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On 3-connected finite *H*-spaces

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§ 1. Introduction

Let X be a finite H-space, i.e., a path connected space admitting a continuous multiplication with homotopy unit and having the homotopy type of a finite CW-complex. Then, on the homotopy groups $\pi_n(X)$ of X, the following results are basic:

(1.1) (W. Browder [6; Th. 6.11]) *The first non-vanishing heigher homotopy group* $\pi_n(X)$ $(n \geqq 2)$ *occurs for odd* n.

(1.2) (A. Clark [9; Th. 1]) *If* X *is simply connected, noncontractible and admits an associative* (*not homotopy associative*) *multiplication, then* $\pi_3(X) \neq 0$.

(1.2) is not true in general, e.g., for $X = S^7$, and we have the following question:

(1.3) *Does there exist a* 3-*connected finite H-space except for the product* $(S^7)^l = S^7 \times \cdots \times S^7$ $(l$-*fold*, $l \geqq 0)$?

In this paper, we study this question under some assumptions. Our main results are stated as follows:

Theorem 1.4. *For a* 3-*connected finite H-space* X, *assume that*

(1.5) $H^*(X; G)$ *are primitively generated for* $G = Z_2$ *and* Q, *and*
(1.6) *the indecomposable module* $QH^n(X; Z_2)$ *vanishes for* $n = 15$.

Then, X *has the homotopy type of* $(S^7)^l$ *for some* $l \geqq 0$.

By this theorem, we have the following

Corollary 1.7. *Let* X *be a homotopy associative finite H-space with* $H^*(X; Z)$ *of* 2-*torsion free and* (1.6). *Then,* X *has the homotopy type of a torus* $(S^1)^t = S^1 \times \cdots \times S^1$ $(t$-*fold*, $t \geqq 0)$ *if and only if* $\pi_3(X) = 0$.

Our method of proof is to study the cohomology of X and the Adams operation ψ^n on the K-ring of the projective plane PX of X.

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§2. Reduction of the main results to Lemma 2.4

PROOF OF COROLLARY 1.7 FROM THEOREM 1.4. Let X be an H-space stated in Corollary 1.7, and $\tilde{X}$ be the universal covering space of X. Then, $\tilde{X}$ is a homotopy associative H-space and so $\tilde{X}$ satisfies (1.5) for $G=Q$ by [4; Th. 6.6]. According to W. Browder [5; Cor.], $\tilde{X}$ is also finite. Assume that $\pi_3(X)=0$. Then $\tilde{X}$ is 3-connected by (1.1). Furthermore, we can prove that

(2.1) $\tilde{X}$ satisfies (1.5) for $G = Z_2$ and (1.6).

Then, $\tilde{X}\simeq(S^7)^l$ by Theorem 1.4. If $l\geqq 1$, then $(S^7)^l$ admits no (mod 2) homotopy associative multiplications by [10; Th. 1]. Thus $l=0$, $\tilde{X}\simeq *$ and $X=K(\pi_1(X), 1)$. If $K(\pi, 1)$ is a finite H-space, then it has the homotopy type of a torus. So, $X\simeq(S^1)^t$. Conversely, if $X\simeq(S^1)^t$, then $\pi_3(X)=0$ clearly Thus, we see the corollary.

To prove (2.1), we consider the map

$$f\colon X \longrightarrow K(\pi_1(X)/\mathrm{tor}, 1) \simeq (S^1)^t$$

inducing the projection $\pi_1(X)\to\pi_1(X)/\mathrm{tor}$ of the fundamental group. Furthermore, we take $g_i\colon S^1\to X$ $(1\leqq i\leqq t)$ so that their homotopy classes form a basis for $\pi_1(X)/\mathrm{tor}$, and consider the composition

$$g\colon (S^1)^t \xrightarrow{g_1\times\cdots\times g_t} X\times\cdots\times X \ (t\text{-fold}) \xrightarrow{\mu_t} X,$$

where μ_t is the t-fold multiplication of X, i.e.,

(2.2) $\mu_2 = \mu\colon X\times X\to X$ is the multiplication of X and $\mu_{s+1} = \mu(\mu_s\times\mathrm{id})$ $(s\geqq 2)$.

Then, for the homotopy fibre $\iota\colon \bar{X}\to X$ of $f\colon X\to(S^1)^t$, we see that

(2.3) $\mu(\iota\times g)\colon \bar{X}\times(S^1)^t\to X\times X\to X$ is homotopy equivalence,

because so is $fg\colon (S^1)^t\to(S^1)^t$ by definition.

Now, since $H^*(X; Z)$ has no 2-torsion by assumption, so is $H^*(\bar{X}; Z)$ by (2.3) and $\pi_1(\bar{X})=\mathrm{tor}\,\pi_1(X)$ has only odd torsion. Thus, $\tilde{X}$ is homotopy equivalent to the universal covering space of $\bar{X}$, which is 2-equivalent to $\bar{X}$; and so

$$H^*(\bar{X}; Z_2)\cong H^*(\tilde{X}; Z_2),\quad \mathrm{Tor}(H^*(\bar{X}; Z), Z_2)\cong \mathrm{Tor}(H^*(\tilde{X}; Z), Z_2)$$

by natural maps. These shows that $QH^{15}(\tilde{X}; Z_2)\cong QH^{15}(\bar{X}; Z_2)\cong Q^{15}H(X; Z_2)=0$ by (2.3) and (1.6), and that $H^*(\tilde{X}; Z)$ has no 2-torsion since so is $H^*(\bar{X}; Z)$. Thus $H^*(X; Z_2)$ is primitively generated by [4; Th. 6.6] since $\tilde{X}$ is a homotopy associative H-space, and (2.1) is valid. Q.E.D.

