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Semilinear elliptic eigenvalue problems in R^N

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1. Introduction

The primary objective is to characterize numbers $\lambda \in \boldsymbol{R}$ such that the semilinear problem

$$(1.1)\qquad \begin{cases} -\Delta u + p(x)u - f(x, u) = \lambda u, & x \in \boldsymbol{R}^N \\ u \in L^2(\boldsymbol{R}^N) \end{cases}$$

has a positive solution $u(x)$ for all $x \in \boldsymbol{R}^N$, $N \geqq 2$, where $p(x)$ is locally Hölder continuous and bounded below in $\boldsymbol{R}^N$ and the nonlinearity satisfies hypotheses $(f_1)-(f_4)$ below. For example, $f(x, t)$ can have the form

$$f(x, t) = \sum_{i=1}^{J} f_i(x) t^{s_i},$$

where

$$1 < s_i < \infty, \quad N = 2,$$

$$1 < s_i < \frac{N+2}{N-2}, \quad N \geqq 3,\ i = 1,\dots, J,$$

and each f_i is a locally Hölder continuous function with $f_i \in L^{s_i+1}(\boldsymbol{R}^N)$.

Let $\lambda^* = \lim_{n\to\infty} \lambda(n)$, where $\lambda(n)$ is the lowest eigenvalue of the linear problem

$$-\Delta v + p(x)v = \lambda v, \quad |x| < n$$
$$v(x) = 0, \quad |x| = n$$

for $n = 1, 2,\dots$. The main Theorem 4.1 establishes, for all $\lambda < \lambda^*$, the existence of a positive solution $u \in W_0^{1,2}(\boldsymbol{R}^N)$ of (1.1) with locally Hölder continuous second partial derivatives in $\boldsymbol{R}^N$. The sharpness of this result is indicated in Examples 4.4 and 4.5: A positive solution of (1.1) does not exist in general if $\lambda \geqq \lambda^*$.

Theorems 4.2 and 4.3 give estimates for the exponential decay at infinity of the positive solution obtained in Theorem 4.1. In the case that $p(x)$ in (1.1) is specialized to $K^2|x|^{2m}$ for positive constants K and m, the estimate is

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$$
(1.2) \qquad \begin{aligned} & 0 < u(x) \leqq C|x|^{-a} \exp\left(-\frac{K}{m+1}|x|^{m+1}\right), \\ & a < (N+m-1)/2, \quad |x| \geqq R \end{aligned}
$$

for some positive constants C and R. This result is essentially the best possible since the asymptotic behavior in (1.2) corresponds to that for Thomé's classical local radial solution of the linearized equation (1.1) (if $a=(N+m-1)/2$).

By our techniques here and in [12], similar conclusions can be obtained for the elliptic eigenvalue problem arising when Δ in (1.1) is replaced by a general linear uniformly elliptic operator of second order. This will not be done to avoid technical questions outside the essential framework. Also, as in [12], an analogue of (1.1) can be treated in which $\boldsymbol{R}^N$ is replaced by an unbounded domain $\Omega \subset \boldsymbol{R}^N$, and the boundary condition $u|_{\partial\Omega}=0$ adjoined. Our procedure applies to a large class of unbounded domains Ω, in particular to exterior domains and quasiconical domains.

Nonlinear eigenvalue problems in *bounded* domains have been extensively investigated [1, 2, 8, 14–16, 18, and References therein], but results for unbounded domains are either limited to special structures or do not aim at positivity and exponential decay of the solutions [4 –7]. Our method is to first construct solutions $u_n(x)$ of Dirichlet problems for the differential equation (1.1) in bounded domains $\{x \in \boldsymbol{R}^N: |x|<n\}$, $n=1, 2,\ldots$ from the critical point theory of Ambrosetti and Rabinowitz [2]. We then prove that there exists a subsequence of $\{u_n\}$ which converges both weakly in $W_0^{1,2}(\boldsymbol{R}^N)$ and locally uniformly in $C^2(\boldsymbol{R}^N)$ to a positive solution of (1.1). Finally, the exponential decay at infinity is established via L^p-estimates, interior estimates, Sobolev embedding, and the maximum principle.

2. Preliminaries

For integers $m \geqq 0$ and $p>1$, and a bounded domain M in $\boldsymbol{R}^N$, $W^{m,p}(M)$ denotes the Banach space of all (equivalence classes of) functions with generalized derivatives up to order m all belonging to $L^p(M)$. The Sobolev space $W_0^{m,p}(\boldsymbol{R}^N)$ is defined as the completion of the set $C_0^\infty(\boldsymbol{R}^N)$ of all infinitely differentiable functions with compact support in $\boldsymbol{R}^N$ with respect to the $W^{m,p}(\boldsymbol{R}^N)$ norm, i.e.

$$
\|u\|_{m,p,\boldsymbol{R}^N} = \left[\int_{\boldsymbol{R}^N} \sum_{|\sigma| \leqq m} |D^\sigma u(x)|^p dx\right]^{1/p}
$$

in multi-index notation.

