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Two-step methods with two off-step nodes

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1. Introduction

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1.1}$$

where $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of this problem,

$$x_n = x_0 + nh \qquad (n = 1, 2,\dots; h > 0), \tag{1.2}$$

where h is a stepsize. Let y_1 be an approximation of $y(x_1)$ obtained by some appropriate method. We are concerned with the case where the approximations y_j $(j=2, 3,\dots)$ of $y(x_j)$ are computed by two-step methods. Conventional two-step methods such as linear two-step methods [1], pseudo-Runge-Kutta methods [1, 3] and so on [4] require starting values y_0 and y_1 to generate y_j $(j=2, 3,\dots)$.

In our previous paper [5], introducing a set of subsidiary off-step nodes

$$x_{n+v} = x_0 + (n+v)h \qquad (n = 0, 1,\dots; 0 < v < 1), \tag{1.3}$$

at the cost of supplying an additional starting value y_v, we proposed two-step methods for computing y_{n+1} together with subsidiary approximations y_{n+v} of $y(x_{n+v})$ $(n=1, 2,\dots)$. It has been shown that for $r=2, 3$ there exists such a method of order $r+3$ with r function evaluations per step.

In this paper we introduce another set of off-step nodes

$$x_{n+\mu} = x_0 + (n+\mu)h \qquad (n = 0, 1,\dots; 0 < \mu < 1, \mu \neq v) \tag{1.4}$$

and at the expense of providing one more starting value y_μ we propose two-step methods of the form

$$y_{n+\mu} = y_n + b_{r+1}(y_n - y_{n-1}) + h \textstyle\sum_{j=0}^{r} c_{r+1j}k_{jn} \qquad (r = 3, 4, 5), \tag{1.5}$$

$$y_{n+v} = y_n + b_{r+2}(y_n - y_{n-1}) + h \textstyle\sum_{j=0}^{r+1} c_{r+2j}k_{jn}, \tag{1.6}$$

$$y_{n+1} = y_n + s(y_n - y_{n-1}) + h \textstyle\sum_{j=0}^{r+2} p_j k_{jn}, \tag{1.7}$$

where

(1.8) $k_{0n} = k_{3n-1},\ k_{1n} = f(x_{n-1+\mu},\ y_{n-1+\mu}),\ k_{2n} = f(x_{n-1+\nu},\ y_{n-1+\nu}),$
$k_{3n} = f(x_n,\ y_n),$

(1.9) $k_{in} = f(x_n + a_i h,\ y_n + b_i(y_n - y_{n-1}) + h\sum_{j=0}^{i-1} c_{ij}k_{jn}) \quad (i = 4, 5,\dots, r+2),$

(1.10) $a_i = b_i + \sum_{j=0}^{i-1} c_{ij} \quad (i = 4, 5,\dots, r+2), \quad a_{r+1} = \mu, \quad a_{r+2} = \nu,$

a_i, b_i, c_{ij} ($j=0, 1,\dots, i-1; j=4, 5,\dots, r+2$), p_j ($j=0, 1,\dots, r+2$) and s are real constants, and $y_{n+\mu}$ ($n=0, 1,\dots$) are subsidiary approximations of $y(x_{n+\mu})$. Convergence of these methods is studied in [6]. A stepsize control is implemented by comparing the method (1.7) with the method

(1.11) $$z_{n+1} = y_n + z(y_n - y_{n-1}) + h\sum_{j=0}^{r+2} w_j k_{jn}.$$

It is shown that for $r=3, 4, 5$ there exist a method (1.7) of order $r+3$ and a method (1.11) of order $r+2$ with r function evaluations per step. Finally these methods are illustrated by numerical examples.

