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Asymptotic nonoscillation under large amplitudes of oscillating coefficients in second order functional equations

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1. Introduction

Our main purpose in this work is to study the nonoscillation property of the solutions of the functional equation

$$y''(t) + a(t)y^{\delta}(g(t)) = f(t) \tag{1}$$

when $a(t)$ oscillates with sufficiently large amplitudes. In the sequel we shall assume that

(i) $a(t)$, $g(t)$, $f(t)$: $R \to R$ and continuous;
(ii) $g(t) \to \infty$ as $t \to \infty$, $g'(t) > 0$ (thus $g(t)$ may be advanced or retarded);
(iii) $\delta > 0$ and the ratio of odd integers. The equation (1) is therefore, allowed to be superlinear, linear or sublinear.

In view of our Theorem 3.1 in [5], we shall assume that solutions of equation (1) under consideration are those which exist continuously on some half line $[t_0, \infty)$. The term "solution", henceforth, applies only to such an entity.

Since the pioneering work of Hammett [1] the asymptotic nature of oscillatory and nonoscillatory solutions of equation (1) has been the subject of numerous studies. A recently published Russian book by Shevelo gives a fairly exhaustive list of references for the interested reader. In regard to obtaining results about the asymptotic nature of the nonoscillatory solutions, the coefficient $a(t)$ has been assumed to be of one sign by majority of authors. When $a(t) > 0$ for sufficiently large t, then a decent account of the nonoscillatory solutions can be found in Kusano and Onose [3–4], Kitamura Kusano and Naito [2], Singh [6] and Kusano and Singh [7]. However when $a(t)$ is oscillatory, nothing seems to be known about the asymptotic nature of the nonoscillatory solutions of equation (1).

In this work, we present an elementary but new technique to study oscillation phenomenon in general. In particular, we not only assume that $a(t)$ be oscillatory, but also utilize the amplitude of oscillation to characterize the nonoscillatory solutions of (1). In what follows we call a solution of (1) oscillatory if it has arbitrarily large zeros in $[t_0, \infty)$; otherwise we call it nonoscillatory.

Our main theorem characterizes the bounded solutions of (1) as either nonmonotic or as monotonic converging to zero as $t\to\infty$.

2. Main results

THEOREM (2.1). *Suppose there exists a function* $\phi(t)\in C^2[[t_0, \infty), (0, \infty)]$ *such that* $\phi'(t)>0$, $\phi''(t)\geq 0$ and

$$\int^{\infty} 1/\phi^2(s)\int^{s}\phi(t)|f(t)|dtds < \infty \tag{2}$$

$$\liminf_{t\to\infty}\left[\int^{t} 1/\phi^2(s)\int^{s}\phi(x)a(x)dxds\right] = -\infty \tag{3}$$

$$\limsup_{t\to\infty}\left[\int^{t} 1/\phi^2(s)\int^{s}\phi(x)a(x)dxds\right] = \infty \tag{4}$$

Let $y(t)$ be a bounded solution of (1). *Then either $y'(t)$ is oscillatory or else $|y(t)|$ monotonically decreases to a limit as $t\to\infty$.*

PROOF. We only need to prove this theorem when $y(t)$ is nonoscillatory. Conditions on $\phi(t)$ imply

$$\int^{\infty} 1/\phi^2(t)dt < \infty\,. \tag{5}$$

Without any loss of generality suppose that there exists a $T>t_0$ such that $y(t)>0$ and $y(g(t))>0$ for $t\geq T$. Suppose first that $y'(t)$ is nonoscillatory. We shall show that $y'(t)$ cannot be eventually positive. Suppose to the contrary that there exists a number $B>T$ such that $y'(t)>0$ and $y'(g(t))>0$ for $t\geq B$. Rewriting (1) as

$$\frac{y''(t)}{y^{\delta}(g(t))} + a(t) = \frac{f(t)}{y^{\delta}(g(t))} \tag{6}$$

and multiplying by $\phi(t)$ we obtain

$$\frac{(\phi(t)y'(t))'}{y^{\delta}(g(t))} - \frac{\phi'(t)y'(t)}{y^{\delta}(g(t))} + a(t)\phi(t) = \frac{f(t)\phi(t)}{y^{\delta}(g(t))}\,. \tag{7}$$