Theorem 1.4 follows from the following

LEMMA 2.4. *Under the assumptions in Theorem* 1.4, $QH^n(X; Q)=0$ *for* $n \neq 7$.

PROOF OF THEOREM 1.4 FROM LEMMA 2.4. First we prove that

(2.5) $H^*(X; Z)$ has no torsion.

In fact, if $H^*(X; Z)$ has p-torsion for a prime p, then $QH^{2i}(X; Z_p) \neq 0$ for some $i \geqq 1$ by [6; Th. 4.9], and $QH^{2ip^k-1}(X; Q) \neq 0$ for some $k \geqq 1$ by [7; Th. 4.7]. Here, $i \geqq 3$ by (1.1) since X is 3-connected, and hence $2ip^k-1 \neq 7$ which contradicts Lemma 2.4. So, (2.5) holds.

Now, we have $H^*(X; Z) \cong H^*((S^7)^l; Z)$ by A. Borel [4: Prop. 6.5], (2.5) and $QH^n(X; Q)=0$ for $n \neq 7$ in Lemma 2.4. Since $\pi_7(X) \cong H_7(X; Z) \cong \mathrm{Hom}(H^7(X; Z), Z)$, there are maps $f_i: S^7 \to X$ $(1 \leqq i \leqq l)$ such that $H_7(X; Z)= Z\{f_{1*}(\xi), \cdots, f_{l*}(\xi)\}$ ($\xi \in H_7(S^7; Z)$ is a generator). Then $f=\mu_l(f_1 \times \cdots \times f_l)$: $(S^7)^l \to X$ (μ_l is given in (2.2)) satisfies $f^*: H^*(X; Z) \cong H^*((S^7)^l; Z)$, and so $X \cong (S^7)^l$. Q. E. D.

§3. Cohomology of X in Theorem 1.4

The rest of this paper is devoted to prove Lemma 2.4.

In this section, assume that X is a 3-connected finite H-space with (1.5). Then, we notice the following results due to E. Thomas [17]:

(3.1) (i) ([17; Th. 1.1]) *Let n and t be positive integers with* $\binom{2n-1-t}{t} \neq 0 \bmod 2$. *Then,*

$$Sq^t PH^{2n-1}(X; Z_2) = 0 \quad \textit{and} \quad PH^{2n-1}(X; Z_2) = Sq^t PH^{2n-1-t}(X; Z_2),$$

where P denotes the primitive module.

(ii) ([17; Th. 1.2]) *If* $u \in PH^{2^s t}(X; Z_2)$, *then*

$$u = v^{2^s} \textit{ for some } v \in PH^t(X; Z_2).$$

REMARK. (3.1) is based on Browder–Thomas [8; Th. 1.1] for $p=2$ which is valid because X is finite (see [14]).

Now, we use the following notation hereafter:

(3.2) $d(n, G) = d(n, G; X) = \dim PH^n(X; G)$ for $G = Z_2$ and Q.

Then, we have the following two lemmas:

LEMMA 3.3. (i) $\dim QH^n(X; Q) = d(n, Q)$, *and* $d(2n, Q)=0$.

(ii) $\dim QH^{2n+1}(X; Z_2)=d(2n+1, Z_2)$, *and* $QH^{2n}(X; Z_2)=0$. *Therefore, the assumption* (1.6) *is equivalent to* $d(15, Z_2)=0$.

PROOF. (i) Since $H^*(X; Q)$ is primitively generated by (1.5), $PH^n(X; Q)\cong QH^n(X; Q)$ by Milnor–Moore [16; Prop. 4.17]. Furthermore, by Hopf's theorem, $QH^{2n}(X; Q)=0$, which implies $d(2n, Q)=0$ by the above fact.

(ii) Since $H^*(X; Z_2)$ is primitively generated by (1.5), we have the exact sequence

$$(3.4) \qquad 0 \longrightarrow P(\xi H^*(X; Z_2)) \longrightarrow PH^*(X; Z_2) \xrightarrow{\pi} QH^*(X; Z_2) \longrightarrow 0$$

by [16; Prop. 4.21], where $\xi: H^*(X; Z_2)\to H^*(X; Z_2)$ is defined by $\xi(x)=x^2$ and is a map of Hopf algebras. Thus $\pi: PH^{2n+1}(X; Z_2)\cong QH^{2n+1}(X; Z_2)$. By (3.1) (ii), $QH^{2n}(X; Z_2)=\pi(PH^{2n}(X; Z_2))=0$. These show (ii). Q.E.D.

LEMMA 3.5. (i) $d(n, Q)=d(n, Z_2)$ *for* $n\leqq 12$, *which is* 0 *if* $n\neq 7, 11$.

(ii) *If* $d(15, Z_2)=0$, *then* $d(n, Z_2)=0$ *for* $n\leqq 30$ *and* $n\neq 7, 11, 13, 14, 28$.

(iii) *If* $d(15, Z_2)=0$, *then* $d(n, Q)=0$ *for* $n\leqq 30$ *and* $n\neq 7, 11, 13, 27$.

(iv) *If* $d(n, Z_2)=0$ *for* $n=11$ *and* 15, *then* $d(n, Q)=d(n, Z_2)$ *for all n, and* $d(n, Q)=d(n, Z_2)=0$ *if* $n\neq 7, 2^r-1$ $(r\geqq 5)$.