2. Preliminaries

Let

(2.1) $t_{n+1} = u(y_n - y_{n-1}) + h\sum_{j=0}^{r+2} v_j k_{jn} \qquad (n = 1, 2,\dots),$

(2.2) $z_{n+1} = y_{n+1} + t_{n+1},$

(2.3) $a_1 = \mu - 1, \quad a_2 = \nu - 1, \quad a_3 = 0,$

(2.4) $y(x) + s(y(x) - y(x-h)) + h\sum_{j=0}^{r+2} p_j y'(x + a_j h) - y(x+h)$
$= \sum_{j=1}^{9} S_j(h^j/j!)y^{(j)}(x) + O(h^{10}),$

(2.5) $y(x) + b_i(y(x) - y(x-h)) + h\sum_{j=0}^{i-1} c_{ij} y'(x + a_j h) - y(x + a_i h)$
$= \sum_{j=1}^{8} e_{ij}(h^j/j!)y^{(j)}(x) + O(h^9) \qquad (i = 4, 5,\dots, r+2),$

(2.6) $u(y(x) - y(x-h)) + h\sum_{j=0}^{r+2} v_j y'(x + a_j h) = \sum_{j=1}^{8} W_j(h^j/j!)y^{(j)}(x) + O(h^9).$
Then we have

(2.7) $(-1)^{k-1} + k\sum_{j=0}^{r+2} a_j^{k-1}p_j - 1 = S_k \qquad (k = 1, 2,\dots, 9),$

(2.8) $(-1)^{k-1}b_i + k\sum_{j=0}^{i-1} a_j^{k-1}c_{ij} - a_i^k = e_{ik}$
$(i = 4, 5,\dots, r+2;\ k = 1, 2,\dots, 8),$

(2.9) $(-1)^{k-1}u + k\sum_{j=0}^{r+2} a_j^{k-1}v_j = W_k \qquad (k = 1, 2,\dots, 8).$
Let

(2.10) $k_{in}^* = y'(x_n + a_i h) \qquad (i = 0, 1, 2, 3),$

(2.11) $k_{in}^* = f(y_n + a_i h,\ y(x_n) + b_i(y(x_n) - y(x_{n-1})) + h \sum_{j=0}^{i-1} c_{ij} k_{jn}^*)$

$(i = 4, 5, \ldots, r+2)$,

(2.12) $g(x) = f_y(x, y(x))$,

(2.13) $T(x_n) = y(x_n) + s(y(x_n) - y(x_{n-1})) + h \sum_{j=0}^{r+2} p_j k_{jn}^* - y(x_{n+1})$,

(2.14) $T_\mu(x_n) = y(x_n) + b_{r+1}(y(x_n) - y(x_{n-1})) + h \sum_{j=0}^{r} c_{r+1\,j} k_{jn}^* - y(x_{n+\mu})$,

(2.15) $T_\nu(x_n) = y(x_n) + b_{r+2}(y(x_n) - y(x_{n-1})) + h \sum_{j=0}^{r+1} c_{r+2\,j} k_{jn}^* - y(x_{n+\nu})$,

(2.16) $R(x_n) = u(y(x_n) - y(x_{n-1})) + h \sum_{j=0}^{r+2} v_j k_{jn}^*$,

(2.17) $A_i = a_i(a_i + 1),\quad B_i = (a_i - a_1)A_i,\quad C_i = (a_i - a_2)B_i,\quad D_{mi} = (a_i - a_m)C_i,$

$E_{mi} = (a_i - a_{m+1})D_{mi},\quad F_{mi} = (a_i - a_{m+2})E_{mi}\quad (i = 0, 1, \ldots, r+2;\ m = 4, 5)$,

(2.18) $X_1 = a_1 + a_2,\quad Y_1 = a_1 a_2,\quad X_{m-2} = a_m + X_1,\quad Y_{m-2} = Y_1 + a_m X_1,$

$Z_{m-2} = a_m Y_1,\quad X_m = a_{m+1} + X_{m-2},\quad Y_m = Y_{m-2} + a_{m+1} X_{m-2},$

$Z_m = Z_{m-2} + a_{m+1} Y_{m-2},\quad U_m = a_{m+1} Z_{m-2},\quad X_{m+2} = a_{m+2} + X_m,$

$Y_{m+2} = Y_m + a_{m+2} X_m,\quad Z_{m+2} = Z_m + a_{m+2} Y_m,\quad U_{m+2} = U_m + a_{m+2} Z_m,$