Integrating (7) between $[B, t]$ we have

$$\begin{aligned}&\frac{\phi(t)y'(t)}{y^{\delta}(g(t))} - k_1 + \int_B^t \frac{\phi(s)y'(s)\delta y'(g(s))g'(s)}{(y^{\delta+1}(g(s)))}\,ds\\ &- \frac{\phi'(t)y(t)}{y^{\delta}(g(t))} + k_2 + \int_B^t \frac{\phi''(s)y(s)}{y^{\delta}(g(s))}\,ds\\ &- \int_B^t \frac{\phi'(s)y(s)\delta y'(g(s))g'(s)ds}{(y^{\delta+1}(g(t)))} \qquad + \int_B^t a(s)\phi(s)ds\end{aligned}$$

$$= \int_B^t \frac{f(s)\phi(s)}{y^\delta(g(s))}\,ds \tag{8}$$

where

$$k_1 = \frac{\phi(B)y'(B)}{y^\delta(g(B))}$$

and

$$k_2 = \frac{\phi'(B)y(B)}{y^\delta(g(B))}\,.$$

Dividing (8) by $\phi^2(t)$ and integrating again we obtain

$$\frac{y(t)}{\phi(t)y^\delta(g(t))} - \frac{y(B)}{\phi(B)y^\delta(g(B))} + \int_B^t \frac{y(s)\phi'(s)ds}{\phi^2(s)y^\delta(g(s))}$$

$$+ \int_B^t \frac{\delta y(s)y'(g(s))g'(s)ds}{\phi(s)y^{\delta+1}(g(s))} - (k_1 - k_2)\int_B^t 1/\phi^2(s)ds$$

$$+ \int_B^t 1/\phi^2(s)\int_B^s \frac{\phi(x)y'(x)\delta y'(g(x))g'(x)}{y^{\delta+1}(g(x))}\,dxds$$

$$- \int_B^t \frac{\phi'(s)y(s)ds}{y^\delta(g(s))\phi^2(s)} + \int_B^t 1/\phi^2(s)\int_B^s \frac{\phi''(x)y(x)dxds}{y^\delta(g(x))}$$

$$- \int_B^t 1/\phi^2(s)\int_B^s \frac{\phi'(x)y(x)\delta y'(g(x))g'(x)dxds}{(y^{\delta+1}(g(x)))}$$

$$+ \int_B^t 1/\phi^2(s)\int_B^s a(x)\phi(x)dxds$$

$$= \int_B^t 1/\phi^2(s)\int_B^s \frac{f(x)\phi(x)dxds}{(y^\delta(g(x))}\,. \tag{9}$$

Now the third and seventh terms on the left hand side of (9) cancel each other. Since $y'(t)>0$, (2) holds, $y(t)$ is bounded, and the right hand side of (9) remains finite, whereas all the terms on the left except

$$\int_B^t 1/\Phi^2(s)\int_B^s a(x)\phi(x)dxds \tag{10}$$

are either positive or finite. Note that ninth term on the left of (9) is of the order of $1/\phi$ and hence bounded. Since (10) oscillates between $-\infty$ and ∞, we reach a contradiction. Hence $y'(t)$ must, eventually, be negative proving the theorem.

EXAMPLE (2.1). The equation

$$y''(t) + t^2 \sin t\, y(\sqrt{t}) = e^{-t} + t^2 \sin t\, e^{-\sqrt{t}}, \qquad t>0 \tag{11}$$

has $y(t)=e^{-t}$ as a nonoscillatory solution satisfying the conclusion of the theorem. By choosing $\phi(t)=t$, it is easily verified that all conditions of Theorem (2.1) hold.

