ and the assertion follows from Corollary 1.4. Q.E.D.

THEOREM 4.2. *If $-a \in P_i$, then there exists a basis $\{c_1, \dots, c_{n(i)}\}$ of $\dot{F}/P_i$ such that the dimension of $D_K(2)/\cap_{j=1,\dots,n(i)} D_K\langle 1, c_j\rangle$ is equal to $n(i) - 1$.*

When P_i is an ordering, we have the following

PROPOSITION 4.3. *If $-a \in P_i$ and P_i is an ordering, then $P_i^c \subseteq R(K)$.*

PROOF. Let c be an element of P_i^c. Since P_i is an ordering, $H_a(P_i) = \dot{F}$ and $c \in P_j$ for any j $(j \neq i)$; therefore, $c \in H_a \subseteq R(K)$ by Proposition 2.6. Q.E.D.

REMARK 4.4. When P_i is an ordering, $D_K\langle 1, c\rangle = D_K(2)$ for any $c \in P_i^c$ by

Proposition 4.3. We note $n(i)-1=0$, and so Theorem 4.2 is valid in this case.

We now proceed to the general case of the proof of Theorem 4.2. Namely, in the rest of this section, we assume that $|X(F/P_i)|>3$. There exists $\alpha\neq 1$ in $\mathrm{gr}(X(F/P_i))$ and we fix it. We can write $\alpha=\sigma_1\sigma_2$ with $\sigma_1, \sigma_2\in X(F/P_i)$ and there exist $\sigma_3,\dots, \sigma_{n(i)}\in X(F/P_i)$ such that $\{\sigma_1,\dots, \sigma_{n(i)}\}$ is basis of $X(F/P_i)$. We take the dual basis $\{a_1,\dots, a_{n(i)}\}$, $a_i\in P_i^c$. We put $c_j=a_j$ $(j\neq 2)$ and $c_2=a_1a_2$. It is clear that $\{c_1,\dots, c_{n(i)}\}$ is a basis of $\dot{F}/P_i$ with $c_j\in P_i^c$. Since $H(c_j)\supseteq X(F)\backslash X(F/P_i)\supseteq H(a)$, we have $D_F\langle 1, c_j\rangle\subseteq D_F\langle 1, a\rangle$ for any j, and so $D_K\langle 1, c_j\rangle\subseteq D_K(2)$ by Corollary 1.4. We put $T=D_F\langle 1, a\rangle$.

LEMMA 4.5. *The dimension of* $D_K(2)/D_K\langle 1, c_j\rangle$ *is eaual to the dimension of* $T/D_F\langle 1, c_j\rangle D_F\langle 1, ac_j\rangle$ *for any* $j=1,\dots, n(i)$. *In particular* $\dim D_K(2)/D_K\langle 1, c_1\rangle =n(i)-2$.

PROOF. Since c_j is P_i-rigid for every $j=1,\dots, n(i)$ (cf. §3), $\dim H_{P_i}(c_j)=n(i)-1$ and hence, moreover, c_j is $R(F)$-rigid. We note that $c_j\notin P_i$ and $-a\in P_i$ by the assumption, so $c_j\notin D_F\langle 1, -a\rangle$. Therefore $D_F\langle 1, -a\rangle\cap D_F\langle 1, c_j\rangle=R(F)$, which implies that $\dot{F}\cdot D_K(2)=\dot{F}\cdot D_K\langle 1, c_j\rangle$ by the norm principle ([2], 2.13). We also note that $D_K(2)\cap\dot{F}=T$ and $D_K\langle 1, c_j\rangle\cap\dot{F}=D_F\langle 1, c_j\rangle D_F\langle 1, ac_j\rangle$; it follows from these relations that $\dim\dot{F}/T$ and $\dim\dot{F}/D_F\langle 1, c_j\rangle D_F\langle 1, ac_j\rangle$ equal $\dim\dot{F}\cdot D_K(2)/D_K(2)$ and $\dim\dot{F}\cdot D_K(2)/D_K\langle 1, c_j\rangle$ respectively. Thus we have

$$\dim T/D_F\langle 1, c_j\rangle D_F\langle 1, ac_j\rangle = \dim D_K(2)/D_K\langle 1, c_j\rangle\,.$$

