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Kaplansky's radical and Hilbert Theorem 90 III

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Let F be a field, $R(F)$ be Kaplansky's radical of F and $K=F(\sqrt{a})$ be a quadratic extension of F. We showed in [4], that if F is a quasi-pythagorean field and K is a radical extension (i.e. $a \in R(F)-\dot{F}^2$), then K is also quasi-pythagorean and the '*H-conjecture*' $N^{-1}(R(F))=\dot{F}\cdot R(K)$ is valid, where $N: K \to F$ is the norm map.

In this paper we generalize the above results and show that the H-conjecture is valid whenever K is a quasi-pythagorean field.

§1. Preliminaries

Throughout the paper, let F be a field of characteristic different from two and $\dot{F}$ be the multiplicative group of F. We introduce in this section some subgroups of $\dot{F}$, and study their properties.

First, we put for $a \in \dot{F}$, $I_a=\{x \in \dot{F};\ D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -x\rangle\}$.

PROPOSITION 1.1. *$I_a = \cap D_F\langle 1, -x\rangle$, where x runs over $D_F\langle 1, -a\rangle$. So I_a is a subgroup of $\dot{F}$.*

PROOF. If $b \in I_a$, then $D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -b\rangle$ and we have $x \in D_F\langle 1, -b\rangle$ for all $x \in D_F\langle 1, -a\rangle$. Then $b \in D_F\langle 1, -x\rangle$ for all $x \in D_F\langle 1, -a\rangle$. So $b \in \cap$ $D_F\langle 1, -x\rangle$, where x runs over $D_F\langle 1, -a\rangle$. Now, all the implications can be reversed and the proposition follows. Q.E.D.

PROPOSITION 1.2. *Let $K=(\sqrt{a})$ be a quadratic extension of F. Then the following statements hold:*

(1) $I_a = \{x \in \dot{F};\ \dot{F}\cdot D_K\langle 1, -x\rangle = \dot{K}\}$.

(2) $I_a \supseteq R(F)$, $I_a \ni a$.

(3) $I_a \supseteq R(K) \cap \dot{F}$.

(4) $R(K) \supseteq R(F)$.

PROOF. Let $N: K \to F$ be the norm map. Then $N(\dot{K})=D_F\langle 1, -a\rangle$ and, by the norm principle ([3], 2.13), we have $N^{-1}(D_F\langle 1, -x\rangle)=\dot{F}\cdot D_K\langle 1, -x\rangle$ for $x \in \dot{F}$. So, $D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -x\rangle$ if and only if $\dot{K}=\dot{F}\cdot D_K\langle 1, -x\rangle$. This shows (1). The assertion (2) is clear and (3) follows from (1). The assertion (4) follows from (1) and (2), since $\dot{F}=D_F\langle 1, -x\rangle \subseteq D_K\langle 1, -x\rangle$ for $x \in R(F)$.

Q. E. D.

PROPOSITION 1.3. *Let F be a quasi-pythagorean field and $x, y \in \dot{F}$. If $x \in D_F\langle 1, y\rangle$, then $D_F\langle 1, x\rangle \subseteq D_F\langle 1, y\rangle$. Moreover, if F is formally real, then $D_F\langle 1, -a\rangle$ is a preordering of F for every $a \notin R(F)$.*

PROOF. $D_F\langle 1, y\rangle \cup \{0\}$ is closed under addition, since we have $D_F\langle 1, y\rangle = D_F\langle r_1, r_2 y\rangle$ for $r_i \in R(F)=D_F(2)$ $(i=1, 2)$. From this, the assertions follow immediately. Q. E. D.

PROPOSITION 1.4. *I_a is a subgroup of $D_F\langle 1, a\rangle$. Moreover we have $I_a = D_F\langle 1, a\rangle$ if F is a quasi-pythagorean field.*

PROOF. By Proposition 1.1, we have $I_a \subseteq D_F\langle 1, a\rangle$, since $-a \in D_F\langle 1, -a\rangle$. Suppose that F is quasi-pythagorean and $x \in D_F\langle 1, a\rangle$. Then $-a \in D_F\langle 1, -x\rangle$ and we have $D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -x\rangle$ by Proposition 1.3. This shows $D_F\langle 1, a\rangle \subseteq I_a$. Q. E. D.