$V_{m+2} = a_{m+2} U_m,$

(2.19) $P_{1j} = X_j - 1,\quad P_{2j} = Y_j - X_j,\quad P_{3j} = Z_j - Y_j,\quad P_{4j} = U_j - Z_j$

$(j = 1, 2, \ldots, 7)$,

(2.20) $F_{j+1} = \sum_{i=4}^{r+2} p_i e_{ij},\quad G_{j+2} = \sum_{i=4}^{r+2} p_i a_i e_{ij},\quad H_{j+2} = \sum_{i=5}^{r+2} p_i \sum_{k=4}^{i-1} c_{ik} e_{kj},$

$K_{j+1} = \sum_{i=4}^{r+2} v_i e_{ij},\quad M_{j+2} = \sum_{i=4}^{r+2} v_i a_i e_{ij},\quad N_{j+2} = \sum_{i=5}^{r+2} v_i \sum_{k=4}^{i-1} c_{ik} e_{kj},$

(2.21) $A = 3 + 5X_1 + 10Y_1,\quad B = 2 + 3X_1 + 5Y_1,\quad Q = 27 - 35X_1 + 50Y_1,$

(2.22) $R_{1i} = 2 + 3X_i + 5Y_i + 10Z_i,\quad R_{2i} = 10 + 14X_i + 21Y_i + 35Z_i,$

$R_{3i} = R_{2i} + 70U_i,\quad R_{4i} = 15 + 20X_i + 28Y_i + 42Z_i + 70U_i,$

$R_{5i} = R_{4i} + 140V_i,\quad R_{6i} = 35 + 45X_i + 60Y_i + 84Z_i + 126U_i + 210V_i$

$(i = 2, 3, \ldots, 7)$,

(2.23) $Q_{1i} = 22 - 27X_i + 35Y_i - 50Z_i,$

$Q_{2i} = 130 - 154X_i + 189Y_i - 245Z_i + 350U_i,$

$Q_{3i} = 225 - 260X_i + 308Y_i - 378Z_i + 490U_i,\quad Q_{4i} = Q_{3i} - 840V_i,$

$Q_{5i} = 595 - 675X_i + 780Y_i - 924Z_i + 1134U_i - 1470V_i,$

(2.24) $d_{1i}=a_i(2a_i+3),\quad d_{2i}=a_i^2(3a_i^2+4(1-a_1)a_i-6a_1),$

$d_{3i}=a_i^2(12a_i^3-15P_{11}a_i^2+20P_{21}a_i+30Y_1),$

$d_{4i}=a_i^3(10a_i^3-12P_{11}a_i^2+15P_{21}a_i+20Y_1),$

(2.25) $g_{1ik}=a_i^2(10a_i^4-12P_{1k}a_i^3+15P_{2k}a_i^2-20P_{3k}a_i-30Z_k),$

$g_{2ik}=a_i^3(60a_i^4-70P_{1k}a_i^3+84P_{2k}a_i^2-105P_{3k}a_i-105Z_k),$

$g_{3ik}=a_i^2(60a_i^5-70P_{1k}a_i^4+84P_{2k}a_i^3-105P_{3k}a_i^2+140P_{4k}a_i+210U_k),$

$g_{4ik}=a_i^3(105a_i^5-120P_{1k}a_i^4+140P_{2k}a_i^3-168P_{3k}a_i^2+210P_{4k}a_i+280U_k).$

Choosing $e_{ik}=c_{ij}=0$ ($i=4, 5, 6;\ k=1, 2,\dots, 5;\ j=4, 5,\dots, m-1$), we have

(2.26) $b_i+\sum_{j=0}^{i-1}c_{ij}=a_i,\quad -b_i+2\sum_{j=0}^{i-1}a_jc_{ij}=a_i^2,$

$-b_i+6\sum_{j=1}^{i-1}A_jc_{ij}=d_{1i},\quad (2a_1+1)b_i+12\sum_{j=2}^{i-1}B_jc_{ij}=d_{2i},$

$-Ab_i+60\sum_{j=m}^{i-1}C_jc_{ij}=d_{3i},$

(2.27) $Bb_i+60\sum_{j=m}^{i-1}a_jC_jc_{ij}-d_{4i}=10e_{i6},$

(2.28) $R_{1m-2}b_i+60\sum_{j=m+1}^{i-1}D_{mj}c_{ij}-g_{1im-2}=10e_{i6}\quad (i\geqq m+1),$