As for the second fissertion, note that $\pm c_1$ are P_i-rigid by Proposition 3.1, (2); so $\dim\dot{F}/J_{P_i}(c_1)=n(i)-2$. It is clear that $H(-c_1)\subseteq X(F/P_i)\subseteq H(-a)$ and hence $J_{P_i}(c_1)\cap T=D_F\langle 1, c_1\rangle D_F\langle 1, ac_1\rangle$ by Lemma 2.11. Since $TP_i=\dot{F}$, we can show that $\dim\dot{F}/J_{P_i}(c_1)$ coincides with $\dim T/D_F\langle 1, c_1\rangle D_F\langle 1, ac_1\rangle$. Our conclusion now follows from the first step. Q.E.D.

The subgroups generated by $\{c_1,\dots, c_{n(i)}\}\cup P$ and $\{c_2,\dots, \check{c}_j,\dots, c_{n(i)}\}\cup P$ are denoted by L and L_j $(j=2,\dots, n(i))$ respectively.

LEMMA 4.6. *If* $x\in L_j\backslash J_{P_i}(c_j)$, *then* $D_K\langle 1, c_1\rangle\cap xD_K\langle 1, c_j\rangle=\phi$ *for every* $j=2,\dots, n(i)$.

PROOF. Assume that there exists an element $x\in L_j\backslash J_{P_i}(c_j)$ such that $D_K\langle 1, c_1\rangle\cap xD_K\langle 1, c_j\rangle\neq\phi$. Then the form $\varphi=\langle 1, c_1\rangle\perp(-ax)\langle 1, c_j\rangle$ is isotropic over K and so it contains a subform which is similar to $\langle 1, -a\rangle$. Hence the form $\langle 1, c_1\rangle\perp x\langle 1, c_j\rangle\cong\langle 1, x, c_jx, c_1\rangle$ contains a subform over P_i which is P_i-similar to $\langle 1, 1\rangle$. But since $x\notin D_{P_i}\langle 1, -c_j\rangle\cdot D_{P_i}\langle 1, c_j\rangle$, the form $\langle 1, x, c_j, c_1x\rangle\cong x\langle 1, x, c_jx, c_1\rangle$ does not contain a subform which is P_i-similar to $\langle 1, 1\rangle$ by Lemma 3.4. This is a contradiction. Q.E.D.

We need a lemma on a vector space over a field. The proof is easy and omitted.

LEMMA 4.7. *Let L be a vector space over a field, and V, W, Z be subspaces of L. Let x be an element of V and W_1 be the subspace generated by $\{x\} \cup W$. If $V \subseteq W_1 + Z$ and $V \nsubseteq W_1$, then there exists $y \in Z \backslash W$ such that $V \cap (y+W) \neq \phi$.*

LEMMA 4.8. *The dimension of $D_K(2)/D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_j\rangle$ is equal to $n(i)-1$ for any $j=2,\dots, n(i)$.*

PROOF. Since $\pm c_1$ are P_i-rigid, $J_{P_i}(c_1)=\{\pm P_i, \pm c_1 P_i\}$ and so $c_j \notin J_{P_i}(c_1)$. By [5], Lemma 2.2, $c_1 \notin J_{P_i}(c_j)$. It follows from Lemma 2.11 that

$$\dot{F} \cap D_K\langle 1, c_j\rangle = D_F\langle 1, c_j\rangle D_F\langle 1, c_j a\rangle = J_{P_i}(c_j) \cap T$$

and hence $c_i \notin D_K\langle 1, c_j\rangle$. Thus $D_K\langle 1, c_1\rangle$ contains $D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_j\rangle$ properly and it follows from Lemma 4.5 that the dimension of $D_K(2)/D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_j\rangle$ is at least $n(i)-1$. As for the reverse inequality, it suffices to show that $D_K\langle 1, c_1\rangle \subseteq c_1 D_K\langle 1, c_j\rangle \cup D_K\langle 1, c_j\rangle$ (cf. Lemma 4.5). Assume on the contrary that $D_K\langle 1, c_1\rangle$ is not contained in $c_1 D_K\langle 1, c_j\rangle \cup D_K\langle 1, c_j\rangle$. By Lemma 4.5 and Lemma 2.11, we have

$$D_K(2)/D_K\langle 1, c_j\rangle \cong T/D_F\langle 1, c_j\rangle D_F\langle 1, ac_j\rangle \cong \dot{F}/J_{P_i}(c_j).$$

