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Study of three-dimensional algebras with straightening laws which are Gorenstein domains I

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Introduction

The concept of ASL (algebra with straightening laws) is an axiomatization of the "straightening formula" appearing in invariant theory. This axiomatization, which is lucid and charming, associates commutative algebras with combinatorics through partially ordered sets (poset for short) and moreover, with topology through simplicial complexes.

Many interesting rings which appeared in classical invariant theory, such as coordinate rings of Grassmann varieties, determinantal and Pfaffian varieties turned out to be ASL, and we can obtain many informations concerning these rings by means of corresponding posets.

On the other hand, as far as the authors know, all the examples known when we started this work, are normal, rational over the base field and are rational singularities in characteristic zero, and D. Eisenbud has proposed a conjecture in [3] that every ASL domain on a wonderful poset should be normal with rational singularities.

However, in the course of classifying Gorenstein ASL domains of dimension 3, we have discovered examples of non-normal ASL domains on wonderful posets (cf. example g)). These are examples of non-normal Del Pezzo surfaces (a Del Pezzo surface is a projective surface X whose anti-canonical sheaf ω_X^{-1} is ample) of arbitrary high degree and we are sure that the theory of ASL will be very helpful to construct interesting examples of rings or varieties with given properties.

Our final goal is to classify all the three dimensional homogeneous Gorenstein ASL domains over a field. Toward this goal, in this first part, we will determine all the posets on which there exist three dimensional homogeneous Gorenstein ASL domains. Moreover, in this process we will show that every three dimensional homogeneous ASL domain over a field is Cohen-Macaulay. Our fundamental method is quite elementary and its origin is in [5].

The main result in this paper is the following

THEOREM. *Let k be a field. The posets on which there exist three dimensional homogeneous Gorenstein ASL domains are among the followings*:

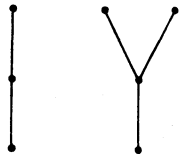
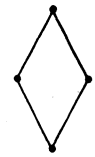


Fig. 1.

Fig. 2.



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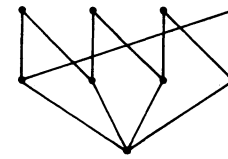

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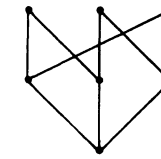

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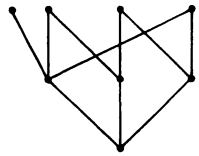

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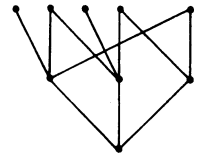

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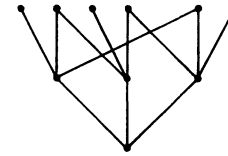

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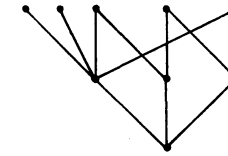

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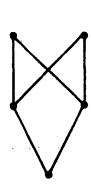
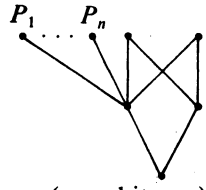
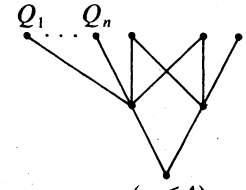


Fig. 10.



(n: arbitrary)

Fig. 11.



($n \leqq 4$)

Fig. 12.

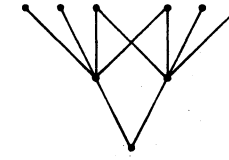


Fig. 13.

Moreover, if k is infinite, there exist examples of homogeneous Gorenstein ASL domains on every poset listed above.

We close this introduction with some remarks concerning Hodge algebras defined in [2]. Though the concept of Hodge algebras is generalized from that of ASL, the gap between them is remarkable. For example, it is shown in [6] that every graded ring over a field has a structure of a Hodge algebra. This fact makes a strong contrast to the fact that the graded rings which can be expressed as an ASL are very limited ones.

1. Notation and preliminaries

We here summarize basic definitions and terminologies on commutative algebras and combinatorics. Consult [2], [3] for further informations.