PROOF. For the simplicity, we denote $PH^n(X; Z_2)$ by PH^n.

(i) Since X is 3-connected, it is clear that $d(n, Q)=0=d(n, Z_2)$ for $n\leqq 4$ by (1.1). Thus (3.1) (i) shows that $PH^5=Sq^2PH^3=0$ and hence $PH^9=Sq^4PH^5=0$. Furthermore, (3.1) (ii) implies

$$(3.6) \qquad PH^n = (PH^t)^{(2^s)} = \{x^{2^s} \mid x\in PH^t\} \qquad \text{for} \quad n = 2^s t.$$

Thus, $PH^{2n}=0$ for $n\leqq 6$. Therefore, in the Bockstein spectral sequence

$$(3.7) \qquad E_1^n = H^n(X; Z_2) \Longrightarrow E_\infty^n = (H^n(X; Z)/\mathrm{tor}) \otimes Z_2,$$

if $n\leqq 12$, then $d_r=0$ on E_r^n and $E_1^n=E_\infty^n$, which implies $d(n, Q)=d(n, Z_2)$ by Lemma 3.3.

(ii) If $n\leqq 7$, then $\binom{15}{2n}\neq 0 \mod 2$ and $PH^{15+2n}=Sq^{2n}PH^{15}=0$ by (3.1) (i) and the assumption. For $n=2^s t\leqq 30$ with odd t, $PH^n=0$ if $t\neq 7, 11, 13$ by (3.6) and (i). On the other hand, by the Adem relation, we have

$$(3.8) \quad PH^{2t} = (PH^t)^{(2)} = Sq^tPH^t = Sq^1Sq^{t-1}PH^t \subset Sq^1PH^{2t-1} \ (t\text{: odd}),$$

which is 0 if $t=11, 13$ by the above argument. Thus, we see (ii).

(iii) By (3.6), (3.8) and $Sq^1(PH^t)^{(2)}=0$, we see that

$$PH^{28} = (PH^7)^{(4)} \subset (Sq^1PH^{13})\cdot(PH^7)^{(2)} = Sq^1(PH^{13}\cdot(PH^7)^{(2)}).$$

Thus in (3.7), $E_2^{2n}=0$ for $n\leqq 15$ and $E_2^{2n+1}=E_1^{2n+1}$ for $n\leqq 14$ with $n\neq 6, 13$.

Therefore, if $n \leqq 30$, then $d_r = 0$ on E_r^n for $r \geqq 2$ and $E_\infty^n = E_2^n$. Hence $d(n, Q)$ ($n \leqq 30$) is 0 if $n \neq 7$, 11, 13, 27 by (ii) and Lemma 3.3 (i).

(iv) Assume $d(11, Z_2) = 0$, in addition to (ii) and (iii). Then, $PH^{13} = Sq^2 PH^{11} = 0$ by (3.1) (i), $PH^{14} \subset Sq^1 PH^{13} = 0$ by (3.8), and $PH^{28} = (PH^{14})^{(2)} = 0$ by (3.6). Thus $d(n, Z_2) = 0$ for $n \leqq 30$ and $n \neq 7$ by (ii). Now, we prove that

(3.9) $d(2n+1, Z_2) = 0$ for $2r' + 1 \leqq 2n + 1 \leqq 4r' - 3$ ($r' = 2^{r-1}$)

by induction on r, which is shown already if $r \leqq 4$. Let $r \geqq 5$.

Case 1) $2r' + 1 \leqq 2n + 1 \leqq 3r' - 3$: Then $\binom{2n+1-r'}{r'} \not\equiv 0$ mod 2 and $PH^{2n+1} = Sq^{r'} PH^{2n+1-r'} = 0$ by (3.1) (i) and the inductive hypothesis.

Case 2) $2n+1 = 3r' - 1$: Take any $x \in PH^{2n+1}$. Then, $x = Sq^{r'} y$ for some $y \in PH^{2r'-1}$ in the same way. Now, $Sq^1 y \in PH^{2r'} = (PH^1)^{(2r')} = 0$ by (3.6), and $Sq^{2^t} y \in PH^{2r'+2^t-1} = 0$ for any t with $1 \leqq t \leqq r-2$ by Case 1). Thus, [1; Th. 4.6.1] and $r \geqq 5$ imply that

$$x = Sq^{r'} y = \sum \alpha_i v_i \text{ for some } v_i \in H^*(X; Z_2) \text{ and } \alpha_i \in \mathscr{A} \text{ with } 0 < \deg \alpha_i < r',$$

where $\mathscr{A}$ is the mod 2 Steenrod algebra. Since $H^*(X; Z_2)$ is primitively generated, we can write as $v_i = w_i + d_i$ where $w_i \in PH^*$ and d_i is decomposable. Here, $w_i = 0$ if $w_i \in PH^{\text{odd}}$ by Case 1) and we can take $w_i = 0$ if $w_i \in PH^{\text{even}}$ by (3.1) (ii). Therefore, $x = \sum \alpha_i d_i \in PH^{2n+1}$ is decomposable, which implies $x = 0$ by the exact sequence (3.4).

Case 3) $3r' + 1 \leqq 2n + 1 \leqq 4r' - 3$: Put $t = 2n + 2 - 3r'$. Then $\binom{2n+1-t}{t} = \binom{3r'-1}{t} \not\equiv 0$ mod 2, and $PH^{2n+1} = Sq^t PH^{3r'-1} = 0$ by (3.1) (i) and Case 2). This completes the inductive proof of (3.9).