From this, it is easy to see that $D_K(2)$ is contained in the subgroup of $\dot{K}$ which is generated by L_j and $c_1 D_K\langle 1, c_j\rangle \cup D_K\langle 1, c_j\rangle$. Since $c_1 \in D_K\langle 1, c_1\rangle \subseteq D_K(2)$, there exists an element $x \in L_j \backslash D_K\langle 1, c_j\rangle$ such that $D_K\langle 1, c_1\rangle \cap xD_K\langle 1, c_j\rangle \neq \phi$ by Lemma 4.7. Then x is not contained in $J_{P_i}(c_j)$, because

$$J_{P_i}(c_j) \cap T = D_F\langle 1, c_j\rangle D_F\langle 1, ac_j\rangle = D_K\langle 1, c_j\rangle \cap \dot{F}.$$

This contradicts Lemma 4.6. Q. E. D.

Combining Lemma 4.8 with the following Lemma 4.9., we can complete the proof of our theorem.

LEMMA 4.9. *$D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_k\rangle = D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_j\rangle$ for any j, k $(j=2,\dots, n(i),\ k=2,\dots, n(i))$.*

PROOF. Assume that $D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_k\rangle \nsubseteq D_K\langle 1, c_j\rangle$ for some j, k. By the proof of Lemma 4.5, $\dot{F}\cdot D_K(2) = \dot{F}\cdot D_K\langle 1, c_m\rangle$ for any $m=1,\dots, n(i)$, so $D_K\langle 1, c_1\rangle \cap D_K\langle 1, c_k\rangle$ is contained in $\dot{F}\cdot D_K\langle 1, c_j\rangle$. Hence we can find an element $x \in \dot{F}\backslash D_K\langle 1, c_j\rangle$ such that $(D_K\langle 1, c_1\rangle \cap D_k\langle 1, c_k\rangle) \cap xD_K\langle 1, c_j\rangle \neq \phi$. In particular $D_K\langle 1, c_1\rangle \cap xD_K\langle 1, c_j\rangle \neq \phi$. On the other hand, by the proof of Lemma 4.8, $D_K\langle 1, c_1\rangle \subseteq c_1 D_K\langle 1, c_j\rangle \cup D_K\langle 1, c_j\rangle$. This implies $xD_K\langle 1, c_j\rangle = c_1 D_K\langle 1, c_j\rangle$

and so $D_K\langle 1, c_k\rangle \cap c_1 D_K\langle 1, c_j\rangle \neq \phi$. Consider the form $\varphi = \langle 1, c_k\rangle \perp (-ac_1)\cdot\langle 1, c_j\rangle$ over F. Then φ is isotropic over K, so φ contains a subform over F which is similar to $\langle 1, -a\rangle$. Hence the form $\langle 1, c_k\rangle \perp c_1\langle 1, c_j\rangle$ contains a subform over P_i which is P_i-similar to $\langle 1, 1\rangle$. This contradicts Lemma 3.3.

Q.E.D.

§5. Quadratic extensions of quasi-pythagorean fields (continued)

In this section, we turn our attention now to the case where a is not contained in $P_i \cup -P_i$ in the situation of §4. In this case, P_i is not an ordering, so there exists an element $\alpha \in \mathrm{gr}(X(F/P_i))$, $\alpha \neq 1$, and we fix it. The bases $\{\sigma_i\}$, $\{c_i\}$ $(i=1,\dots, n(i))$ and the group L will continue to have the previous meanings.

The main purpose of this section is to prove the following Theorem 5.1.