(1.1) All posets (partially ordered sets) to be considered are finite.

The *length* of a chain (totally ordered set) X is the cardinality $\sharp(X)$ as a set.

The *rank* of a poset H, denoted by $rank(H)$, is the supremum of length of chains contained in H.

The *height* of an element α in a poset is the supremum of length of chains descending from α, and written by $ht(\alpha)$. Note that $rank(H)$ and $ht(\alpha)$ in this paper are one more than those in [1] or [2].

An *ideal* in a poset H is a subset I such that $\alpha \in I$, $\beta \in H$, and $\beta \leq \alpha$ together imply $\beta \in I$.

(1.2) Suppose R is a ring and H, a subset of R, is a poset. A *monomial* is a product of the form $\alpha_1\alpha_2\cdots\alpha_p$ where $\alpha_i \in H$. A monomial $\alpha_1\alpha_2\cdots\alpha_p$ is called *standard* if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$. Now let k be a field, R a k-algebra, H a poset contained in R which generates R as a k-algebra. Then we call R an *algebra with straightening laws* on H over k if the following conditions are satisfied:

(ASL–1) The set of standard monomials is a basis of the algebra R as a vector space over k.

(ASL–2) If α and β in H are incomparable (written as $\alpha \sim \beta$) and if

$$(*) \qquad \alpha\beta = \sum r_i\gamma_{i1}\gamma_{i2}\cdots\gamma_{ip_i},$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots$, is the linear combination of standard monomials, then $\gamma_{i1} \leq \alpha, \beta$ for every i.

Note that the right-hand side of the relation in (ASL–2) is allowed to be the empty sum ($=0$), but that, though 1 is a standard monomial, no $\gamma_{i1}\gamma_{i2}\cdots\gamma_{ip_i}$ can be 1. The relations $(*)$ are called the *straightening relations* for R.

(1.3) We denote by $[\alpha\beta]$ the set of standard monomials which appear in the right-hand side of the relation for $\alpha\beta$ with $\alpha \sim \beta$. More generally, for a monomial $\alpha_1\alpha_2\cdots\alpha_p$, we denote by $[\alpha_1\alpha_2\cdots\alpha_p]$ the set of standard monomials which appear in the standard monomial expression of $\alpha_1\alpha_2\cdots\alpha_p$.

It is well known that the dimension of R as a k-algebra coincides with the rank of H (see [2]).

(1.4) An ASL R on a poset H over a filed k is called *graded* if there is a grading $R = \oplus_{n\geq 0} R_n$ such that $R_0 = k$ and each element of H is homogeneous of positive degree.

When $H \subset R_1$ we say that R is *homogeneous*.

(1.5) Throughout the remainder of this paper, we fix a field k. For convention, unless otherwise stated, we use small letters of Roman alphabet, for example, t, a_1, b_2, p_3^*,... to denote the elements of the field k, and use capital letters, for example, A, B, T,... or Greek letters, for example, α, β,... to denote the elements of the poset H.

2. Every three dimensional homogeneous ASL domain over a field is Cohen-Macaulay

In this section, unless otherwise stated, let H be a poset of rank 3 with a unique minimal element T, and R be a homogeneous ASL (not necessarily a domain) on H over a field k.

LEMMA 1. *If* $A, B \in H$, $ht(A) = ht(B) = 2$, $A \neq B$ *and* $T\alpha \in [AB]$, $\alpha \neq T, A, B$,

then $ht(\alpha)=3$ *and* $\alpha>A$, $\alpha>B$ (*for the definition of* $[AB]$, *cf.* (1.3)).

PROOF. In general, if α, β, $\gamma\in H$, $\alpha\sim\beta$, then $\alpha^2\notin[\alpha\beta]$, and moreover, $\gamma^2\in[\alpha\beta]$ implies $\gamma<\alpha$, $\gamma<\beta$.