Hölder spaces on bounded domains $M \subset \boldsymbol{R}^N$ are denoted by $C^{m+\alpha}(\overline{M})$, with norms $\|\cdot\|_{m+\alpha,\overline{M}}$, $0<\alpha<1$; $m=0, 1, 2,\ldots$. The notation $C_{\mathrm{loc}}^{m+\alpha}(\boldsymbol{R}^N)$ is used for the set of all $u \in C^{m+\alpha}(\overline{M})$ for every bounded subdomain M of $\boldsymbol{R}^N$.

The conditions (p), (f_1)–(f_4) below are to be imposed on equation (1.1) throughout the sequel:

(p) $p \in C^{\alpha}_{loc}(\boldsymbol{R}^N)$ for fixed $\alpha \in (0, 1)$, and $p(x)$ is bounded from below in $\boldsymbol{R}^N$; without loss of generality $p(x) \geqq 0$ for all $x \in \boldsymbol{R}^N$ since λ in (1.1) can be translated if necessary.

(f_1) $f \in C^{\alpha}_{loc}(\boldsymbol{R}^N \times \boldsymbol{R})$ and $f(x, t)$ is locally Lipschitz continuous with respect to t for all $x \in \boldsymbol{R}^N$.

(f_2) There exist constants $s_i > 1$ and nonnegative bounded functions $f_i \in L^{s_i+1}(\boldsymbol{R}^N)$, $i = 1, \ldots, J$, such that

$$|f(x, t)| \leqq \sum_{i=1}^{J} f_i(x)|t|^{s_i}, \quad x \in \boldsymbol{R}^N,\ t \in \boldsymbol{R},$$

where

$$1 < s_i < \infty \qquad \text{if} \quad N = 2,$$

$$1 < s_i < \frac{N+2}{N-2} \qquad \text{if} \quad N \geqq 3.$$

(f_3) $\lim_{t\to\infty} \dfrac{f(x, t)}{t} = +\infty$ locally uniformly in $\boldsymbol{R}^N$.

(f_4) There exists a positive constant ε such that $(2+\varepsilon)F(x, t) \leqq tf(x, t)$ for all $t \geqq 0$, $x \in \boldsymbol{R}^N$, where

$$F(x, t) = \int_0^t f(x, s)ds.$$

Condition (f_2) implies in particular that $f(x, t) = o(t)$ as $t \to 0$ uniformly in $\boldsymbol{R}^N$.

For functions $\phi \in W_0^{1,2}(\boldsymbol{R}^N)$ with compact support in $\boldsymbol{R}^N$, define $I(\phi) = I_1(\phi) - I_2(\phi)$, where

$$I_1(\phi) = \frac{1}{2}\int_{\boldsymbol{R}^N} [|\nabla \phi|^2 + p(x)\phi^2(x) - \lambda\phi^2(x)]dx, \tag{2.1}$$

$$I_2(\phi) = \int_{\boldsymbol{R}^N} F(x, \phi(x))dx. \tag{2.2}$$

3. Local solutions and a priori bounds

We use the notation

$$\Omega_n = \{x \in \boldsymbol{R}^N : |x| < n\}; \quad \Omega_n^c = \boldsymbol{R}^N \backslash \Omega_n; \tag{3.1}$$

$$S_n = \{x \in \boldsymbol{R}^N : |x| = n\}, \quad n = 1, 2, \ldots. \tag{3.2}$$

Let $\lambda(n)$ denote the smallest eigenvalue of the linear problem

$$(3.3) \qquad -\Delta v + p(x)v = \lambda v, \quad x \in \Omega_n$$
$$v|_{S_n} = 0,$$

as guaranteed by the Krein–Rutman theorem. Since $\bar{\Omega}_n \subset \Omega_{n+1}$ for $n=1, 2,\dots$ it is well known that $\lambda(n) > \lambda(n+1)$ for $n=1, 2,\dots$ and $\{\lambda(n)\}$ is bounded below. Then we can define

$$\lambda^* = \lim_{n\to\infty} \lambda(n).$$

LEMMA 3.1. *If $\lambda < \lambda^*$, there exists a constant C such that*

$$(3.4) \qquad I_1(\phi) \geqq C\|\phi\|^2_{1,2,\boldsymbol{R}^N}$$

for all $\phi \in W_0^{1,2}(\boldsymbol{R}^N)$ with compact support in $\boldsymbol{R}^N$.