THEOREM 5.1. *If $a \notin \pm P_i$, then there exists a basis $\{d_1,\dots, d_{n(i)}\}$ of $\dot{F}/P_i$ such that the dimension of $D_K(2)/D_K(2) \cap (\bigcap_{j=1,\dots,n(i)} D_K\langle 1, -d_j\rangle)$ is at most $n(i)-m(i)-1$, where $m(i)=\dim \dot{F}/D_{P_i}\langle 1, a\rangle$.*

First we suppose that $\alpha(a)=-1$. Before proceeding with the next proposition, observe that from Proposition 3.1, a is P_i-rigid, so $n(i)-m(i)-1=0$.

PROPOSITION 5.2. *If $\alpha(a)=-1$, then there exists a basis $\{d_1,\dots, d_{n(i)}\}$ of $\dot{F}/P_i$ such that $D_K(2) \subseteq D_K\langle 1, -d_j\rangle$ for any $j=1,\dots, n(i)$. In particular, the dimension of $D_K(2)/D_K(2) \cap (\bigcap_{j=1,\dots,n(i)} D_K\langle 1, -d_j\rangle)$ is equal to 0.*

PROOF. Let d_1 be an element of P_i^c such that $d_1 \in -aP_i$. For $j \geqq 2$, we put $d_j = c_j$. Since $d_1 \in aL = c_1 L$, it is clear that $\{d_1,\dots, d_{n(i)}\}$ is a basis of $\dot{F}/P_i$. Note that $H(-d_1) = H_{P_i}(a)$, and so $D_F\langle 1, -d_1\rangle = D_{P_i}\langle 1, a\rangle$. By Corollary 1.4, $D_K(2) \subseteq D_K\langle 1, -d_1\rangle$, thus it suffices to show that $D_K\langle 1, -d_1\rangle \subseteq D_K\langle 1, -d_j\rangle$ for any $j \geqq 2$. Since $D_F\langle 1, -a\rangle = D_{P_i}\langle 1, -a\rangle \cap D_{P_i^c}\langle 1, -a\rangle$, we have

$$D_F\langle 1, -d_1\rangle \cap D_F\langle 1, -a\rangle = P_i \cap D_{P_i^c}\langle 1, -a\rangle.$$

Similarly, we have

$$D_F\langle 1, -d_j\rangle \cap D_F\langle 1, -a\rangle = D_{P_i}\langle 1, -d_j\rangle \cap D_{P_i}\langle 1, -a\rangle \cap D_{P_i^c}\langle 1, -a\rangle$$

for $j \geqq 2$. Therefore

$$D_F\langle 1, -d_1\rangle \cap D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -d_j\rangle \cap D_F\langle 1, -a\rangle$$

and this shows that $\dot{F}\cdot D_K\langle 1, -d_1\rangle \subseteq \dot{F}\cdot D_K\langle 1, -d_j\rangle$ by the norm principle. On the other hand, since $D_K\langle 1, -d_j\rangle \cap \dot{F} = D_{P_i}\langle 1, -d_j\rangle D_F\langle 1, -ad_j\rangle$, Lemma 2.8 shows that $D_K\langle 1, -d_j\rangle \cap \dot{F} = D_{P_i}\langle 1, -d_j\rangle D_{P_i}\langle 1, -ad_j\rangle$. This implies $aP_i \subseteq D_K\langle 1, -d_j\rangle$, so $-d_1 \in D_K\langle 1, -d_j\rangle$. By noting that $-d_1 \in c_1 L$, we can show that $\dot{F}\cdot D_K\langle 1, -d_j\rangle = L\cdot D_K\langle 1, -d_j\rangle$, so $D_K\langle 1, -d_1\rangle \subseteq L\cdot D_K\langle 1, -d_j\rangle$. We now suppose on the contrary that $D_K\langle 1, -d_1\rangle \nsubseteq D_K\langle 1, -d_j\rangle$ for some $j \geqq 2$. There exists $y \in L\backslash D_K\langle 1, -d_j\rangle$ such that $D_K\langle 1, -d_1\rangle \cap yD_K\langle 1, -d_j\rangle \neq \phi$. So the form $\langle 1, -d_1\rangle \perp (-y)\langle 1, -d_j\rangle$ is isotropic over K, and it contains a subform over F which is similar to $\langle 1, -a\rangle$. However, we have

$$y \notin D_K\langle 1, -d_j\rangle \cap \dot{F} = D_{P_i}\langle 1, -d_j\rangle D_{P_i}\langle 1, -ad_j\rangle .$$

Therefore the form $\langle 1, -d_j, -y, d_1 y\rangle \cong -y\langle 1, -d_1, -y, yd_j\rangle$ does not contain a subform which is P-similar to $\langle 1, -a\rangle$ by Lemma 3.4. This is a contradiction.