Now for $a \in \dot{F}$, we put $H_a = \{x \in \dot{F};\ D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle = \dot{F}\}$. We note that $H_a = H_{-1}$ for $a \in -R(F)$. Moreover, if F is formally real and quasi-pythagorean, then H_{-1} is the group $H(P)$ for $P=D_F(\infty)$ defined in §2 of [5]. In this case H_{-1} is denoted by H. By Remark 2.3, (1) of [5], we have $H=\dot{F}$ if and only if the space $X(F)$ of orderings in F satisfies SAP.

PROPOSITION 1.5. *If $K=F(\sqrt{a})$ is a quadratic extension of F, then we have $H_a = \{x \in \dot{F};\ D_K\langle 1, -x\rangle \supseteq \dot{F}\}$.*

PROOF. Since we have $D_K\langle 1, -x\rangle \cap \dot{F} = D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle$ ([1], Lemma 3.5.), the assertion follows immediately. Q. E. D.

PROPOSITION 1.6. *$H_a = \bigcap_{x \in \dot{F}} D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle$. So H_a is a subgroup of $\dot{F}$.*

PROOF. If $a \in \dot{F}^2$, then we have $H_a = R(F)$ by the definition of H_a. So the assertion is valid. Suppose $a \notin \dot{F}^2$. Then for $x \in \dot{F}$, $x \in H_a$ is equivalent to $x \in D_K\langle 1, -y\rangle \cap \dot{F}$ for all $y \in \dot{F}$. Hence we have the desired equality which clearly implies that H_a is a subgroup of $\dot{F}$. Q. E. D.

PROPOSITION 1.7. *The following statements hold:*

(1) *$H_a \supseteq R(F)$, $H_a \ni a$.*

(2) *If $K = F(\sqrt{a})$ is a quadratic extension, then we have $H_a \supseteq R(K) \cap \dot{F}$.*

PROOF. The assertion (1) follows from the definition of H_a, and (2) follows from Proposition 1.5. Q. E. D.

PROPOSITION 1.8. *If F is a quasi-pythagorean field, then $H_a \subseteq D_F\langle 1, a\rangle$.*

PROOF. For $x=-1$, $D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle = D_F\langle 1, 1\rangle D_F\langle 1, a\rangle = D_F\langle 1, a\rangle$, since $D_F\langle 1, 1\rangle = R(F) \subseteq D_F\langle 1, a\rangle$. From this, the desired inclusion follows by Proposition 1.6. Q.E.D.

PROPOSITION 1.9. *If $K=F(\sqrt{a})$ is a quadratic extension of F, then we have $R(K) \cap \dot{F} = I_a \cap H_a$.*

PROOF. By (1) of Proposition 1.2 and Proposition 1.5, we have $I_a \cap H_a \subseteq R(K) \cap \dot{F}$. The other inclusion follows from (3) of Proposition 1.2 and (2) of Proposition 1.7. Q.E.D.

For a quadratic extension $K=F(\sqrt{a})$ of F, we defined the set $\bar{R}(K)=\{x \in \dot{K};\ \dot{F}\cdot D_K\langle 1, -x\rangle = \dot{K}\}$ and the subgroup $I_K(\dot{F}) = \bigcap_{x\in\dot{F}} D_K\langle 1, -x\rangle$ of $\dot{K}$ in §2 of [4] and showed $R(K)=\bar{R}(K) \cap I_K(\dot{F})$.

We note that $\bar{R}(K) \cap \dot{F} = I_a$ and $I_K(\dot{F}) \cap \dot{F} = H_a$. Proposition 1.9 follows again from these relations.

Now the following result is essentially contained in the proof of Theorem 2.13 of [4]. But we state and prove it for completeness.

PROPOSITION 1.10. *The notation being as above, if F is a quasi-pythagorean field, then we have $D_K(2) \subseteq \bar{R}(K)$ and therefore $R(K)=I_K(\dot{F})$.*