PROOF. For any such ϕ, there exists an integer n such that supp $\phi \subset \Omega_n$. It follows from the variational characterization of $\lambda(n)$ that

$$\lambda(n)\int_{\boldsymbol{R}^N} \phi^2(x)dx = \lambda(n)\int_{\Omega_n} \phi^2(x)dx \leqq \int_{\Omega_n} [|\nabla\phi|^2 + p(x)\phi^2(x)]dx$$

and hence that

$$\lambda(n)\int_{\boldsymbol{R}^N} \phi^2(x)dx \leqq \int_{\boldsymbol{R}^N} [|\nabla\phi|^2 + p(x)\phi^2(x)]dx.$$

Therefore, since $\lambda^* \leqq \lambda(n)$ for all n,

$$(\lambda^*+1)\int_{\boldsymbol{R}^N} \phi^2(x)dx \leqq [\lambda(n)+1]\int_{\boldsymbol{R}^N} \phi^2(x)dx$$
$$\leqq \int_{\boldsymbol{R}^N} [|\nabla\phi|^2 + (p(x)+1)\phi^2(x)]dx.$$

Then the definition (2.1) gives

$$I_1(\phi) = \frac{1}{2}\int_{\boldsymbol{R}^N}\Big[|\nabla\phi|^2 + (p(x)+1)\phi^2(x)\Big]dx - \frac{1}{2}(\lambda+1)\int_{\boldsymbol{R}^N} \phi^2(x)dx$$
$$\geqq \frac{1}{2}\Big[1 - \frac{\lambda+1}{\lambda^*+1}\Big]\int_{\boldsymbol{R}^N} [|\nabla\phi|^2 + (p(x)+1)\phi^2(x)]dx,$$

which implies the conclusion (3.4) since $p(x) \geqq 0$ throughout $\boldsymbol{R}^N$ and $\lambda < \lambda^*$.

For $\rho > 0$ define

$$B_\rho = \{\phi \in W_0^{1,2}(\boldsymbol{R}^N) \colon \|\phi\|_{1,2,\boldsymbol{R}^N} < \rho\},$$
$$E_\rho = \{\phi \in W_0^{1,2}(\boldsymbol{R}^N) \colon \|\phi\|_{1,2,\boldsymbol{R}^N} = \rho\}.$$

LEMMA 3.2. *If $\lambda < \lambda^*$, there exist positive constants ν and ρ such that*

$$I(\phi) > 0 \quad \textit{for all} \quad \phi \in B_\rho \backslash \{0\};$$
$$I(\phi) \geqq \nu \quad \textit{for all} \quad \phi \in E_\rho.$$

PROOF. For arbitrary $\varepsilon > 0$ and arbitrary $\phi \in W_0^{1,2}(\boldsymbol{R}^N)$, assumption ($f_2$) easily leads to the estimate

$$(3.5) \qquad |I_2(\phi)| \leqq \int_{\boldsymbol{R}^N} [\varepsilon|\phi|^2 + C_1|\phi|^{s+1}]dx$$

for some positive constant C_1, where

$$s = \max\{s_1, \dots, s_J\}, \quad \text{so} \quad s > 1.$$

Since $2 < s+1 < 2N/(N-2)$, $N \geqq 3$, by (f_2), an embedding theorem of Aronszajn and Smith [3] (see also Berger and Schechter [6, p. 264]) shows that there exist positive constants C_2 and C_3, independent of ϕ, such that

$$\int_{\boldsymbol{R}^N} |\phi|^{s+1}dx \leqq C_2\|\phi\|_{1,2,\boldsymbol{R}^N}^{s+1}$$

and

$$\int_{\boldsymbol{R}^N} |\phi|^2 dx \leqq C_3\|\phi\|_{1,2,\boldsymbol{R}^N}^2.$$

Then (3.5) implies that

$$|I_2(\phi)| \leqq (\varepsilon C_3 + C_1C_2\rho^{s-1})\,\|\phi\|_{1,2,\boldsymbol{R}^N}^2$$

for all $\phi \in B_\rho \cup E_\rho$. Let C be as in Lemma 3.1 and choose ε and ρ such that

$$\varepsilon C_3 = \frac{C}{4} = C_1C_2\rho^{s-1}.$$

Then

$$(3.6) \qquad |I_2(\phi)| \leqq \frac{1}{2} C\|\phi\|_{1,2,\boldsymbol{R}^N}^2, \quad \phi \in B_\rho \cup E_\rho.$$

Let $\nu = C\rho^2/2$. Then the conclusions of Lemma 3.2 follow from (3.4) and (3.6).