Finally, we prove that

(3.10) $d(2n, Z_2) = 0$ for any $n = r't$ with $r' = 2^{r-1}$ and odd t.

If $t \neq 2^s - 1$ ($s \geqq 3$), then $PH^{2n} = 0$ by (3.6) and (3.9). Assume $t = 2^s - 1$ ($s \geqq 3$). If $r' = 1$, then $PH^{2n} \subset Sq^1 PH^{2t-1} = 0$ by (3.8) and (3.9). If $r' \geqq 2$, then $PH^{2n} = (PH^{2t})^{(r')}$ by (3.6), which is 0 as is shown. Thus, we see (3.10), and (iv) is proved for Z_2.

Now, consider the Bockstein spectral sequence (3.7). Then, $PE_1^{2n} = PH^{2n} = 0$ and $d_r = 0$ on E_r^n for any $r \geqq 1$, since $E_1^n = H^n(X; Z_2)$ is primitively generated. Thus, $E_\infty^n = E_1^n$ which means $d(n, Q) = d(n, Z_2)$ for any $n \geqq 1$, and (iv) is proved completely. Q. E. D.

§ 4. *K*-ring of *X* and the projective plane of *X*

We continue to assume that X is a 3-connected finite H-space with (1.5).

Furthermore, we regard X to be a finite CW-complex and the multiplication μ a cellular map.

Let Y be a CW-complex with the n-skeleton Y^n, and $K^*(Y)$ be the Z_2-graded complex K-ring with $K^0(Y)=K(Y)$ and $K^1(Y)=K(\Sigma Y)$, where Σ denotes the suspension. We filter $K^*(Y)$ by

$$F_pK^j(Y) = \mathrm{Ker}\,(K^j(Y)\to K^j(Y^{p-1})) \quad (j=0, 1). \tag{4.1}$$

Then, for any $y\in K^j(Y)$, we write

$$\deg y = p \quad \text{if} \quad y\in F_pK^j(Y) - F_{p+1}K^j(Y). \tag{4.2}$$

Now, we prove the following key lemmas.

PROPOSITION 4.3. *Under the above assumption on X, $K^*(X)$ is torsion free and has the structure of primitively generated Hopf algebra. Moreover, there exist $x_i\in PK^1(X)$, $1\leqq i\leqq l$, such that*

$$K^*(X) \cong \Lambda_Z(x_1,\cdots, x_l) \quad \textit{and} \quad \sharp\{i\mid \deg x_i=n\} = d(n, Q).$$

Here, $\sharp A$ denotes the number of elements in a finite set A.

PROOF. Since $H^*(X; Z_2)$ is primitively generated by (1.5), the Pontrjagin ring $H_*(X; Z_2)$ is associative by [16; Prop. 4.20]. Thus $H_*(\Omega X; Z)$ (ΩX is the loop space of X) is torsion free by J. Lin [6; Th. 8.1], and then so is $K^*(X)$ by R. Kane [13; Th. 1.4]. This implies that $K^*(X\times X)\cong K^*(X)\otimes K^*(X)$ and $K^*(X)$ has the structure of Hopf algebra. Furthermore, the Chern character

$$ch\colon K^*(X) \longrightarrow K^*(X)\otimes Q \xrightarrow[\cong]{} H^*(X; Q)$$

is monomorphic and is a map of Hopf algebras. Here, $H^*(X; Q)$ is an exterior algebra over primitive elements by assumption (1.5) and Hopf's theorem. Thus, by L. Hodgikin [11; Th. 2.2], we see that

$$K^*(X) = \Lambda_Z(x_1,\cdots, x_l) \qquad \text{for} \quad x_i\in PK^*(X).$$

Here $x_i\in PK^1(X)$, because $PH^{\mathrm{even}}(X; Q)=0$ by Lemma 3.3 (i) and $ch(K^0(X))\subset H^{\mathrm{even}}(X; Q)$. On the other hand, by the Atiyah-Hirzebruch spectral sequence for $K^*(\ )\otimes Q$, we see that

$$(F_{2p-1}K^1(X)/F_{2p}K^1(X))\otimes Q \cong H^{2p-1}(X; Q),$$

which implies $\sharp\{i\mid \deg x_i=2p-1\}=d(2p-1, Q)$. Q.E.D.

Let PX be the projective plane of X, i.e.,

$$PX = \Sigma X\cup_{H(\mu)} C(X*X)$$

is the mapping cone of the Hopf construction $H(\mu): X*X \to \Sigma X$ of μ. Then, PX is a finite CW-complex containing ΣX as a subcomplex. By definition, we have the exact sequence

$$(4.4)\quad \cdots \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(X\wedge X) \longrightarrow \tilde{K}(PX) \xrightarrow{\tau} \tilde{K}^1(X) \longrightarrow \tilde{K}^1(X\wedge X) \longrightarrow \cdots$$
$$(\tilde{K}(Y)=\tilde{K}^0(Y)),$$

where $\tilde{K}(X\wedge X)\cong(\tilde{K}^*(X)\otimes\tilde{K}^*(X))^0$ by the above proposition.

PROPOSITION 4.5. *For x_i $(1\leqq i\leqq l)$ in the above proposition, there exist elements y_i and an ideal S in $K(PK)$ such that*

$$\tau y_i = x_i,\quad \deg y_i = \deg x_i + 1;\quad \tau S = 0,\quad S\cdot K(PX) = 0,$$

$$K(PX) \cong T^3A \oplus S \text{ (as rings)},\quad \text{and}\quad \psi^n(S)\subset S \text{ for all } n,$$

where τ is the homomorphism in (4.4),

$$T^3A = A/D^3A,\quad A = Z[y_1,\cdots, y_l],\quad D^3A = (\tilde{A}\cdot\tilde{A})\cdot\tilde{A}$$

and ψ^n is the Adams operation on K.