Now, $T\alpha^2\in[\alpha(AB)]$ since $T\alpha\in[AB]$. However, if we assume $\alpha\sim A$ then $T\alpha^2\notin[(\alpha A)B]$ because $ht(B)=2$, $B\neq\alpha$, a contradiction. So, α is comparable with A and, consequently, $\alpha>A$ because $\alpha\neq A$, T and $ht(A)=2$. By the same argument, $\alpha>B$. Q. E. D.

NOTATION. Suppose that B, $C\in H$, $ht(B)=ht(C)=2$, $B\neq C$. Then $B\wedge C$ means that there exists $X\in H$ such that $X>B$, $X>C$. On the other hand, if Y, $Z\in H$, $ht(Y)=ht(Z)=3$, $Y\neq Z$, then $Y\vee Z$ means $Y>A$, $Z>A$ for some element $A\in H$ with $ht(A)=2$. Moreover we write the negation of $B\wedge C$ (resp. $Y\vee Z$) as $B\not\wedge C$ (resp. $Y\not\vee Z$).

LEMMA 2. *If* A, $X\in H$, $ht(A)=2$, $ht(X)=3$, $A\sim X$ *and if* $T\alpha\in[AX]$, $\alpha\neq T$, A, X, *then*

(i) $ht(\alpha)=2$, $\alpha<X$, $\alpha\wedge A$, *or*

(ii) $ht(\alpha)=3$, $\alpha>A$, $\alpha\vee X$.

PROOF. As $T\alpha\in[AX]$, we have $T\alpha^2\in[\alpha(AX)]$.

(i) If $ht(\alpha)=2$, $\alpha\neq A$, then as $T\alpha^2\in[(\alpha A)X]$, there exists $\beta\in H$, with $T\beta\in[\alpha A]$ and $\alpha^2\in[\beta X]$. This means $\beta>\alpha$ and $X>\alpha$. Moreover, as $T\beta\in[\alpha A]$, $\beta>A$ by Lemma 1.

(ii) If $ht(\alpha)=3$, $\alpha\neq X$ and if $\alpha\sim A$, then $T\alpha^2\notin[(\alpha A)X]$. So, $\alpha>A$. If $\alpha\not\vee X$, every standard monomial of $[\alpha X]$ is of the form $T\beta$ ($\beta\in H$), and $T\alpha^2\notin[(T\beta)A]$. This contradicts the fact that $T\alpha^2\in[(\alpha X)A]$. Thus we have $\alpha\vee X$. Q. E. D.

LEMMA 3. *If* X, $Y\in H$, $ht(X)=ht(Y)=3$, $X\neq Y$, $X\not\vee Y$ *and if* $T\alpha\in[XY]$, $\alpha\neq T$, X, Y, *then either*

(i) $ht(\alpha)=2$, $\alpha<X$, $\beta>\alpha$, $\beta\vee Y$ *for some* $\beta\in H$ *with* $ht(\beta)=3$, *or* $ht(\alpha)=2$, $\alpha<Y$, $\gamma>\alpha$, $\gamma\vee X$ *for some* $\gamma\in H$ *with* $ht(\gamma)=3$, *or*

(ii) $ht(\alpha)=3$, $X\vee\alpha$, $Y\vee\alpha$.

PROOF. As $T\alpha\in[XY]$, $X\not\vee Y$, we have $T\alpha^2\in[\alpha(XY)]$.

(i) If $ht(\alpha)=2$, $\alpha\sim X$, as $T\alpha^2\in[(\alpha X)Y]$, there exists $\gamma\in H$ with $T\gamma\in[\alpha X]$ and $\alpha^2\in[\gamma Y]$. This means $\gamma>\alpha$, $Y>\alpha$. Moreover, as $T\gamma\in[\alpha X]$, $\gamma\vee X$ by Lemma 2.

(ii) If $ht(\alpha)=3$, $\alpha\not\vee Y$, then every standard monomial of $[\alpha Y]$ is of the form $T\beta$ ($\beta\in H$). Hence $T\alpha^2\notin[X(\alpha Y)]$. This contradicts the fact that $T\alpha^2\in[\alpha(XY)]$. Thus we have $\alpha\vee Y$. Similarly, $\alpha\vee X$. Q. E. D.