THEOREM 3.3 (Ambrosetti and Rabinowitz [2, p. 365]). *If* $\lambda < \lambda^*$, *there exists a sequence of nonnegative functions* $u_n \in W_0^{1,2}(\Omega_n)$, $n = 1, 2, \dots$, *with the following properties*:

(A) $u_n \in C^{2+\alpha}(\bar{\Omega}_n)$, α *as in* (p), (f_1);

(B) $-\Delta u_n(x) + p(x)u_n(x) = \lambda u_n(x) + f(x, u_n(x))$, $x \in \Omega_n$;

(C) $u_n(x) = 0$ if $|x| \geqq n$;

(D) $\quad u_n(x) > 0 \quad \text{if} \quad x \in \Omega_n.$

Furthermore, the sequence $v_n = I(u_n)$, $n = 1, 2, \ldots$, *is nonincreasing and satisfies* $v_n \geqq v > 0$ *for all n, where v is as in Lemma* 3.2.

A slight modification of the proof given by Ambrosetti and Rabinowitz [2, p. 364] shows that there exists an element $e_1 \in W_0^{1,2}(\Omega_1)$ such that $I(e_1) = 0$ and $\|e_1\|_{1,2,\Omega_1} > 0$. Therefore $\|e_1\|_{1,2,\Omega_1} > \rho$ by Lemma 3.2 above. We can then define an element $e_n \in W_0^{1,2}(\Omega_n)$ to be the extension of e_1 to Ω_n which is identically zero outside Ω_1, and consequently $\|e_n\|_{1,2,\Omega_n} > \rho$. The nonincreasing property of $\{v_n\}$ follows from the variational characterization of v_n in [2] since $\bar{\Omega}_n \subset \Omega_{n+1}$ for each $n = 1, 2, \ldots$. The property $v_n \geqq v > 0$ is implied by Lemma 3.2 and the above fact that E_ρ separates e_n and the zero element in $W_0^{1,2}(\Omega_n)$.

LEMMA 3.4. *The sequence* $\{u_n\}$ *in Theorem* 3.3 *is uniformly bounded in the* $W^{1,2}(\boldsymbol{R}^N)$ *norm.*

This can be proved routinely from Lemma 3.1, Theorem 3.3, Green's theorem, and Assumption (f_4).

LEMMA 3.5. *For any bounded domain G in* $\boldsymbol{R}^N$ *there exists a positive integer* $m = m(G)$ *and positive constants K and* α, $0 < \alpha < 1$, *independent of n, such that the sequence* $\{u_n\}$ *in Theorem* 3.3 *satisfies*

$$\|u_n\|_{2+\alpha,\bar{G}} \leqq K \qquad \text{for all} \quad n \geqq m. \tag{3.7}$$

PROOF. Let m be a positive integer for which $\bar{G} \subset \Omega_m$, so also $\bar{G} \subset \Omega_n$ for all $n \geqq m$. Let s be as in Lemma 3.2 and define

$$p = \frac{2N}{(N-2)s}, \quad N \geqq 3. \tag{3.8}$$

The proof will be given for the case $p \geqq N/2$, $N \geqq 3$.

Let M and Q be smooth bounded domains such that $\bar{G} \subset M$, $\bar{M} \subset Q$, and $\bar{Q} \subset \Omega_m$. In view of (3.8), a standard embedding theorem [9, p. 43] states that there exists a constant C, independent of n, such that

$$\|u_n\|_{0,ps,\Omega_m} \leqq C\|u_n\|_{1,2,\Omega_m}, \; n \geqq m. \tag{3.9}$$

Then Lemma 3.4 shows that $\|u_n\|_{0,ps,\Omega_m}$ is uniformly bounded with respect to n. Define

$$F_n(x) = \lambda u_n(x) + f(x, u_n(x)), \quad n = 1, 2, \ldots.$$

It follows from the growth hypothesis (f_2) that $\|F_n\|_{0,p,\Omega_m}$ is uniformly bounded for $n \geqq m$. Since u_n satisfies the differential equation $(-\Delta + p)u_n = F_n$ in Ω_m for

$n \geqq m$ by Theorem 3.3, application of the a priori estimate [10, Theorem 37I, p. 169]

$$\|u_n\|_{2,p,Q} \leqq C_1(\|F_n\|_{0,p,\Omega_m} + \|u_n\|_{0,2,\Omega_m})$$

yields the uniform estimate

$$\|u_n\|_{2,p,Q} \leqq C_2, \quad n \geqq m$$

for some positive constant C_2, independent of n. Use of $L^r(Q)$-estimates again [9, p. 43], as in (3.9), shows that $\|u_n\|_{0,r,Q}$ is uniformly bounded for arbitrary r in $1<r<\infty$, and hence also $\|F_n\|_{0,r,Q}$ is uniformly bounded by (f_2). Another application of the a priori estimate for the differential equation $(-\Delta+p)u_n = F_n$ [10, p. 169] gives

$$\|u_n\|_{2,r,M} \leqq C_3, \quad n \geqq m$$

for another positive constant C_3 independent of n, and for arbitrary $r>1$. Sobolev embedding [9, p. 43] then implies that $\|u_n\|_{1+\alpha,M}$ is uniformly bounded for any α in $0<\alpha<1$. Since $f \in C^{\alpha}_{loc}$ from (f_1), the conclusion (3.7) follows from an interior Schauder estimate [9, p. 110].