Q. E. D.

For the rest, we suppose $\alpha(a)=1$. We claim that $\dot{F}\cdot D_K\langle 1, -c_1\rangle \subseteq \dot{F}\cdot D_K\langle 1, -c_j\rangle$. To see this, it suffices to show by the norm principle that

$$D_F\langle 1, -c_1\rangle \cap D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -c_j\rangle \cap D_F\langle 1, -a\rangle .$$

Since $-c_1$ is P_i-rigid, $D_{P_i}\langle 1, -a\rangle \subseteq L$ implies $D_{P_i}\langle 1, -c_1\rangle \cap D_{P_i}\langle 1, -a\rangle = P_i$. Thus we can see that

$$D_F\langle 1, -c_1\rangle \cap D_F\langle 1, -a\rangle = P_i \cap D_{P_i^c}\langle 1, -a\rangle ,$$

because $D_F\langle 1, -a\rangle = D_{P_i}\langle 1, -a\rangle \cap D_{P_i^c}\langle 1, -a\rangle$. Similarly, for $j=2,\dots, n(i)$ we have

$$D_F\langle 1, -c_j\rangle \cap D_F\langle 1, -a\rangle = D_{P_i}\langle 1, -c_j\rangle \cap D_{P_i}\langle 1, -a\rangle \cap D_{P_i^c}\langle 1, -a\rangle .$$

These establish the claim.

Lemma 5.3. *$D_K\langle 1, -c_1\rangle \subseteq D_K\langle 1, -c_j\rangle \cup (-c_1)D_K\langle 1, -c_j\rangle$ for any $j=2,\dots, n(i)$.*

Proof. Assume that $D_K\langle 1, -c_1\rangle \nsubseteq D_K\langle 1, -c_j\rangle \cup (-c_1)D_K\langle 1, -c_j\rangle$. By Lemma 4.7, there exists $y \in L$ such that $y \notin D_K\langle 1, -c_j\rangle$ and $D_K\langle 1, -c_1\rangle \cap yD_K\langle 1, -c_j\rangle \neq \phi$. The form $\langle 1, -c_1\rangle \perp (-y)\langle 1, -c_j\rangle$ is isotropic over K, so it contains a subform over F which is similar to $\langle 1, -a\rangle$. However, we have

$$y \notin D_K\langle 1, -c_j\rangle \cap \dot{F} = D_{P_i}\langle 1, -c_j\rangle D_{P_i}\langle 1, -ac_j\rangle .$$

So, by Lemma 3.4, the form $\langle 1, -c_j, -y, yc_1\rangle \cong (-y)\langle 1, -c_1, -y, c_j y\rangle$ does not contain a subform which is P_i-similar to $\langle 1, -a\rangle$. This is a contradiction.

Q. E. D.

Let b be an element of P_i^c such that $b \in -aP_i$. Then $H(a) \supseteq H(-b) = H_{P_i}(a)$, so $a \in D_F\langle 1, -b\rangle$. This shows $D_K(2) \subseteq D_K\langle 1, -b\rangle$ by Corollary 1.4. Since $D_F\langle 1, -b\rangle = D_{P_i}\langle 1, a\rangle$, we have

$$D_F\langle 1, -b\rangle \cap D_F\langle 1, -a\rangle = P_i \cap D_{P_i^c}\langle 1, -a\rangle.$$

Therefore

$$D_F\langle 1, -b\rangle \cap D_F\langle 1, -a\rangle = D_F\langle 1, -c_1\rangle \cap D_F\langle 1, -a\rangle,$$

which implies $\dot{F}\cdot D_K\langle 1, -b\rangle = \dot{F}\cdot D_K\langle 1, -c_1\rangle$ by the norm principle. We fix the element b.