PROOF. Let x be an element of $D_K(2)$. For any $y \in \dot{K}-(\dot{F} \cup x\dot{F})$, we can write $x=(b_1+c_1y)^2+(b_2+c_2y)^2$ $(b_i, c_i \in \dot{F})$. Then $x=(b_1^2+b_2^2)+(c_1^2+c_2^2)y^2+2(b_1c_1+b_2c_2)y$. By Lemma 2.11 of [4], we have $f_y(y^2)=Im(y\cdot\bar{y}^2)/Im(y)=N(y)Im(\bar{y})/Im(y)=-N(y)$ (see §2 of [4] for notation involved), and this implies that there exists $\alpha \in F$ such that $y^2=-N(y)+\alpha y$, and hence there exists $\beta \in F$ such that $x=(b_1^2+b_2^2)+(c_1^2+c_2^2)(-N(y))+\beta y$. Namely $f_y(x)=(b_1^2+b_2^2)+(c_1^2+c_2^2)(-N(y)) \in D_F\langle\langle 1, -N(y)\rangle\rangle$. Since F is quasi-pythagorean, we have $D_F\langle\langle 1, -N(y)\rangle\rangle = D_F\langle 1, -N(y)\rangle$ and $x \in \bar{R}(K)$ by Lemma 2.12 of [4]. Hence we have $D_K(2) \subseteq \bar{R}(K)$. Then $R(K)=D_K(2) \cap R(K) = D_K(2) \cap \bar{R}(K) \cap I_K(\dot{F}) = D_K(2) \cap I_K(\dot{F}) = I_K(\dot{F})$. Q.E.D.

§2. The main theorem

In this section we show the theorem stated in the beginning of the paper, and deduce several consequences from it.

THEOREM 2.1. *For a quadratic extension $K=F(\sqrt{a})$ of F, K is quasi-pythagorean if and only if F is quasi-pythagorean and $H_a = D_F\langle 1, a\rangle$. Furthermore, if these conditions are satisfied, then we have $N^{-1}(R(F))=\dot{F}\cdot R(K)$, N being the norm map, and $I_a=H_a=D_F\langle 1, a\rangle = R(K) \cap \dot{F}$.*

PROOF. Suppose that K is quasi-pythagorean. Then we have $D_K\langle 1, a\rangle = D_K(2) = R(K)$. It follows that $D_F\langle 1, a\rangle \subseteq \dot{F} \cap D_K\langle 1, a\rangle = \dot{F} \cap R(K)$. By Proposition 1.9, we have $\dot{F} \cap R(K) = I_a \cap H_a$. Hence $D_F\langle 1, a\rangle \subseteq I_a \cap H_a \subseteq I_a$. But $I_a \subseteq D_F\langle 1, a\rangle$ by Proposition 1.4. Therefore we have $D_F\langle 1, a\rangle = I_a \subseteq H_a$, and $\dot{F} \cap R(K) = D_F\langle 1, a\rangle$. By Proposition 1.8 we see that $H_a = D_F\langle 1, a\rangle$. On the other hand, we have, by Lemma in §2 of [2], $D_F\langle 1, a\rangle \cap D_F\langle 1, -a\rangle = D_F\langle 1, a\rangle \cap D_F\langle 1, a^2\rangle = D_F(2)$ since $D_F(2) \subseteq D_K(2) \cap \dot{F} = D_F\langle 1, a\rangle$. So in particular $D_F(2) \subseteq D_F\langle 1, -a\rangle$.

Let $N: \dot{K} \to \dot{F}$ be the norm map. The image of N is $D_F\langle 1, -a\rangle$. Since $N(R(K)) \subseteq R(F)$ always holds, we have $N(\dot{F} \cdot R(K)) \subseteq R(F)$ and therefore $N^{-1}(R(F)) \supseteq \dot{F} \cdot R(K) = \dot{F} \cdot D_K(2)$. However, by the norm principle (2.13 of [3]), $\dot{F} \cdot D_K(2) = N^{-1}(D_F(2))$. So we have $N^{-1}(R(F)) \supseteq N^{-1}(D_F(2))$, which implies $R(F) = D_F(2)$ and $N^{-1}(R(F)) = \dot{F} \cdot R(K)$.