The proof of Lemma 3.5 for the case $N=2$ is essentially the same. The proof for $p<N/2$, $N \geqq 3$ is a modification with more steps in the bootstrap procedure.

4. Existence of positive solutions in $\boldsymbol{R}^N$

For $\lambda<\lambda^*$, a positive solution of (1.1) with exponential decay at ∞ will be obtained as the limit in $C^2_{loc}(\boldsymbol{R}^N)$ of a convergent subsequence of the sequence $\{u_n\}$ guaranteed by Theorem 3.3.

THEOREM 4.1. *Suppose that* (p) *and* (f_1)–(f_4) *are satisfied and* $\lambda<\lambda^*$, *where* λ^* *is as in* §3. *Then* (1.1) *has a positive solution* $u(x)$ *in* $\boldsymbol{R}^N$ *with the following properties*:

(i) $u \in W^{1,2}_0(\boldsymbol{R}^N) \cap C^{2+\alpha}_{loc}(\boldsymbol{R}^N)$;

(ii) $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} (\nabla u)(x) = 0$

uniformly in $\boldsymbol{R}^N$.

PROOF. Let $\{u_n(x)\}$ be the sequence in Theorem 3.3, and let G be any bounded domain in $\boldsymbol{R}^N$. The procedure in [11, 12] shows in view of Lemma 3.5 and the compactness of the injection $C^{2+\alpha}(\bar{G}) \to C^2(\bar{G})$ that $\{u_n\}$ has a subsequence $\{u_n^*\}$ which converges in the $C^2(\bar{G})$ norm to a function $u \in C^2(\bar{G})$. It follows from Theorem 3.3(B) that u satisfies the differential equation (1.1) on $\bar{G}$, and hence $u \in C^{2+\alpha}(\bar{G})$ by a standard Schauder estimate. Then $u \in C^{2+\alpha}_{loc}(\boldsymbol{R}^N)$, and weak convergence in $W_0^{1,2}(\boldsymbol{R}^N)$ of a subsequence of $\{u_n^*\}$ to $\tilde{u} \in W_0^{1,2}(\boldsymbol{R}^N)$ follows from the uniform boundedness of $\|u_n^*\|_{1,2,\boldsymbol{R}^N}$ (Lemma 3.4). Evidently $\tilde{u}=u$ in any bounded domain G by the convergence of $\{u_n^*\}$ in $C^2(\bar{G})$.

The next step is to show that $u(x)$ is not the zero function. By Theorem 3.3 (B, C), Green's theorem applied to u_n^* gives, since u_n^* has support Ω_n,

$$0 < \nu \leqq I(u_n^*) = \int_{\boldsymbol{R}^N} f(x, u_n^*(x)) u_n^*(x) dx - I_2(u_n^*) \tag{4.1}$$

for $n=1, 2,\dots$. Then assumption (f_2) and Hölder's inequality lead to

$$0 < \nu \leqq \sum_{i=1}^{J} \|f_i u_n^*\|_{0,s_i+1,\boldsymbol{R}^N} \|u_n^*\|_{0,s_i+1,\boldsymbol{R}^N}^{s_i} - I_2(u_n^*)\,. \tag{4.2}$$

Since

$$2 < s_i + 1 < \frac{2N}{N-2}\,,\ N \geqq 3,\ i = 1,\dots, J$$

by (f_2), there exists a positive constant C, independent of u, such that

$$\|u\|_{0,s_i+1,\boldsymbol{R}^N} \leqq C\|u\|_{1,2,\boldsymbol{R}^N}$$

for all $u \in W_0^{1,2}(\boldsymbol{R}^N)$ [3, 6, p. 264], and clearly this also holds if $N=2$. Then Lemma 3.4 implies that the sequence of norms $\|u_n^*\|_{0,s_i+1,\boldsymbol{R}^N}$ is uniformly bounded. It can also be shown without difficulty [12] because of the compactness of the multiplication operator $u \to f_i u$ from $W_0^{1,2}(\boldsymbol{R}^N)$ into $L^{s_i+1}(\boldsymbol{R}^N)$ [6, p. 264] that $\{u_n^*\}$ has a subsequence $\{\tilde{u}_n\}$ such that both

$$\lim_{n\to\infty} \|f_i \tilde{u}_n\|_{0,s_i+1,\boldsymbol{R}^N} = \|f_i u\|_{0,s_i+1,\boldsymbol{R}^N}$$

and

$$\lim_{n\to\infty} I_2(\tilde{u}_n) = I_2(u)\,,$$

where u is the solution of (1.1) constructed above. Then (4.2) implies that there is a positive constant K such that

$$0 < \nu \leqq K \sum_{i=1}^{J} \|f_i u\|_{0,s_i+1,\boldsymbol{R}^N} - I_2(u)\,,$$

showing that $u(x)$ is not identically zero.