(2.29) $-R_{2m-2}b_i+420\sum_{j=m+1}^{i-1}a_jD_{mj}c_{ij}-g_{2im-2}=60e_{i7}-70P_{1m-2}e_{i6},$

(2.30) $-R_{3m}b_i+420\sum_{j=m+2}^{i-1}E_{mj}c_{ij}-g_{3im}=60e_{i7}-70P_{1m}e_{i6}\quad (i\geqq m+2),$

(2.31) $R_{4m}b_i+840\sum_{j=m+2}^{i-1}a_jE_{mj}c_{ij}-g_{4im}=105e_{i8}-120P_{1m}e_{i7}+140P_{2m}e_{i6},$

(2.32) $T(x)=\sum_{j=1}^{9}S_j(h^j/j!)y^{(j)}(x)+\sum_{j=6}^{8}F_{j+1}(h^{j+1}/j!)g(x)y^{(j)}(x)$

$+\sum_{j=6}^{7}[G_{j+2}g'(x)+H_{j+2}g^2(x)](h^{j+2}/j!)y^{(j)}(x)+O(h^{10}),$

(2.33) $T_\mu(x)=\sum_{j=6}^{8}e_{r+1\,j}(h^j/j!)y^{(j)}(x)$

$+\sum_{j=6}^{7}(\sum_{k=4}^{r}c_{r+1k}e_{kj})(h^{j+1}/j!)g(x)y^{(j)}(x)+O(h^8),$

(2.34) $T_\nu(x)=\sum_{j=6}^{8}e_{r+2j}(h^j/j!)y^{(j)}(x)$

$+\sum_{j=6}^{7}(\sum_{k=4}^{r+1}c_{r+2k}e_{kj})(h^{j+1}/j!)g(x)y^{(j)}(x)+O(h^8),$

(2.35) $R(x)=\sum_{j=1}^{8}W_j(h^j/j!)y^{(j)}(x)+\sum_{j=6}^{7}K_{j+1}(h^{j+1}/j!)g(x)y^{(j)}(x)$

$+[M_8g'(x)+N_8g^2(x)](h^8/6!)y^{(6)}(x)+O(h^9).$

If we put $S_i=0$ ($i=1, 2,\dots, 6$), we have

(2.36) $\sum_{j=0}^{r+2}p_j=1-s,\quad 2\sum_{j=0}^{r+2}a_jp_j=1+s,\quad 6\sum_{j=1}^{r+2}A_jp_j=5+s,$

$12\sum_{j=2}^{r+2}B_jp_j=7-10a_1-(2a_1+1)s,\quad 60\sum_{j=m}^{r+2}C_jp_j=As+Q,$

$60\sum_{j=m+1}^{r+2}D_{mj}p_j=-R_{1m-2}s+Q_{1m-2},$

(2.37) $420\sum_{j=m+2}^{r+2} E_{mj}p_j - R_{3m}s - Q_{2m} = 60S_7,$

(2.38) $840\sum_{j=m+2}^{r+2} a_jE_{mj}p_j + R_{4m}s - Q_{3m} = 105S_8 - 120P_{1m}S_7 \ (r \geqq m-1),$

(2.39) $840\sum_{j=m+3}^{r+2} F_{mj}p_j + R_{5m+2}s - Q_{4m+2} = 105S_8 - 120P_{1m+1}S_7 \ (r \geqq m),$

(2.40) $$2520\sum_{j=m+3}^{r+2} a_jF_{mj}p_j - R_{6m+2}s - Q_{5m+2} = 280S_9 - 315P_{1m+2}S_8 + 360P_{2m+2}S_7.$$

Setting $W_i = 0$ $(i = 1, 2,\ldots, 5)$, we have

(2.41) $$\sum_{j=0}^{r+2} v_j = -u, \quad 2\sum_{j=0}^{r+2} a_jv_j = u, \quad 6\sum_{j=1}^{r+2} A_jv_j = u,$$
$$12\sum_{j=2}^{r+2} B_jv_j = -(2a_1+1)u, \quad 60\sum_{j=m}^{r+2} C_jv_j = Au,$$