PROOF. The proof of the corresponding results for $H^*(PX; Z_p)$ and $K(PK)\otimes Z_{(2)}$ are given in [8; Th. 1.1] and [12; Lemmas 6.3–4]. This proposition can be also proved by the same method, and we omit the details. Q. E. D.

T^3A in the above is called the *filtered truncated polynomial algebra of height* 3 *on* $\{y_i\}$.

Let B be a filtered algebra over Z by a filtration

$$B = F_0B \supset F_1B \supset \cdots \supset F_pB \supset \cdots \text{ with } F_pB\cdot F_qB \subset F_{p+q}B \text{ for any } p,\, q \geqq 0.$$

Then, we say that B is a *ψ-algebra* if there are maps ψ^n: $B\to B$ $(n\in Z)$ of filtered algebras, i.e., algebra homomorphisms ψ^n with $\psi^n F_pB\subset F_pB$, such that

(4.6.1) $\psi^1=\mathrm{id}$ and $\psi^m\psi^n=\psi^n\psi^m=\psi^{nm}$ for any $m,\, n\in Z$,
(4.6.2) if $x\in F_{2r}B$, then $\psi^n x\equiv n^r x \bmod F_{2r+1}B$ for any $r\geqq 0$ and $n\in Z$, and
(4.6.3) $\psi^2 x\equiv x^2 \bmod 2$ for any $x\in B$.

By [2; Th. 5.1], [3; (1.1–5)] and the definition, we see that

LEMMA 4.7. (i) *The K-ring $K(Y)$ of a finite CW-complex Y filtered by* (4.1) *is a ψ-algebra by the Adams operations ψ^n.*
(ii) *If I is an ideal in a ψ-algebra B with $\psi^n I\subset I$ for all n, then B/I is also a ψ-algebra.*

Now, according to Proposition 4.5, we can prove Lemma 2.4 and hence the main results in §1 (see §2) by the following

PROPOSITION 4.8. *Assume that a filtered truncated polynomial algebra*

$$T^3A = A/D^3A, \quad A = Z[y_1,\cdots, y_l] \quad \textit{with} \quad \deg y_i = 8,\ 12,\ 14 \textit{ or even} \geqq 28,$$

of height 3 *is a* ψ*-algebra. Then*:

(i) *There is no* i *with* $\deg y_i = 12$.

(ii) *If* $\deg y_i$ *is* 8 *or* 2^r $(r \geqq 5)$, *then* $\deg y_i = 8$ *for all* i.

PROOF OF LEMMA 2.4 FROM PROPOSITION 4.8. Let X be an H-space in Theorem 1.4. Then, X is regarded as an H-space in this section satisfying (1.6), i.e., $d(15, Z_2)=0$ (see Lemma 3.3 (ii)). Thus, $T^3A = K(PX)/S$ in Proposition 4.5 is a ψ-algebra by Lemma 4.7, and the generators $y_1,\cdots, y_l$ satisfy $\#\{i|\deg y_i = n+1\} = d(n, Q)$ by Proposition 4.3. Therefore, $d(11, Q)=0$ by Lemma 3.5 (iii) and Proposition 4.8 (i), and hence $QH^n(X; Q)=0$ for $n \neq 7$ by Lemma 3.3 (i), 3.5 (iv) and Proposition 4.8 (ii). Q. E. D.

The above proposition is proved algebraically in the next section.

§5. Proof of Proposition 4.8

Let T^3A be a ψ-algebra in Proposition 4.8. Then, the ideal I in T^3A generated by $\{y_i|\deg y_i \geqq 28\}$ satisfies $\psi^n I \subset I$ for all n. In fact, if $\deg y_i = 2r \geqq 28$, then $\psi^n y_i \equiv n^r y_i \mod F_{2r+1}T^3A$ by (4.6.2) and $F_{2r+1}T^3A \subset I$ by assumption, which show $\psi^n y_i \in I$. Therefore, we have a ψ-algebra T^3A/I by Lemma 4.7 (ii), which is isomorphic to

(5.1.1) a ψ-algebra $T^3A_1 = A_1/D^3A_1$, $A_1 = Z[y_1,\cdots, y_t]$, with $\deg y_i = 2\varepsilon(s)$ if $t_{s-1} < i \leqq t_s$, and $\varepsilon(s) = 4$, 6 or 7 according to $s = 1$, 2 or 3, respectively $(t_0=0,\ t_3=t)$.