To prove properties (ii) of the theorem, we use the notation

$$M(x) = \{y \in \boldsymbol{R}^N\colon |y-x|<1\},\quad x \in \boldsymbol{R}^N;$$

$$N(x) = \left\{y \in \boldsymbol{R}^N\colon |y-x| < \frac{1}{2}\right\},\quad x \in \boldsymbol{R}^N;$$

$$\sigma = \min\{s_i : i=1,\dots, J\}.$$

The proof of (ii) will be given in the case $N=2$. The case $N \geqq 3$ is similar, along the lines presented in Lemma 3.5.

A standard estimate in $L^r(M(x))$ is, if $N=2$ [9, p. 43],

$$\|u\|_{0,r,M(x)} \leqq C\|u\|_{1,2,M(x)}, \quad x \in \boldsymbol{R}^N \tag{4.3}$$

for some positive constant C independent of u, and for arbitrary $r>1$. Since the constant K in Lemma 3.5 depends only on N, s, and the volume of G (not on its location), we can take G in Lemma 3.5 to be $M(x)$ and conclude that $\{u_n(x)\}$ is uniformly bounded in $\boldsymbol{R}^N$, from which $u(x)$ also is bounded in $\boldsymbol{R}^N$. Then

$$|u(x)|^{rs_i/\sigma} \leqq \text{constant}\, |u(x)|^r, \quad x \in \boldsymbol{R}^N$$

for $i=1,\dots, J$ since each $s_i \geqq \sigma$. By assumption (f_2) there exists a positive constant C_1 such that

$$\int_{M(x)} |f(y, u(y))|^{r/\sigma} dy \leqq C_1 \int_{M(x)} |u(y)|^r dy.$$

Let $F(y)=f(y, u(y))$. Then

$$\|F\|_{0,r/\sigma,M(x)} \leqq C_2\|u\|^{\sigma}_{0,r,M(x)}$$

for another positive constant C_2, and (4.3) yields

$$\|F\|_{0,r/\sigma,M(x)} \leqq C_2 C^{\sigma}\|u\|^{\sigma}_{1,2,M(x)}. \tag{4.4}$$

Then a standard a priori interior estimate for equation (1.1) gives [10, p. 169]

$$\begin{aligned}\|u\|_{2,r/\sigma,N(x)} &\leqq C_3[\|F\|_{0,r/\sigma,M(x)} + \|u\|_{0,2,M(x)}] \\ &\leqq C_4\|u\|_{1,2,M(x)}\end{aligned}$$

for another constant C_4, upon use of (4.3), (4.4), $\sigma>1$, and the finiteness of $\|u\|_{1,2,\boldsymbol{R}^N}$. The Sobolev embedding lemma [9, p. 43] therefore shows that

$$\|u\|_{1+\alpha,\overline{N(x)}} \leqq C_5\|u\|_{1,2,M(x)}$$

for a positive constant C_5 and for arbitrary $\alpha \in (0, 1)$, proving property (ii) of Theorem 4.1 in the case $N=2$.

To prove the positivity of $u(x)$ throughout $\boldsymbol{R}^N$, notice from (1.1), assumption (f_2), and property (ii) that $u(x)$ satisfies a linear elliptic inequality $-\Delta u+\gamma u \geqq 0$ in $\boldsymbol{R}^N$ for some constant $\gamma>0$. Since $u(x)$ is a nontrivial nonnegative solution of this inequality in Ω_n, the strong maximum principle [13] applied to Ω_n shows that $u(x)>0$ throughout Ω_n, $n=1, 2,\dots$, and therefore throughout $\boldsymbol{R}^N$.