LEMMA 4. *If* A, X, $Y\in H$, $ht(X)=ht(Y)=3$, $ht(A)=2$, $X\neq Y$, $X>A$, $Y>A$

and if $A\alpha \in [XY]$, $A < \alpha$, *then* $\alpha = X$ *or* $\alpha = Y$.

PROOF. Let I be the subset $\{\alpha \in H;\ \alpha \not\geq A\}$ of H and $H' = H - I$ be the complement of I in H. Note that H' is a subposet of H with a unique minimal element A. Since I is an ideal of H, [2, Prop. 1.2b)] says that the quotient algebra $R' = R/I$ is an ASL on H' and that $A\alpha \in [XY]$ in R implies $A\alpha \in [XY]$ in R'. Now, in R', as $A\alpha \in [XY]$, we have $A\alpha^2 \in [\alpha(XY)]$. If $\alpha \neq Y$, every standard monomial of $[Y\alpha]$ is of the form $A\beta$ $(\beta \in H')$. As $A\alpha^2 \in [X(Y\alpha)]$, there exists $\beta \in H'$ with $\alpha^2 \in [\beta X]$. This means $\alpha = X = \beta$. Q. E. D.

PROPOSITION A. *Let* H *be a poset of rank* 3 *with a unique minimal element* T. *Assume that there exist three elements* A, B, *and* Y *of* H *which satisfy the following conditions*:

(i) $ht(A) = ht(B) = 2$, $ht(Y) = 3$, $Y > B$ *and* $A \sim Y$,

(ii) $A \not\sim B$, *and*

(iii) $X \not\sim Y$ *for any element* $X \in H$ *with* $X > A$.

Then any homogeneous ASL R *on* H *over a field* k *cannot be an integral domain.*

PROOF. By Lemma 1 we have

$$AB = T(tT + aA + bB) \tag{1}$$

and moreover

$$AY = T(cT + dA + eY)$$

by Lemma 2. We calculate the standard monomial expression of ABY in two ways, namely

$$\begin{aligned} ABY = (AB)Y &= TY(tT + aA + bB) \\ &= tT^2Y + aT^2(cT + dA + eY) + bTBY \\ = (AY)B &= TB(cT + dA + eY) \\ &= cT^2B + dT^2(tT + aA + bB) + eTBY. \end{aligned}$$

Now we have $b = e$ from the coefficients of TBY, also $t + ae = 0$ from T^2Y, hence $t = -ab$. If we substitute $t = -ab$ in (1), then we get

$$(A - bT)(B - aT) = 0,$$

which means that the algebra R is not an integral domain. Q. E. D.

Note that Prop. A implies that $H - \{T\}$ is connected if R is an integral domain. Since H is of rank 3 with a unique minimal element T, the connectedness of $H - \{T\}$ is equivalent to *Cohen-Maculayness* of the poset H over the field k (see Baclawski [1]). Combining this result with the fundamental theorem on ASL [2, Cor. 7.2] we have the following

COROLLARY. *Every three dimensional homogeneous ASL domain over a field is Cohen-Macaulay.*

Let H be a poset of rank 2. An element $P \in H$ is called a *branch* if 1) $ht(P)=2$ and there exists a unique element A such that $P>A$, or 2) $ht(P)=1$ and there exists a unique element X such that $P<X$. Moreover, $\{P_1, P_2, \ldots, P_n\}$ $(P_i \in H)$ is called a *branch sequence* of length n if P_n is a branch of H and P_i is a branch of the subposet $H-\{P_{i+1}, \ldots, P_n\}$ for all i $(1 \leq i \leq n-1)$, in such a way that P_{i-1} is a unique element of $H-\{P_{i+1}, \ldots, P_n\}$ which is comparable with P_i $(2 \leq i \leq n)$.

Suppose that a poset H is of rank 3 with a unique minimal element T. Then an element $P \in H$, $P \neq T$, is called a branch of H when P is a branch of the rank 2 poset $H-\{T\}$.