(2.42) $60\sum_{j=m+1}^{r+2} D_{mj}v_j + R_{1m-2}u = 10W_6,$

(2.43) $420\sum_{j=m+2}^{r+2} E_{mj}v_j - R_{3m}u = 60W_7 - 70P_{1m}W_6 \qquad (r \geqq m-1),$

(2.44) $$840\sum_{j=m+3}^{r+2} F_{mj}v_j + R_{5m+2}u = 105W_8 - 120P_{1m+2}W_7 + 140P_{2m+2}W_6 \qquad (r \geqq m).$$

From (1.5), (1.6) and (1.7) we have

(2.45) $$y_{n+2+\sigma} - (1+s)y_{n+1+\sigma} + sy_{n+\sigma} = h\Phi_\sigma(x_n; y_{n-1}, y_n, y_{n+1}, y_{n-1+\mu}, y_{n+\mu}, y_{n+1+\mu}, y_{n-1+\nu}, y_{n+\nu}, y_{n+1+\nu}; h) \qquad (\sigma = \mu, \nu, 0).$$

Hence the method (1.5)–(1.7) is stable if and only if $-1 \leqq s < 1$ [6].

3. Construction of the methods

We shall show the following

THEOREM. *For $r = 3, 4, 5$ there exists a method* (1.7) *of order $r+3$ which embeds a method* (2.2) *of order $r+2$.*

3.1. Case $r = 3$

We choose $S_i = e_{5i} = 0$ $(i = 1, 2,\ldots, 6)$, $e_{4j} = W_j = 0$ $(j = 1, 2,\ldots, 5)$ and $s = v_5 = 0$. Then for any given a_4, a_5 and $u \neq 0$ such that $A \neq 0$ and $R_{12} \neq 0$ other constants are determined uniquely by (2.26), (2.28), (2.36) and (2.41), and we have $F_7 = p_4e_{46}$ and $K_7 = v_4e_{46}$.

For instance the choice

(3.1) $$\mu = 0.475, \quad \nu = 0.72, \quad u = -0.5$$

yields

(3.2) $b_4 = -10.57084022,\quad c_{40} = 1.535351271,\quad c_{41} = 7.817720652,$

$c_{42} = -1.668025015,\quad c_{43} = 3.360793310,\quad e_{46} = -5.06E-1,$

(3.3) $b_5 = 2.820015690,\quad c_{50} = -0.3866898256,\quad c_{51} = -2.321160150,$

$c_{52} = 0.8538960019,\quad c_{53} = -0.8839560779,\quad c_{54} = 0.6378943610,$

$e_{57} = -2.73E-1,$

(3.4) $p_0 = -0.03316404542,\quad p_1 = 0.5131534954,\quad p_2 = -1.295834612,$

$p_3 = 1.466226744,\quad p_4 = -0.4966636240,\quad p_5 = 0.8462820415,$

$S_7 = -3.76E-1,\quad F_7 = -2.50E-1,$

(3.5) $v_0 = -0.07330178082,\quad v_1 = 0.3607658602,\quad v_2 = -0.05726365496,$

$v_3 = 0.1302064686,\quad v_4 = -0.007010454636,\quad v_5 = 0,$

$W_6 = -2.66\text{E}{-2}.$

3.2 Case $r=4$

We set $S_i=0$ $(i=1, 2,\dots, 7)$, $e_{5j}=e_{6j}=W_j=0$ $(j=1, 2,\dots, 6)$, $e_{4k}=0$ $(k=1, 2,\dots, 5)$ and $c_{64}=s=p_4=v_4=0$, and we have

(3.6) $$14(25v^2-60v+31)\mu^2-14(60v^2-149v+80)\mu+434v^2-1120v+627=0.$$

For any a_4 $(0<a_4\leqq 1)$, $u\neq 0$ and μ and v satisfying (3.6) such that $A\neq 0$, $R_{1i}\neq 0$ $(i=2, 3)$ and $a_i\neq a_4$ $(i=5, 6)$, other constants are determined uniquely by (2.26), (2.28), (2.36), (2.37), (2.41) and (2.42), and we have $K_7=0$.