THEOREM 4.2. *If the hypotheses of Theorem* 4.1 *are satisfied and in addition* $\lim_{|x|\to\infty} p(x) = +\infty$, *there exist positive constants* C_0 *and* δ *such that the solution* $u(x)$ *of* (1.1) *in Theorem* 4.1 *satisfies*

$$0 < u(x) \leqq C_0 e^{-\delta|x|} \qquad \textit{for all} \quad x \in \boldsymbol{R}^N. \tag{4.5}$$

PROOF. Choose a positive number ρ large enough so that $\gamma > 0$, where

$$\gamma = \inf_{|x| \geqq \rho} p(x) - \lambda. \tag{4.6}$$

Define $L = -\Delta + \frac{1}{2}\gamma$, $v(x) = C\exp(-\delta|x|)$ for positive constants C and δ to be determined. An easy calculation gives

$$\frac{Lv}{v} = -\delta^2 + \frac{\gamma}{2} + \frac{(N-1)\delta}{r},$$

where $r = |x|$. Therefore there exists a sufficiently small positive number δ such that

$$(Lv)(x) \geqq 0 \qquad \text{for all} \quad x \in \boldsymbol{R}^N \quad \text{with} \quad |x| \geqq \rho. \tag{4.7}$$

By assumption (f_2) and Theorem 4.1(ii) there exists a number $R \geqq \rho$ such that

$$|f(x, u(x))| \leqq \frac{\gamma}{2} u(x) \qquad \text{for} \quad |x| \geqq R.$$

Then Theorem 4.1 and (4.6) show that $u(x)$ satisfies the differential inequality

$$\begin{aligned}(Lu)(x) &= \left[\lambda - p(x) + \frac{\gamma}{2}\right] u(x) + f(x, u(x)) \\ &\leqq [\lambda - p(x) + \gamma] u(x) \leqq 0\end{aligned} \tag{4.8}$$

for all $|x| \geqq R$. We can assume that $u(x) \leqq 1$ for all $|x| \geqq R$ by Theorem 4.1(ii). Let $C = e^{\delta R}$ in the definition of $v(x)$. Then on $|x| = R$,

$$v(x) = Ce^{-\delta|x|} = 1 \geqq u(x).$$

It follows from (4.7) and (4.8) that $L(v-u) \geqq 0$ for $|x| \geqq R$ and $v - u \geqq 0$ on $|x| = R$. Since $v(x) - u(x) \to 0$ as $|x| \to \infty$ uniformly in $\boldsymbol{R}^N$, the maximum principle shows that $v - u \geqq 0$ throughout $\{x \in \boldsymbol{R}^N : |x| \geqq R\}$. This proves (4.5), where

$$C_0 = \max\{C, \sup_{|x| \leqq R} e^{\delta|x|} u(x)\}.$$

Sharper estimates for the exponential decay at ∞ of positive solutions of (1.1) will now be obtained when (1.1) is specialized to the form

(4.9) $$L_0u = -\Delta u + k^2|x|^{2m}u = \lambda u + f(x, u), \quad x \in \boldsymbol{R}^N$$
$$u \in L^2(\boldsymbol{R}^N),$$

where k and m are positive constants and $f(x, u)$ satisfies hypotheses (f_1)–(f_4).

THEOREM 4.3. *If $\lambda<\lambda^*$, problem (4.9) has a positive solution $u(x)$ in $\boldsymbol{R}^N$ such that*

$$u(x) \leqq C|x|^{-a} \exp\left(-\frac{k}{m+1}|x|^{m+1}\right), \quad |x| \geqq R$$

for some positive constants C and R, and for any $a<(N+m-1)/2$.

PROOF. Let $\rho_0=(2|\lambda|/k^2)^{1/2m}$ and let L_1 be the linear elliptic operator defined in $\Omega^c_{\rho_0}$ by

(4.10) $$L_1u = L_0u - 2\lambda u = -\Delta u + k^2|x|^{2m}u - 2\lambda u.$$

For a constant a to be determined, define

$$v(x) = r^{-a} \exp\left(-\frac{k}{m+1} r^{m+1}\right), \quad r = |x|.$$

Calculation gives

$$\frac{L_1 v}{v} = k(N+m-1-2a)r^{m-1} - 2\lambda + a(N-2-a)r^{-2},$$

showing, if $a<(N+m-1)/2$, that there exists a number $\rho \geqq \rho_0$ such that

(4.11) $$(L_1v)(x) \geqq 0 \qquad \text{for all} \quad x \in \Omega^c_\rho.$$

By assumption (f_2) and Theorem 4.1(ii), there exists $R \geqq \rho$ such that both

(4.12) $$0 < u(x) \leqq 1 \quad \text{and} \quad |f(x, u(x))| \leqq \lambda u(x)$$

for all $|x| \geqq R$. Therefore, by Theorem 4.1, $u(x)$ satisfies the inequality

(4.13) $$(L_1u)(x) = -\lambda u(x) + f(x, u(x)) \leqq 0, \; x \in \Omega^c_R.$$

Define

(4.14) $$V(x) = Cv(x), \quad C = R^a \exp\left(\frac{k}{m+1} R^{m+1}\right).$$

Then $V(x)$ satisfies (4.11) and $V(x)=1 \geqq u(x)$ on $|x|=R$. We conclude from (4.11)–(4.14) that

$$L_1(V-u) \geqq 0 \qquad \text{in} \quad \Omega^c_R$$

$$V(x) - u(x) \geqq 0 \quad \text{on} \quad |x| = R$$

$$V(x) - u(x) \to 0 \quad \text{as} \quad |x| \to \infty .$$

Hence $V(x)-u(x) \geqq 0$ throughout Ω_R^c by the maximum principle, i.e., $u(x) \leqq Cv(x)$ for all $|x| \geqq R$, completing the proof of Theorem 4.3.