Conversely, suppose that F is quasi-pythagorean and $H_a = D_F\langle 1, a\rangle$. We shall prove that K is quasi-pythagorean. By Proposition 1.10, we have only to show that $D_K(2) \subseteq D_K\langle 1, -x\rangle$ for all $x \in \dot{F}$. So let γ be any element of $D_K(2)$. Since $D_F(2) = R(F) \subseteq D_F\langle 1, -x\rangle$, we have, by the norm principle (2.13 of [3]), $\dot{F} \cdot D_K(2) \subseteq \dot{F} \cdot D_K\langle 1, -x\rangle$ for all $x \in \dot{F}$. So there exists $f \in \dot{F}$ such that $f\gamma \in D_K\langle 1, -x\rangle$. Then $f \in D_K\langle 1, 1\rangle D_K\langle 1, -x\rangle \cap \dot{F} \subseteq D_K\langle\langle 1, -x\rangle\rangle \cap \dot{F}$, and we can write $f = (b_1 + c_1\sqrt{a})^2 + (b_2 + c_2\sqrt{a})^2 - x(b_3 + c_3\sqrt{a})^2 - x(b_4 + c_4\sqrt{a})^2$ $(b_i, c_i \in F)$. Then we have $f = b_1^2 + b_2^2 + a(c_1^2 + c_2^2) - x(b_3^2 + b_4^2) - ax(c_3^2 + c_4^2)$. Now for a moment, we assume, in the last equality, each of the four sums of two squares is not zero. Then as these sums are in $D_F(2) = R(F)$, we have $f = b^2 + ac^2 - x(b'^2 + ac'^2)$ for some $b, c, b', c' \in F$. Since $b^2 + ac^2, b'^2 + ac'^2 \in D_F\langle 1, a\rangle \cup \{0\}$, and $D_F\langle 1, a\rangle = R(K) \cap \dot{F}$, we have $f \in D_K\langle 1, -x\rangle$. If some of the four sums are equal to zero, we see readily $f \in D_K\langle 1, -x\rangle$. So we have $\gamma \in D_K\langle 1, -x\rangle$ for all $x \in \dot{F}$.

Considering Proposition 1.4, Proposition 1.8 and Proposition 1.9, we see easily that the last statement holds. Q. E. D.

REMARK 2.2. In the notation of Theorem 2.1, the assertion that F is quasi-pythagorean if K is so, has already been proved in Proposition 4.10 of [3], in a more general form.

In all the rest of the paper, let K denote a quadratic extension $F(\sqrt{a})$ of F. The following proposition is a strengthenning of Theorem 2.13 of [4].

COROLLARY 2.3. *Suppose K is a radical extension of F. Then, K is quasi-pythagorean if and only if F is so.*

PROOF. Suppose F is quasi-pythagorean, then $R(F) = D_F(2) = D_F\langle 1, a\rangle$ since we have $a \in R(F)$. On the other hand, recalling the definitions of I_a and

H_a, we see easily that $I_a = R(F) = H_a$. So the assertion follows from Theorem 2.1. Q. E. D.

COROLLARY 2.4. *The following statements are equivalent:*

(1) *K is a quasi-pythagorean field which is not formally real.*

(2) $R(K) \supseteq \dot{F}$.

(3) *F is a quasi-pythagorean field, $a \in -R(F)$, and $H = \dot{F}$.*

PROOF. (1)⇒(2): This is clear since $R(K) = \dot{K}$ in this case.

(2)⇒(3): Since we have $R(K) \cap \dot{F} = I_a \cap H_a$, (2) implies that $I_a = H_a = \dot{F}$. In particular $D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -x\rangle$ for all $x \in \dot{F}$. It follows that $-a \in R(F)$ and $D_F(2) = D_F\langle 1, -a\rangle = R(F)$. So F is quasi-pythagorean. Since $H = H_a$ for $a \in -R(F)$, we have $H = \dot{F}$.

(3)⇒(1): We have $D_F\langle 1, a\rangle = \dot{F}$ and $H_a = H = \dot{F}$. So (1) follows from Theorem 2.1. Q. E. D.

Using the above corollary and the theorem of Tsen–Lang, we can prove the following result (see Theorem 17.9 and Corollary 17.8 of [6]): Let k be a real closed field and L is a formally real field over k with tr. $\deg_k L = 1$. Then L is a quasi-pythagorean field which is a SAP field. In particular the rational function field $\boldsymbol{R}(X)$ in one variable X over the real number field $\boldsymbol{R}$ is a quasi-pythagorean SAP field.

If F is a quasi-pythagorean field which is not formally real, then K is also such a field by Corollary 2.4 and (4) of Proposition 1.2.

In the following, we consider the case in which F is formally real.