Hereafter, consider this ψ-algebra T^3A_1. Then, we have

(5.1.2) $\psi^n y_i = n^{\varepsilon(s)} y_i + \sum_{t_s<j} A(i, j; n) y_j + \sum_{j\leqq k} B(i, j, k; n) y_j y_k \ (t_{s-1} < i \leqq t_s)$

for some integers A and B by (4.6.2). Therefore,

(5.1.3) for any $j > t_2$, the coefficient of y_j^2 in $\psi^m \psi^n y_j$ is equal to

$$n^7 B(j, j, j; m) + m^{14} B(j, j, j; n) + m^7 \textstyle\sum_{i\leqq t_2} B(j, i, j; n) A(i, j; m)$$
$$+ \textstyle\sum_{i\leqq k\leqq t_2} B(j, i, k; n) A(i, j; m) A(k, j; m).$$

Thus, by comparing them in $\psi^2\psi^{-1}y_j = \psi^{-1}\psi^2 y_j$ of (4.6.1), we have

$$2B(j,j,j;2) \equiv \textstyle\sum_{i\leqq t_2} B(j,i,j;2)A(i,j;-1)$$
$$\qquad - \textstyle\sum_{i\leqq k\leqq t_2} B(j,i,k;2)A(i,j;-1)A(k,j;-1) \bmod 4,$$

because $A(i,j;2)\equiv 0 \bmod 2$ by (4.6.3). Here, (4.6.3) also shows that $B(j,j,j;2)\not\equiv 0$ and $B(j,i,j;2)\equiv 0\equiv B(j,i,k;2) \bmod 2$. Therefore,

(*) for any $j > t_2$, there is $i \leqq t_2$ such that $A(i,j:-1)$ is odd.

Then, by changing the generators y_i $(1\leqq i\leqq t)$ if necessary, we may assume that

(5.1.4) $A(i,j;-1)$ $(i\leqq t_2<j)$ is odd when and only when $i=i(j)$,

where

$$i(j)=\begin{cases} j-t_2 & \text{if } j\leqq t_2+r, \\ t_1+j-t_2-r & \text{if } j>t_2+r, \end{cases} \quad \text{for some } r\geqq 0 \text{ with } d_3-d_2\leqq r\leqq d_1$$

($d_s=t_s-t_{s-1}=\#\{i \mid \deg y_i=2\varepsilon(s)\}$). In fact, for $j_0>t_2$, take $i_0\leqq t_2$ with odd $A(i_0,j_0;-1)$ by (*), and with $i_0>t_1$ if it exists; and replace $y_j(j_0\neq j>t_2)$ with odd $A(i_0,j;-1)$ by $y_j+y_{j_0}$ and y_i $(i_0\neq i\leqq t_2)$ with odd $A(i,j_0;-1)$ by $y_i+y_{i_0}$. Repeat these replacements for all $j_0>t_2$ and change the order if necessary. Then, $\{y_i\}$ is replaced with the new $\{y_i\}$ so that $A(i,j;-1)$ turns out to satisfy (5.1.4).

Here, we notice that

(5.1.5) $\qquad A(i,j;-1)=0$ for any i,j with $i\leqq t_1<j\leqq t_2$.

This is seen by the following equalities of (5.1.1) and (5.4.2) for $n=-1$:

$$y_i=\psi^1 y_i=\psi^{-1}\psi^{-1}y_i\equiv y_i+2\textstyle\sum_{t_1<j\leqq t_2} A(i,j;-1) \quad \bmod F_{13}T^3A_1.$$

Now, we put

(5.1.6)
$$\bar{y}_i = y_i+\textstyle\sum_{t_2<j}[A(i,j;-1)/2]y_j \text{ for } i\leqq t_2,$$
$$\bar{y}_j=\psi^{-1}\bar{y}_{i(j)}-\bar{y}_{i(j)} \text{ for } j\geqq t_2 \quad (\text{by } i(j) \text{ in (5.1.4)}).$$

Then, by (5.1.2), (5.1.4–5) and (4.6.1), we see the following $(i\leqq t_2<j)$:

(5.1.7) $\psi^{-1}\bar{y}_i\equiv\begin{cases}\bar{y}_i+y_j & \text{if } i=i(j)\\ \bar{y}_i & \text{otherwise}\end{cases} \bmod D^2A_1,\quad \psi^{-1}\bar{y}_j=-\bar{y}_j;$

(5.1.8) $\qquad \bar{y}_i\equiv y_i \bmod F_{14}T^3A_1,\quad \bar{y}_j\equiv y_j \bmod F_{15}T^3A_1.$

LEMMA 5.2. (i) *T^3A_1 in (5.1.1) is equal to $T^3\bar{A}_1=\bar{A}_1/D^3\bar{A}_1$ with $\bar{A}_1=Z[\bar{y}_1,\cdots,\bar{y}_t]$, where* $\deg\bar{y}_i=\deg y_i$ $(1\leqq i\leqq t)$.

(ii) *Let I be the ideal in $T^3\bar{A}_1$ generated by $\{\bar{y}_j \mid j>t_2\}$. Then, $\psi^nI\subset I$ for all n, and we have a ψ-algebra*

$$T^3\bar{A}_1/I \cong T^3A_2 = A_2/D^3A_2,\ A_2 = Z[\bar{y}_1,\cdots,\bar{y}_{t_2}].$$

PROOF. (i) is clear by (5.1.6–8). By (5.1.2) for $T^3\bar{A}_1$,

$$\psi^n\bar{y}_j = n^7\bar{y}_j + \sum_{i\leqq k}\bar{B}(j, i, k; n)\bar{y}_i\bar{y}_k \qquad \text{for} \quad j > t_2 .$$

Now, compare the coefficients of $\bar{y}_i\bar{y}_k$ in $\psi^{-1}\psi^n\bar{y}_j=\psi^n\psi^{-1}\bar{y}_j$. Then, by (5.1.7) and $D^2A_1=D^2\bar{A}_1$, we see that

$$\bar{B}(j, i, k; n) = 0 \text{ for any } i \leqq k \leqq t_2, \text{ and } \psi^n\bar{y}_j \in I \text{ for any } j > t_2 .$$

This implies that $\psi^nI\subset I$, and we see (ii) by Lemma 5.2 (ii). Q.E.D.