LEMMA 5.4. $D_K\langle 1, -c_1\rangle \subseteq D_K\langle 1, -b\rangle \cup (-c_1)D_K\langle 1, -b\rangle$.

The proof is similar to that of Lemma 5.3, and is omitted.

LEMMA 5.5. $D_K\langle 1, -b\rangle \cap D_K\langle 1, -c_1\rangle \subseteq D_K\langle 1, -c_j\rangle$ *for any* $j=2,\dots, n(i)$.

PROOF. First assume $c_j \in aP_i$. Since c_j is in P_i^c, Proposition 2.6 implies that $c_j \in H_a \subseteq R(K)$. So the assertion is clear in this case. Next assume $c_j \notin aP_i$. We suppose that there exists $\alpha \in D_K\langle 1, -b\rangle \cap D_K\langle 1, -c_1\rangle$ such that $\alpha \notin D_K\langle 1, -c_j\rangle$. By Lemma 5.3, $\alpha D_K\langle 1, -c_j\rangle = -c_1 D_K\langle 1, -c_j\rangle$ and this shows that $D_K\langle 1, -b\rangle \cap (-c_1)D_K\langle 1, -c_j\rangle \neq \phi$. So the form $\langle 1, -b\rangle \perp c_1\langle 1, c_j\rangle$ is isotropic over K, and it contains a subform over F which is similar to $\langle 1, -a\rangle$. This contradicts Lemma 3.3. Q.E.D.

To simplify the notation, we write $A_i = \cap_{j=1,\dots,n(i)} D_K\langle 1, -c_j\rangle$. By Lemma 5.5, we have $D_K\langle 1, -b\rangle \cap A_i = D_K\langle 1, -b\rangle \cap D_K\langle 1, -c_1\rangle$. So the next lemma shows that the dimension of $D_K\langle 1, -b\rangle / D_K\langle 1, -b\rangle \cap A_i$ is at most $n(i)-m(i)-1$.

LEMMA 5.6. $\dim (D_K\langle 1, -b\rangle / D_K\langle 1, -c_1\rangle \cap D_K\langle 1, -b\rangle) \leqq n(i)-m(i)-1$.

PROOF. By Lemma 5.4, $\dim (D_K\langle 1, -c_1\rangle D_K\langle 1, -b\rangle / D_K\langle 1, -b\rangle) \leqq 1$, and so we also have $\dim (D_K\langle 1, -c_1\rangle / D_K\langle 1, -c_1\rangle \cap D_K\langle 1, -b\rangle) \leqq 1$. On the other hand,

$$\dim (\dot{F}\cdot D_K\langle 1, -c_1\rangle / D_K\langle 1, -c_1\rangle) = \dim (\dot{F}/D_{P_i}\langle 1, -c_1\rangle D_{P_i}\langle 1, -ac_1\rangle),$$

because $D_K\langle 1, -c_1\rangle \cap \dot{F} = D_{P_i}\langle 1, -c_1\rangle D_{P_i}\langle 1, -ac_1\rangle$. Since the elements $-c_1$, $-ac_1$ are P_i-rigid by Proposition 3.1, $\dim (\dot{F}\cdot D_K\langle 1, -c_1\rangle / D_K\langle 1, -c_1\rangle) = n(i)-2$, and hence

$$\dim (\dot{F}\cdot D_K\langle 1, -c_1\rangle / D_K\langle 1, -c_1\rangle \cap D_K\langle 1, -b\rangle) \leqq n(i) - 1.$$

Therefore, we have only to show that $\dim (\dot{F}\cdot D_K\langle 1, -c_1\rangle / D_K\langle 1, -b\rangle) = m(i)$. However since we have

$$D_K\langle 1, -b\rangle \cap \dot{F} = D_{P_i}\langle 1, -b\rangle D_{P_i}\langle 1, -ab\rangle \quad \text{and} \quad b \in -aP_i,$$

$D_K\langle 1, -b\rangle \cap \dot{F} = D_{P_i}\langle 1, a\rangle$, and the claim is proved. We have seen that $\dot{F}\cdot D_K\langle 1, -b\rangle = \dot{F}\cdot D_K\langle 1, -c\rangle$, and the assertion follows. Q.E.D.