PROPOSITION B. *Let H be a poset of rank 3 with a unique minimal element T, and R be a homogeneous ASL on H over a field k. Suppose that P is a branch of H with $ht(P)=3$ and that $A \in H$ is a unique element with $P>A$, $ht(A)=2$. If $\alpha, \beta \in H-\{P\}$, $\alpha \nsim \beta$, then P does not appear in the right-hand side of the straightening relation of $\alpha\beta$, that is, TP, AP, $P^2 \notin [\alpha\beta]$.*

Consequently, $k[H-\{P\}]$ is an ASL subring of R with the same straightening relations as those of R.

PROOF. Thanks to Lemma 1, Lemma 2 and Lemma 3, we have only to consider the case of $ht(\alpha)=ht(\beta)=3$ and $\alpha \vee \beta$. In this case it is obvious that $P^2 \notin [\alpha\beta]$ and moreover, by Lemma 4, $AP \notin [\alpha\beta]$ even if $\alpha>A$ and $\beta>A$.

Now we shall prove $TP \notin [\alpha\beta]$. In general, if $\gamma, \delta \in H$, $\gamma \leq \delta$, $ht(\gamma) \leq 2$, $\delta \neq P$, then $TP^2 \notin [(\gamma\delta)P]$. In fact, it is easy to see that TP^2 does not belong to $[T^2P]$, $[TBP]$, $[TZP]$ $(B, Z \in H, ht(B)=2, ht(Z)=3, Z \neq P)$, $[A^2P]$, and $[AZP]$ $(Z \in H, ht(Z)=3, Z>A, Z \neq P)$. Moreover, $TP^2 \notin [B^2P]$ and $[BZP]$ $(B, Z \in H, ht(B)=2, ht(Z)=3, B \neq A, Z \neq P, Z>B)$, since every standard monomial of $[BP]$ is of the form $T\beta$ $(\beta \in H)$.

Accordingly, $TP \in [\alpha\beta]$ implies $TP^2 \in [(\alpha\beta)P]$. We may assume $\alpha>A$ since $TP^2 \notin [\beta(\alpha P)]$ if $\alpha \nsim A$, and similarly $\beta>A$. By Lemma 4, the standard monomials of $[\alpha P]$ are A^2, AP, $A\alpha$ and of the form $T\gamma$ $(\gamma \in H)$. But TP^2 does not belong to $[(T\gamma)\beta]$, $[A^2\beta]$, $[A(P\beta)]$ and $[A(\alpha\beta)]$, hence $TP^2 \notin [(\alpha P)\beta]$. This shows $TP \notin [\alpha\beta]$. Q.E.D.

REMARK. The same result holds for a branch of height 2.

3. Calculation of Poincaré series and Gorenstein homogeneous ASL of dimension 2

Let $R=\oplus_{n \geq 0} R_n$ be a graded ASL on a poset H over a field $k=R_0$. We

denote by $\mathscr{P}_R(\theta)$ the *Poincaré series of* R, namely

$$\mathscr{P}_R(\theta) = \sum_{n\geq 0} (\dim_k R_n)\theta^n.$$

LEMMA 5. $\mathscr{P}_R(\theta) = \sum_{\alpha_1<\cdots<\alpha_s} (\prod_{1\leq i\leq s} \theta^{\deg(\alpha_i)}/(1-\theta^{\deg(\alpha_i)}))$,

where $\alpha_1<\cdots<\alpha_s$ *ranges over the set of chains of* H *including the empty set.*

PROOF. The k-vector space R has a k-basis consisting of the standard monomials. If we fix a chain $\alpha_1<\cdots<\alpha_s$ and count all the standard monomials of the form $\alpha_1^{n_1}\cdots\alpha_s^{n_s}$ $(n_i\geq 1)$,

$$\begin{aligned}&\sum_{n_i\geq 1,\,1\leq i\leq s} \theta^{n_1\deg(\alpha_1)+\cdots+n_s\deg(\alpha_s)}\\ &\quad = \prod_{1\leq i\leq s} \Big(\sum_{n_i\geq 1} (\theta^{\deg(\alpha_i)})^{n_i}\Big)\\ &\quad = \prod_{1\leq i\leq s} \theta^{\deg(\alpha_i)}/(1-\theta^{\deg(\alpha_i)}).\end{aligned}$$

Summing up these terms, we get the result. Q.E.D.