For instance the choice

(3.7) $$\mu = 0.5,\quad v = 0.8944214639,\quad a_4 = 0.675,\quad u = -0.5$$

yields

(3.8) $b_4 = -22.90457102,\quad c_{40} = 3.535669047,\quad c_{41} = 17.18938358,$

$c_{42} = -8.580227199,\quad c_{43} = 11.43474559,\quad e_{46} = -1.63E+0,$

(3.9) $b_5 = -1.452588224,\quad c_{50} = 0.2070869290,\quad c_{51} = 1.268152211,$

$c_{52} = -1.943565301,\quad c_{53} = 2.369551210,\quad c_{54} = 0.05136317476,$

$e_{57} = 1.62E-1,$

(3.10) $b_6 = 9.665320921,\quad c_{60} = -1.399600243,\quad c_{61} = -8.108142987,$

$c_{62} = 8.663023327,\quad c_{63} = -9.313405398,\quad c_{64} = 0,$

$c_{65} = 1.387225844,\quad e_{67} = -1.33E+0,$

(3.11) $p_0 = -0.0002604862769, \quad p_1 = 0.007475908655,$

$p_2 = -0.2075555104, \quad p_3 = 0.4457409447, \quad p_4 = 0,$

$p_5 = 0.4902512337, \quad p_6 = 0.2643479096, \quad S_7 = 6.72E-4,$

$F_7 = -2.71E-1,$

(3.12) $v_0 = 0.07255003032, \quad v_1 = 0.4178452993, \quad v_2 = -0.4423239876,$

$v_3 = 0.4873012654, \quad v_4 = 0, \quad v_5 = -0.04160721900,$

$v_6 = 0.006234611543, \quad W_6 = 7.13E-2.$

3.3 Case $r=5$

We choose $S_i=0$ $(i=1, 2,\dots, 8)$, $e_{5j}=e_{6j}=e_{7j}=W_j=0$ $(j=1, 2,\dots, 7)$, $e_{4k}=0$ $(k=1, 2,\dots, 6)$ and $p_4=v_4=c_{74}=0$, and we have

(3.13) $10Aa_4^4 - 12(AP_{11}-B)a_4^3 + 15(AP_{21}-BP_{11})a_4^2 + 20(AY_1+BP_{21})a_4 + 30BY_1 = 0,$

(3.14) $60Ca_5^5 - 10(7P_{12}C-D)a_5^4 + 12(7P_{22}C-P_{12}D)a_5^3 - 15(7P_{32}C+P_{22}D)a_5^2 - 20(7CZ_2+P_{32}D)a_5 - 30Z_2D = 0,$

where $C=R_{12}$ and $D=R_{22}$. For any μ, v, $u\neq 0$ and a_i $(i=4, 5)$ satisfying (3.13) and (3.14) such that $A\neq 0$, $C\neq 0$, $R_{35}\neq 0$, $R_{57}\neq 0$, $a_i\neq \mu$, v and $0<a_i\leqq 1$ $(i=4, 5)$, other constants are determined uniquely by (2.26), (2.28), (2.30), (2.36), (2.37), (2.39), (2.41), (2.42) and (2.43), and we have $F_7=F_8=K_7=K_8=0$.

For instance the choice

(3.15) $\mu = 0.904, \quad v = 0.342, \quad a_4 = 0.5076061751, \quad a_5 = 0.6570915471,$

$u = 1.0$

yields

(3.16) $b_4 = 34.53590888, \quad c_{40} = -3.565512499, \quad c_{41} = -22.20711780,$

$c_{42} = -17.78022895, \quad c_{43} = 9.524556536, \quad e_{47} = -5.33E-1,$

(3.17) $b_5 = -1.337705905, \quad c_{50} = 0.1350142014, \quad c_{51} = 0.4412783792,$

$c_{52} = 0.7057437510, \quad c_{53} = 0.3408428475, \quad c_{54} = 0.3719182732,$