We can now prove Theorem 5.1. We have a homomorphism $D_K(2) \to D_K\langle 1, -b\rangle / D_K\langle 1, -b\rangle \cap A_i$, whose kernel is $D_K(2) \cap A_i$. So $\dim(D_K(2)/D_K(2) \cap A_i) \leqq n(i) - m(i) - 1$ by Lemma 5.6. Thus we complete the proof of Theorem 5.1.

§6. Main theorem

In this section, we state the main theorem (Theorem 6.1) of this paper.

THEOREM 6.1. *Let F be a quasi-pythagorean field where its Kaplansky's radical $R(F)$ is of finite index. Let $K = F(\sqrt{a})$ be a quadratic extension of F. Then we have $N^{-1}(R(F)) = \dot{F} \cdot R(K)$.*

PROOF. If F is not formally real, then the assertion follows from [4], Theorem 2.13. So we may assume that F is formally real. Let $X(F)$ be the space of orderings of F and $X(F/P_1), \dots, X(F/P_n)$ be the connected components of $X(F)$. We write $n(i) = \dim X_i$, $P_i^c = \cap_{j \neq i} P_j$ and $m(i) = \dim \dot{F}/D_{P_i}\langle 1, a\rangle$. We define the subgroups A_i $(i = 1, \dots, n)$ of $\dot{K}$ as follows. If $-a \in P_i$, then we put $A_i = \cap_{j=1,\dots,n(i)} D_K\langle 1, c_j\rangle$, where $\{c_j\}$ $j = 1, \dots, n(i)$ is the basis of $\dot{F}/P_i$ given in Theorem 4.2. In this case, $m(i) = 0$, and so we have $\dim D_K(2)/A_i = n(i) - m(i) - 1$ by Theorem 4.2. If $a \in P_i$, then we put $A_i = D_K(2)$. It is clear that $\dim D_K(2)/A_i = n(i) - m(i) = 0$. If $a \notin \pm P_i$, then we put $A_i = D_K(2) \cap (\cap_{j=1,\dots,n(i)} D_K\langle 1, -d_j\rangle)$, where $\{d_j\}$ $j = 1, \dots, n(i)$ is the basis of $\dot{F}/P_i$ given in Theorem 5.1. By Theorem 5.1, we have $\dim D_K(2)/A_i \leqq n(i) - m(i) - 1$.

By the way, let $\{b_{i1}, \dots, b_{in(i)}\}$ $(i = 1, \dots, n)$ be a basis of $\dot{F}/P_i$, consisting of elements in P_i^c. Then we can easily see that $\cup_{i=1,\dots,n} \{b_{i1}, \dots, b_{in(i)}\}$ is a basis of $\dot{F}/R(F)$. Therefore we have $R(K) = \cap_{i=1,\dots,n} A_i$ by Proposition 1.2 and Proposition 4.1. From this equation, it is easy to see that $\dim D_K(2)/R(K) \leqq \sum_{i=1,\dots,n} \dim D_K(2)/A_i$. On the other hand, from the above observation, we have

$$\sum_{i=1,\dots,n} \dim D_K(2)/A_i \leqq \sum_{i=1,\dots,n} n(i) - \sum_{i=1,\dots,n} m(i) - |I|,$$

where $I = \{i;\, a \notin P_i\}$. Since $\sum_{i=1,\dots,n} n(i) = \dim \dot{F}/R(F)$ and $\sum_{i=1,\dots,n} m(i) = \dim \dot{F}/D_F\langle 1, a\rangle$, we have $\dim D_K(2)/R(K) \leqq \dim D_F\langle 1, a\rangle / R(F) - |I|$. By Corollary 2.7, $\dim H_a/R(F) = |I|$, and it implies $\dim D_K(2)/R(K) \leqq \dim D_F\langle 1, a\rangle / H_a$. This proves the assertion by Proposition 1.2, (1). Q.E.D.

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