COROLLARY. *If* R *is homogeneous, then*

$$\mathscr{P}_R(\theta) = \sum_{0\leq i\leq d} c_i(\theta/(1-\theta))^i,$$

where $d=rank(H)$, c_i *is the number of the chains of length* i in H, *and* $c_0=1$.

DEFINITION. A rank 2 poset H is called a *cycle* of degree $2n$ if H is of the form

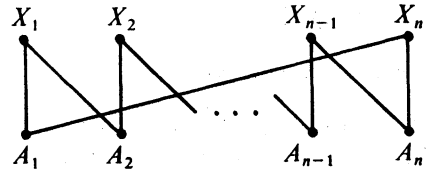


Fig. 14.

where n is a positive integer.

If R is a homogeneous ASL on a poset H with a unique minimal element T, then R is Gorenstein if and only if so is $R/(T)$, since T is a non-zero divisor on R ([2, Th. 5.4]) and $R/(T)$ is an ASL on $H-\{T\}$. So, for a while, we will study homogeneous ASL (not necessarily a domain) on a poset H of rank 2.

By Stanley [7, Th. 4.1], if R is Gorenstein, then

$$(*) \qquad \mathscr{P}_R(1/\theta) = \theta^\rho \mathscr{P}_R(\theta)$$

for some integer ρ. By the calculation of Poincaré series as above,

$$\mathscr{P}_R(\theta) = (1+(c_1-2)\theta+(1-c_1+c_2)\theta^2)/(1-\theta)^2$$

and if $\mathscr{P}_R(\theta)$ satisfies (*), then $0 \leq \rho \leq 2$ and

(a) if $\rho=2$, then $c_1=2$ and $c_2=1$,

(b) if $\rho=1$, then $c_1=3$ and $c_2=2$,

(c) if $\rho=0$, then $c_1=c_2 \geq 3$.

It is obvious that if R satisfies (a) (resp. (b)), then

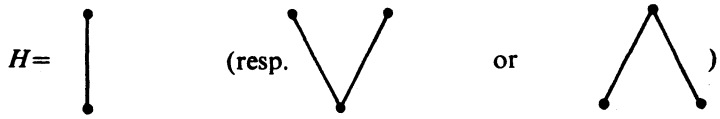


Fig. 15. Fig. 16. Fig. 17.

and it is easy to check that H is a cycle with branch sequences if R satisfies (c).

PROPOSITION C. *If R is a Gorenstein homogeneous ASL on a poset H of rank 2 over a field k, then H is*

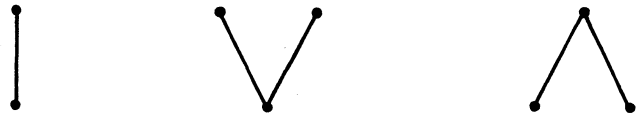


Fig. 18. Fig. 19. Fig. 20.

or a cycle with branch sequences which satisfies the following conditions:

(i) *the length of branch sequences of H from a cycle is at most* 1,

(ii) *if $P \in H$ is a branch of H, then $ht(P)=2$.*

To prove this, we need some lemmas.

SUBLEMMA 1. *If ρ is as in (*), then $\rho=-a(R)$, where $a(R)$ is defined in* [4, (3.1.4)], *that is, if R is Gorenstein, then the canonical module K_R of R is isomorphic to $R(-\rho)$ as graded R-modules.*

PROOF. We may assume k is infinite. If we take a regular sequence (x_1, x_2) from R_1, then $\mathscr{P}_{R/(x_1,x_2)}(\theta)=(1-\theta)^2\mathscr{P}_R(\theta)$ and on the other hand, $a(R/(x_1, x_2))=a(R)+2=\deg(\mathscr{P}_{R/(x_1,x_2)}(\theta))$ (cf. [4, (3.1.6), (3.1.4)]). From these equalities, we get $a(R)+2=2-\rho$. Q.E.D.