$e_{58} = -5.28E-2,$

(3.18) $b_6 = -11.03438741, \quad c_{60} = 1.120778577, \quad c_{61} = 5.568320667,$

$c_{62} = 5.773473673, \quad c_{63} = -0.9740570107, \quad c_{64} = -0.3350867960,$

$c_{65} = 0.7849582964, \quad e_{68} = -5.36E-1,$

(3.19) $b_7 = -3.031199895,\quad c_{70} = 0.3074472541,\quad c_{71} = 1.385552776,$

$c_{72} = 1.589075508,\quad c_{73} = 0.04113356034,\quad c_{74} = 0,$

$c_{75} = 0.06576373415,\quad c_{76} = -0.01577293821,\quad e_{78} = -1.42E-1,$

(3.20) $s = 0.2428733357,\quad p_0 = -0.02419657518,\quad p_1 = -0.1180080624,$

$p_2 = -0.1296951316,\quad p_3 = 0.1489507863,\quad p_4 = 0,$

$p_5 = 0.2289030122,\quad p_6 = 0.2267983033,\quad p_7 = 0.4243743317,$

$S_9 = -3.32E-2,\quad F_9 = -1.94E-1,$

(3.21) $v_0 = -0.1015527525,\quad v_1 = -0.5035064634,\quad v_2 = -0.5233496733,$

$v_3 = 0.09675621105,\quad v_4 = 0,\quad v_5 = -0.02669845199,$

$v_6 = 0.005931997435,\quad v_7 = 0.05241913276,\quad W_8 = 4.84E-2.$

4. Numerical examples

The following six problems are tested:

Problem 1. $y' = y$, $y(0) = 1$. $y(x) = \exp(x)$.

Problem 2. $y' = 2xy$, $y(0) = 1$. $y(x) = \exp(x^2)$.

Problem 3. $y' = -5y$, $y(0) = 1$. $y(x) = \exp(-5x)$.

Problem 4. $y' = -y^2$, $y(0) = 1$. $y(x) = 1/(1+x)$.

Problem 5. $y' = y - 2x/y$, $y(0) = 1$. $y(x) = (1+2x)^{1/2}$.

Problem 6. $y' = 1 - y^2$, $y(0) = 0$. $y(x) = \tanh(x)$.

Computation by methods (3.1), (3.7) and (3.15) is carried out by the following program.

(i) Compute $y_\mu, y_\nu, y_1, f_0, f_\mu, f_\nu$ and f_1.
(ii) Compute $y_{1+\mu}, y_{1+\nu}, y_2, f_{1+\mu}, f_{1+\nu}, f_2$ and t_2.
(iii) If $|t_2| > \varepsilon \max(1, |t_2|)$, then halve the stepsize and go to (i).
(iv) If $|t_2| < \varepsilon_1 \max(1, |t_2|)$, then replace y_0, f_0 and h by y_2, f_2 and $2h$ respectively and go to (i).
(v) Replace $y_0, y_\mu, y_\nu, y_1, f_0, f_\mu, f_\nu$, and f_1 by $y_1, y_{1+\mu}, y_{1+\nu}, y_2, f_1, f_{1+\mu}, f_{1+\nu}$, and f_2 respectively and go to (ii).

Here $\varepsilon = 10^{-r-5}/2$ and $\varepsilon_1 = 2^{-r-6}\varepsilon$. Butcher (6,8) formula [1], Shanks (7,9) formula and Shanks (8,12) formula [2] are used for computing starting values for methods of order 6,7 and 8 respectively. The errors at $x=3$ are listed in Table 1.

Table 1.

Meth \ Prob	1	2	3	4	5	6
(3.1)	$2.86E-6$	$2.04E-3$	$-4.16E-10$	$-3.67E-8$	$-3.44E-6$	$9.97E-9$
(3.7)	$-2.06E-7$	$-7.64E-5$	$1.12E-10$	$-8.18E-11$	$2.58E-8$	$1.43E-10$
(3.15)	$1.47E-8$	$-3.76E-7$	$1.62E-9$	$3.32E-11$	$7.21E-9$	$6.32E-10$

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