From now on, we omit the bars of generators and consider the above ψ-algebra

$$T^3A_2 = A_2/D^3A_2,\ A_2 = Z[y_1,\cdots, y_{t_2}], \text{ with}$$
$$\deg y_k = 8 \text{ if } k \leqq t_1 ,\ = 12 \text{ otherwise},$$

where (5.1.2) is written as follows:

(5.3.1) $\psi^ny_i = n^4y_i + \sum_{t_1<k}A(i, k; n)y_k + \sum_{k\leqq k'}B(i, k, k'; n)y_ky_{k'}$ for $i \leqq t_1$,

(5.3.2) $\psi^ny_j = n^6y_j + \sum_{k\leqq k'}B(j, k, k'; n)y_ky_{k'}$ for $j > t_1$.

Then, for $i\leqq i'\leqq t_1\leqq j$, the coefficient of y_j in $\psi^m\psi^ny_i$ is $n^4A(i, j; m)+m^6A(i, j; n)$ and that of $y_iy_{i'}$ in $\psi^m\psi^ny_j$ is $n^6B(j, i, i'; m)+m^8B(j, i, i'; n)$. Thus by comparing them in $\psi^2\psi^3y_k=\psi^3\psi^2y_k$ of (4.6.1), we see that

(5.3.3) $3^3A(i, j; 2) = 2A(i, j; 3)$ for any $i \leqq t_1 < j$,

(5.3.4) $3^5B(j, i, i'; 2) = 2^3B(j, i, i'; 3)$ for any $i \leqq i' \leqq t_1 < j$.

To study A and B more precisely, we prepare the following (5.3.6–7) for $i\leqq t_1<j$ and $n, m\in Z$, where

(5.3.5)
$$\begin{aligned} C(l) &= m^{12}B(l, j, j; n) + m^6\sum_{k\leqq t_1}B(l, k, j; n)A(k, j; m) \\ &\quad + \sum_{k\leqq k'\leqq t_1}B(l, k, k'; n)A(k, j; m)A(k', j; m), \\ D(l) &= m^{10}B(l, i, j; n) + m^4\sum_{k\leqq i}B(l, k, i; n)A(k, j; m) \\ &\quad + m^4\sum_{i\leqq k}B(l, i, k; n)A(k, j; m), \\ E(l, l') &= n^4B(i, l, l'; m) + \sum_{t_1<k}A(i, k; n)B(k, l, l'; m). \end{aligned}$$

(5.3.6) The coefficients of y_j^2 and y_iy_j in $\psi^m\psi^ny_j$ are equal to

$n^6B(j, j, j; m)+C(j)$ and $n^6B(j, i, j; m) + D(j)$, respectively.

(5.3.7) Those of y_i^2, y_j^2 and y_iy_j in $\psi^m\psi^ny_i$ are equal to

$E(i, i) + m^8B(i, i, i; n)$, $E(j, j) + C(i)$ and $E(i, j) + D(i)$, respectively.

LEMMA 5.4. *$A(i, j; 3)$ is even for any $i \leqq t_1 < j$.*

PROOF. Suppose contrarily that $A(a, b; 3)$ is odd for some $a \leqq t_1 < b$. Then, by changing the generators y_k, $1 \leqq k \leqq t_2$, we may assume that

(5.5.1) $A(a, j; 3) \equiv 0 \equiv A(i, b; 3) \mod 2^7$ for any i, j with $a \neq i \leqq t_1 < j \neq b$.

In fact, there are integers λ and μ with $\lambda A(a, b; 3) + \mu = 1$ and $\mu \equiv 0 \mod 2^7$ by assumption. Then, we see (5.5.1) by replacing y_i $(a \neq i \leqq t_1)$ and y_b with

$$\tilde{y}_i = y_i - \lambda A(i, b; 3) y_a \text{ and } \tilde{y}_b = y_b + \textstyle\sum_{t_1 < j \neq b} A(a, j; 3) y_j, \text{ respectively,}$$

because (5.3.1) turns out to

$$\left.\begin{aligned} \psi^3 y_a &\equiv 3^4 y_a + \textstyle\sum_{t_1 < j \neq b} \mu A(a, j; 3) y_j + A(a, b; 3) \tilde{y}_b \\ \psi^3 \tilde{y}_i &\equiv 3^4 \tilde{y}_i + \textstyle\sum_{t_1 < j \neq b} \tilde{A}(i, j; 3) y_j + \mu A(i, b; 3) \tilde{y}_b \end{aligned}\right\} \mod D^2 A_2.$$

We now consider the coefficients in $\psi^2 \psi^3 y_k = \psi^3 \psi^2 y_k$ given in (5.3.6–7) $(k = b$ or $a)$ and compare them by taking $\mod 2^r$ and by using (5.3.3–4) and (5.5.1). Then, in the first place, we see that

$$\begin{aligned} &(5.5.2) & \alpha &= A(a, b; 2) B(b, a, a; 3) = 0 \mod 2^4, \\ & & \beta &= A(a, b; 3) B(b, a, a; 2) \equiv 0 \mod 2^6. \end{aligned}$$