THEOREM (2.2). *In addition to the conditions of Theorem* (2.1) *suppose*

$$\limsup_{t\to\infty} \int^t (h(s)a(s)-f(s))ds > 0 \tag{12}$$

$$\liminf_{t\to\infty} \int^t (h(s)a(s)-f(s))ds < 0 \tag{13}$$

for every continuous and monotonic function $h(t)$ *such that* $\lim_{t\to\infty} h(t)\neq 0$. *Let* $y(t)$ *be a bounded solution of equation* (1). *Then either* $y'(t)$ *is oscillatory or else* $|y(t)|$ *decreases monotonically to zero as* $t\to\infty$.

PROOF. Without any loss of generality suppose $y(t)$ is eventually positive. From Theorem (2.1), if $y'(t)$ is nonoscillatory then $y'(t)<0$ and

$$\lim_{t\to\infty} y(t) = \lambda\,. \tag{14}$$

We shall show that $\lambda=0$. Suppose to the contrary that $\lambda>0$. Due to (12) and (13) there exist a $T>t_0$ and $B>0$ such that

$$\liminf_{t\to\infty} \int_T^t (h(s)a(s)-f(s))ds < -B\,. \tag{15}$$

Let $h(t)\equiv y^\delta(g(t))$. Since $y'(t)<0$, and $y'(t)$ is eventually positive, it follows that

$$\limsup_{t\to\infty} y'(t) = 0\,. \tag{16}$$

Let T be large enough so that

$$|y'(B)| < B/2\,. \tag{17}$$

From equation (1) we have

$$y'(t) = y'(T) - \int_T^t (a(s)h(s)-f(s))ds \tag{18}$$

which implies in view of (15) and (17) that $\limsup_{t\to\infty} y'(t)\geq B/2$. This contradicts (16). Hence $\lambda=0$. The proof is complete.

THEOREM (2.3). *In addition to the hypotheses of Theorem* (2.1) *suppose that* $\delta>1$ *and* $g(t)\geq t$ *for sufficient large* t. *Then, for every solution* $y(t)$ *of* (1), *either* $y'(t)$ *is oscillatory or else* $y(t)$ *tends monotonically to a finite limit as* $t\to\infty$

PROOF. Let $y(t)$ be any solution of equation (1). We proceed as in Theorem 2.1. From the proof of Theorem (2.1), the only place we use the boundedness of $y(t)$ is in the seventh expression on the left in equation (8). We designate this by J, as

$$(19)\qquad J = -\int_B^t \frac{1}{\phi^2(s)} \int_B^s \frac{\phi'(x)y(x)\delta y'(g(x))g'(x)}{y^{\delta+1}(g(x))} dxds$$

$$= \int_B^t \frac{1}{\phi^2(s)} \int_B^s \phi'(x)y(x) \frac{d}{dx}\left[\frac{1}{y^{\delta}(g(x))}\right] dxds.$$

From (19) J can be rewritten as

$$(20)\qquad J = -\int_B^t \frac{1}{\phi^2(s)} \int_B^s \phi'(x) \frac{y(x)}{y(g(x))} \frac{\delta y'(g(x))g'(x)}{y^{\delta}(g(x))} dxds$$

$$= \int_B^t \frac{1}{\phi^2(s)} \int_B^s \phi'(x) \frac{y(x)}{y(g(x))} \frac{\delta}{\delta-1} \frac{d}{dx}\left[\frac{1}{y^{\delta-1}(g(x))}\right] dxds.$$

Now $\delta > 1$ and $g(t) \geq t$. Thus, if $y(t) > 0$ and $y'(t) > 0$, then

$$y(x) \leq y(g(x))$$

so that $y(x)/y(g(x)) \leq 1$, provided t is large enough.

Hence J in (20) is bounded. From now on the contradiction follows as in the proof of Theorem (2.1).

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