COROLLARY 2.5. *Let F be a formally real, quasi-pythagorean field and $a \in H$. We assume $\pm a \notin R(F)$ and denote the preorderings $D_F\langle 1, a\rangle$, $D_F\langle 1, -a\rangle$ by T, T' respectively. Then K is quasi-pythagorean if and only if the preordering T' is SAP.*

PROOF. Since a is an element of H, we have $TT' = \dot{F}$. So any $b \in \dot{F}$ can be written as $b = xy$ where $x \in T$, $y \in T'$. Then $D_{T'}\langle 1, -b\rangle = D_{T'}\langle 1, -x\rangle$ and $D_{T'}\langle 1, b\rangle = D_{T'}\langle 1, x\rangle$. But by Lemma 2.6 below, $D_F\langle 1, -x\rangle D_F\langle 1, x\rangle = D_{T'}\langle 1, -x\rangle D_{T'}\langle 1, x\rangle$ for any $x \in T$. Hence $T = H_a$ i.e. $D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle = \dot{F}$ for all $x \in T$ if and only if $D_{T'}\langle 1, -b\rangle D_{T'}\langle 1, b\rangle = \dot{F}$ for all $b \in \dot{F}$. The last condition is equivalent to T' being SAP, by Remark 2.3, (1) of [5]. Q. E. D.

In the following lemma and its proof, we use freely the notation of [5], §1, $X(P)$ being an abbreviation for $X(F/P)$.

LEMMA 2.6. *Let F be a formally real, quasi-pythagorean field. We assume $\pm a \notin R(F)$ and denote the preorderings $D_F\langle 1, a\rangle$, $D_F\langle 1, -a\rangle$ by T, T' respectively. Then for $x \in T$, the following statements hold:*

(1) $D_F\langle 1, x\rangle = D_{T'}\langle 1, x\rangle \cap T$.
(2) $D_F\langle 1, -x\rangle = D_{T'}\langle 1, -x\rangle$.
(3) *If we further assume* $a \in H$, *then we have*

$$D_F\langle 1, x\rangle D_F\langle 1, -x\rangle = D_{T'}\langle 1, x\rangle D_{T'}\langle 1, -x\rangle .$$

PROOF. We first note that, for any preordering P of F, and for any $z \in \dot{F}$, we have $D_F\langle 1, z\rangle = (H(z))^{\perp}$, $D_P\langle 1, z\rangle = (H(z) \cap X(P))^{\perp}$. Now for $x \in T$, we have $H(x) \supseteq X(T)$, $H(-x) \subseteq X(T')$.

(1) $D_F\langle 1, x\rangle = (H(x))^{\perp} = (X(T) \cup (H(x) \cap X(T')))^{\perp}$
$= (X(T))^{\perp} \cap (H(x) \cap X(T'))^{\perp} = T \cap D_{T'}\langle 1, x\rangle$.
(2) $D_F\langle 1, -x\rangle = (H(-x))^{\perp} = (H(-x) \cap X(T'))^{\perp} = D_{T'}\langle 1, -x\rangle$.
(3) We have $T' \cdot D_F\langle 1, x\rangle = T' \cdot (T \cap D_{T'}\langle 1, x\rangle)$
$= (T' \cdot T) \cap D_{T'}\langle 1, x\rangle$ (since $T' \subseteq D_{T'}\langle 1, x\rangle$)
$= D_{T'}\langle 1, x\rangle$ (since $T \cdot T' = \dot{F}$).

Therefore we have $D_F\langle 1, x\rangle D_F\langle 1, -x\rangle = D_F\langle 1, x\rangle D_{T'}\langle 1, -x\rangle = D_{T'}\langle 1, x\rangle D_{T'}\langle 1, -x.\rangle$ Q.E.D.

As an application of Corollary 2.5, we have the following result.

COROLLARY 2.7. *Let F be a formally real field. Then F is quasi-pythagorean and SAP if and only if every quadratic extension K of F is quasi-pythagorean. Moreover, when these conditions are satisfied, K is also SAP if it is formally real.*

PROOF. The first assertion is obvious. To show the second assertion, let F be a formally real, quasi-pythagorean and SAP field. Then, by Corollary 2.4, $F(\sqrt{-1})$ is a quasi-pythagorean field which is not formally real. So, for any formally real quadratic extension K of F, $K(\sqrt{-1})$ is quasi-pythagorean by Corollary 2.3. Theorefore K is SAP by Corollary 2.4. Q.E.D.

LEMMA 2.8. *Let F be a formally real, quasi-pythagorean field. We assume $\pm a \notin R(F)$ and denote the preorderings $D_F\langle 1, a\rangle$, $D_F\langle 1, -a\rangle$ by T, T' respectively. Then we have $H(T') \cap T = H_a$.*

PROOF. For any $x \in T$, $D_F\langle 1, -x\rangle = D_{T'}\langle 1, -x\rangle$ and $D_F\langle 1, -ax\rangle = D_{T'}\langle 1, -ax\rangle$ by Lemma 2.6, (2). So we have $H(T') \cap T = H_a$ by Proposition 1.8 Q.E.D.