In fact, (5.3.7) for y_a^2 implies $\alpha \equiv \beta \mod 2^4$ by (5.5.1) and (5.3.3). On the other hand, $2^2 \alpha = 3^3 \beta$ by (5.3.3–4). These show (5.5.2). In the second place, by (5.3.6) for $y_a y_b$ taking mod 2^7, we see that

$$2^6 B(b, a, b; 3) + 2 \cdot 3^4 \beta + 3^6 (3^4 - 1) B(b, a, b; 2) \equiv 2^5 \alpha \mod 2^7,$$

which together with (5.5.2) implies that

(5.5.3) $\qquad B(b, a, b; 2) \equiv 2^2 B(b, a, b; 3) \mod 2^3.$

In the third place, by (5.3.6) for y_b^2 taking mod 2^3 and (5.5.2), we have

$$B(b, a, b; 2) A(a, b; 3) \equiv \alpha A(a, b; 2) - \beta A(a, b; 3) \equiv 0 \mod 2^3.$$

Since $A(a, b; 3)$ is odd by assumption, this shows that

(5.5.4) $\qquad B(b, a, b; 2) \equiv 0 \mod 2^3$, and hence $B(b, a, b; 3)$ is even,

by (5.5.3). Finally, taking mod 2^2, (5.3.7) for $y_a y_b$ implies that

$$2B(a, a, a; 2) A(a, b; 3) \equiv A(a, b; 3) B(b, a, b; 2) - A(a, b; 2) B(b, a, b; 3) \equiv 0$$

mod 2^2 by (5.5.4) and (5.3.3). Thus

(5.5.5) $\qquad B(a, a, a; 2)$ is even, since $A(a, b; 3)$ is odd.

This contradicts (4.6.3); and the lemma is proved. Q. E. D.

LEMMA 5.6. *$t_2=t_1$, i.e., there exists no y_j with* $\deg y_j=12$.

PROOF. Compare the coefficients of y_j^2 in $\psi^2\psi^3 y_i=\psi^3\psi^2 y_i$ taking mod 2^3 for any $i\leqq t_1<j$ by using (5.3.7), Lemma 5.4 and (5.3.3). Then, we see that

$$(5.7.1)\quad \sum_{t_1<k} A(i, k; 3)B(k, j, j; 2)$$
$$\equiv \sum_{t_1<k} A(i, k; 2)B(k, j, j; 3) + \sum_{k\leqq t_1} B(i, k, j; 2)A(k, j; 3)$$
$$+ \sum_{k\leqq k'\leqq t_1} B(i, k, k'; 2)A(k, j; 3)A(k', j; 3) \mod 2^3.$$

We notice by (4.6.3) that

$$(5.7.2)\quad B(k, k', k''; 2) \equiv 1 \mod 2 \text{ if and only if } k = k' = k''.$$

Here, (5.7.1) implies firstly by taking mod 2^2 that $A(i, j; 3)\equiv 0 \mod 2^2$ and then

$$(5.7.3)\qquad A(i, j; 3) \equiv 0 \mod 2^3 \text{ for any } i \leqq t_1 < j.$$

Compare now the coefficients of y_j^2 in $\psi^2\psi^3 y_j=\psi^3\psi^2 y_j$ taking $\mod 2^4$ using (5.3.6). Then, by (5.7.2–3) and (5.3.3), we see that

$$(5.7.4)\qquad 3^6(3^6-1)B(j, j, j; 2) \equiv 0 \mod 2^4.$$

Thus $B(j, j, j; 2)$ is even, which contradicts (5.7.2) if $j(>t_1)$ exists; and we have $t_2=t_1$. Q. E. D.

Now, we are ready to prove Proposition 4.8.

PROOF OF PROPOSITION 4.8. (i) is already proved by Lemma 5.6.

(ii) Suppose that (ii) is not valid, and let $r\geqq 5$ be the least integer with $\sharp\{i \mid \deg y_i=2^r\}\neq 0$. Consider the ideal I in T^3A generated by $\{y_i \mid \deg y_i\geqq 2^{r+1}\}$. Then, by Lemma 4.7 (ii), we have a ψ-algebra T^3A/I, which is isomorphic to

$$T^3B = B/D^3B,\ B = Z[y_1,\cdots, y_s], \text{ with } \deg y_i = 8 \text{ if } i \leqq s_1, = 2^r \text{ if } i > s_1.$$

In this ψ-algebra, (4.6.2) implies that

$$\psi^n y_i \equiv n^4 y_i + \sum_{s_1<k} A(i, k; n) y_k \mod D^2B \text{ for } i \leqq s_1,$$
$$\psi^n y_j = n^{r'} y_j + \sum_{k\leqq k', k'>s_1} B(j, k, k'; n) y_k y_{k'} \text{ for } j > s_1,$$

where $r'=2^{r-1}$. Consider $\psi^2\psi^3 y_j=\psi^3\psi^2 y_j$ $(j>s_1)$. Then, by comparing the coefficients of $y_i y_j$ $(i\leqq s_1)$ taking mod $2^{r'}$, we see that

$$3^{r'}(3^4-1)B(j, i, j; 2) \equiv 0 \mod 2^{r'} \text{ and } B(j, i, j; 2) \equiv 0 \mod 2^{r+2},$$

since $r'-4\geqq r+2$. Therefore, by comparing those of y_j^2 taking mod 2^{r+2}, we have

$$3^{r'}(3^{r'}-1)B(j, j, j; 2) \equiv 0 \mod 2^{r+2}$$

in the same way as (5.7.4). Here, $3^{r'}-1 \equiv 2^{r+1} \mod 2^{r+2}$ by [2; Lemma 8.1]. Thus,

$$B(j, j, j; 2) \equiv 0 \mod 2,$$

which contradicts (4.6.3); and (ii) is valid. Q. E. D.

Thus, the main results in §1 are proved completely as noted at the end of §4

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