Example 4.4. An example of a problem (1.1) which does not have a positive solution in $\boldsymbol{R}^N$ for any $\lambda > \lambda^*$ is

$$(4.15) \qquad \begin{aligned} &-\Delta u + p(x)u - q(x)u^3 = \lambda u, \quad x \in \boldsymbol{R}^3 \\ &u \in L^2(\boldsymbol{R}^3), \end{aligned}$$

where $p(x)$ satisfies hypothesis (p), $q(x)$ is positive, bounded, and locally Hölder continuous, and $q \in L^4(\boldsymbol{R}^3)$.

To prove this, suppose to the contrary that (4.15) has a positive solution in $\boldsymbol{R}^3$ for some $\lambda > \lambda^*$. Let $\lambda(n)$ denote the smallest eigenvalue of the linear problem (3.3), $\lambda(n) > \lambda(n+1) > \lambda^*$ for every $n = 1, 2, \dots$, and let $v_n(x)$ be a positive normalized eigenfunction of (3.3) in Ω_n corresponding to $\lambda(n)$. Then

$$(4.16) \qquad \int_{\Omega_n} (|\nabla v_n|^2 + p(x)v_n^2)dx = \lambda(n) \int_{\Omega_n} v_n^2 dx.$$

Integration of Picone's identity over Ω_n gives

$$\int_{\Omega_n} \left(u^2 \left| \nabla \left(\frac{v_n}{u} \right) \right|^2 + \nabla \cdot \frac{v_n^2}{u} \nabla u \right) dx = \int_{\Omega_n} \left(|\nabla v_n|^2 + \frac{v_n^2}{u} \Delta u \right) dx,$$

$$n = 1, 2, \dots .$$

By (4.15), (4.16), and the divergence theorem, this reduces to

$$(4.17) \qquad 0 < \int_{\Omega_n} u^2 \left| \nabla \left(\frac{v_n}{u} \right) \right|^2 dx = \int_{\Omega_n} [(\lambda(n) - \lambda)v_n^2 - q(x)u^2 v_n^2] dx .$$

If $\lambda > \lambda^*$, we can choose an integer n such that $\lambda(n) < \lambda$, for which the right side of (4.17) is negative, a contradiction.

Example 4.5. If $\lim_{|x| \to \infty} p(x) = +\infty$ in addition to the other hypotheses, (4.15) is an example of a problem (1.1) with no positive solution in $\boldsymbol{R}^N$ for any $\lambda \geqq \lambda^*$.

In view of Example 4.4, it is enough to show this if $\lambda = \lambda^*$. Let the normalized eigenfunction $v_n(x)$ of (3.3) corresponding to $\lambda(n)$ be extended to $\boldsymbol{R}^3$ by defining Ω_n to be its support. Since $\lim_{|x| \to \infty} p(x) = +\infty$, it is known [17] that λ^* is the smallest eigenvalue of the linear problem

$$(4.18) \qquad -\Delta v + p(x)v = \lambda^* v, \qquad v \in L^2(\boldsymbol{R}^3)$$

and that

$$\lambda^* = \lim_{n\to\infty} \lambda(n), \quad \lim_{n\to\infty} \|v_n - v\|_{L^2(\boldsymbol{R}^3)} = 0, \tag{4.19}$$

where v is a normalized eigenfunction for (4.18) corresponding to λ^*. Let

$$Q = \sup_{x\in\boldsymbol{R}^3} q(x)u^2(x).$$

Then by the Schwarz inequality

$$\int_{\boldsymbol{R}^3} q(x)u^2(x)\,[v_n^2(x) - v^2(x)]dx \leqq Q\|v_n + v\|_{L^2(\boldsymbol{R}^3)}\|v_n - v\|_{L^2(\boldsymbol{R}^3)}.$$

Since $\|v_n + v\|_{L^2(\boldsymbol{R}^3)}$ is uniformly bounded in n, this implies that

$$\lim_{n\to\infty} \int_{\boldsymbol{R}^3} q(x)u^2(x)v_n^2(x)dx = \int_{\boldsymbol{R}^3} q(x)u^2(x)v^2(x)dx. \tag{4.20}$$

If $\lambda = \lambda^*$, it follows from (4.17), (4.19), and (4.20) that

$$0 \leqq -\int_{\boldsymbol{R}^3} q(x)u^2(x)v^2(x)dx,$$

which is a contradiction.

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