In the proof of the following lemma, we use the translation group gr $(X(P))$ of the space $X(P)$ defined in [7]. Namely gr $(X(P)) = \{\alpha \in \chi(F/P);\ \alpha X(P) = X(P)\}$, where $\chi(F/P) = \mathrm{Hom}(\dot{F}/P, \{\pm 1\})$ is the character group of $\dot{F}/P$. For a preordering P of finite index, $X(P)$ is connected if and only if $|X(P)|=1$ or, $|X(P)|\geqq 3$ and $\mathrm{gr}(X(P))\neq 1$.

LEMMA 2.9. *Let F be a formally real,quasi-pythagorean field and assume that $X(F)$ be finite and connected. Then we have $H_a=R(F)\cup aR(F)$.*

PROOF. If $a\in R(F)$, we have $H_a=R(F)$ and the assertion holds. If $-a\in R(F)$, we have $H_a=H$. Hence $\dim H_a/R(F)=1$ by Theorem 2.5 of [5]. So the assertion holds by Proposition 1.7 and Proposition 1.8. Since the case $|X(F)|=1$ is included in the above ones, we suppose that $\pm a\notin R(F)$, $|X(F)|\geqq 3$ and $\mathrm{gr}(X(F))\neq 1$. Let α be an element of $\mathrm{gr}(X(F))$ such that $\alpha\neq 1$. We consider two cases.

Case 1: $\alpha(a)=1$. We have $\alpha X(T')=X(T')$. First we assume $|X(T')|\geqq 3$. Then $X(T')$ is connected. By Theorem 2.5 of [5], $\dim H(T')/T'=1$. So we have $H(T')=T'\cup aT'$. Thus we have $H_a=R(F)\cup aR(F)$ by Lemma 2.8, by noting $T\cap T'=R(F)$. Next we assume $|X(T')|\leqq 2$. Since $\alpha X(T')=X(T')$, we see that $X(T')$ consists of even number of orderings and so $|X(T')|=2$. If we write $X(T')=\{\sigma_1, \sigma_2\}$, we see that $\alpha=\sigma_1\sigma_2$. Since $\alpha X(T)=X(T)$, we take a set $\{\tau_1,\dots, \tau_n, \alpha\tau_1\}$ as a basis of $X(T)$ and it is easy to see that $\{\tau_1,\dots, \tau_n, \alpha\tau_1, \sigma_1\}$ is a basis of $X(F)$. This implies that $\dim X(T)=\dim X(F)-1$, and so $\dim T/R(F)=1$. From this the assertion follows by Proposition 1.7 and Proposition 1.8.

Case 2: $\alpha(a)=-1$. We have $\alpha X(T')\subseteq X(T)$ and $\alpha X(T)\subseteq X(T')$. Hence $\alpha X(T)=X(T')$. If we take a basis $\{\sigma_1,\dots, \sigma_n\}$ of $X(T)$, then $\{\sigma_1,\dots, \sigma_n, \alpha\sigma_1\}$ is a basis of $X(F)$. Thus $\dim X(T)=\dim X(F)-1$, and so $\dim T/R(F)=1$. The assertion follows similarly to case 1. Q.E.D.

COROLLARY 2.10. *Let F be a formally real, quasi-pythagorean field. If a is $R(F)$-rigid, then K is quasi-pythagorean. Conversely if $X(F)$ is finite and connected, and if K is quasi-pythagorean, then a is $R(F)$-rigid.*

PROOF. If a is $R(F)$-rigid, we have $D_F\langle 1, a\rangle=R(F)\cup aR(F)$. So we see easily that $D_F\langle 1, a\rangle=H_a$, which implies that K is quasi-pythagorean by Theorem 2.1.

Conversely assume that K is quasi-pythagorean. In case $a\in R(F)$, the assertion is trivial. If $-a\in R(F)$, then $H_a=D_F\langle 1, a\rangle=\dot{F}$ by Theorem 2.1; also $H_a=H$ and, since $X(F)$ is finite and connected, we have $\dim H_a/R(F)=1$ by Theorem 2.5 of [5]. This implies that a is $R(F)$-rigid. Hence we may suppose $\pm a\notin R(F)$. Then we have $D_F\langle 1, a\rangle=H_a=R(F)\cup aR(F)$ by Theorem 2.1 and Lemma 2.9. So a is $R(F)$-rigid. Q.E.D.

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