SUBLEMMA 2. *Let H be a poset of rank 2, I be an ideal of H and R be a Gorenstein ASL of $a(R)=\rho=0$. If*

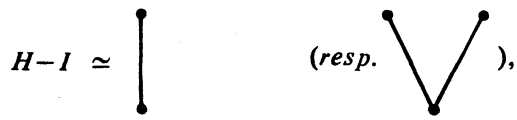


Fig. 21. Fig. 22.

then $[0: I]_1=(0)$ (resp. $\dim_k [0: I]_1=1$), where $[0: I]_1$ is the vector space of homogeneous elements of degree 1 of R which is annihilated by I.

PROOF. By [4, (2.2.9)],

$$K_{R/I} \simeq \underline{\mathrm{Hom}}_R(R/I, K_R) \simeq \underline{\mathrm{Hom}}_R(R/I, R) \simeq [0: I]$$

as graded R-modules. On the other hand, if

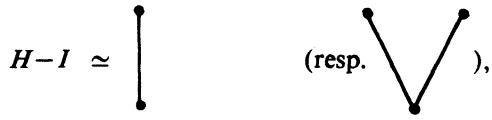


Fig. 23. Fig. 24.

then $K_{R/I} \simeq (R/I)(-2)$ (resp. $K_{R/I} \simeq (R/I)(-1)$), which implies $[0: I]_1=(0)$ (resp. $\dim_k [0: I]_1=1$). Q. E. D.

PROOF OF PROPOSITION C. Suppose that R is a Gorenstein ASL on H with $rank(H)=2$. We may assume $a(R)=0$, since if $a(R)<0$, our assertion is obvious.

(i) Let $\{P_1, \ldots, P_n\}$ be a branch sequence of length $n \geq 2$.

1) If $P_{n-1} > P_n$, then $I=H-\{P_{n-1}, P_n\}$ forms an ideal and

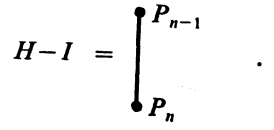


Fig. 25.

But as $P_n \in [0: I]_1$, this contradicts Sublemma 2.

2) In case $P_{n-1} < P_n$, let Q be a unique element of H such that $Q > P_{n-1}$. Then $I=H-\{Q, P_{n-1}, P_n\}$ is an ideal of H and

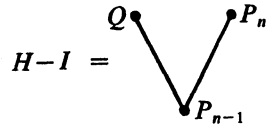


Fig. 26.

But as $P_{n-1}, P_n \in [0: I]_1$, this contradicts Sublemma 2.

(ii) Let P be a branch of height 1 and let Q be a unique element of H such that $Q > P$. Then $I=H-\{Q, P\}$ is an ideal and

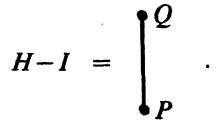


Fig. 27.

But as $P \in [0: I]_1$, this contradicts Sublemma 2. Q. E. D.

REMARK. (1) Conversely, if a poset H of rank 2 satisfies the condition of Prop. C, there exists a homogeneous Gorenstein ASL R on H over k, if k is infinite ([8]).

(2) It should be noted that the Poincaré series of an ASL on a poset H is determined by H. If R is an ASL domain on a poset H, such that $H-\{T\}$ satisfies one of the conditions (**a**), (**b**), (**c**), then R is necessarily Gorenstein by Stanley [7, Th. 4.4]. For example, if R is an ASL on one of the following posets, then R cannot be a domain.

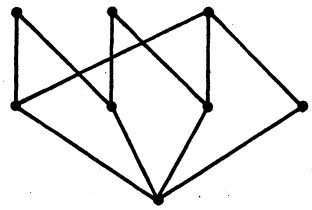


Fig. 28.

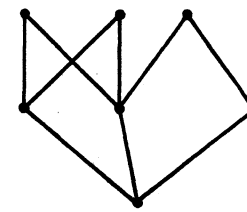


Fig. 29.

EXAMPLE a) We put

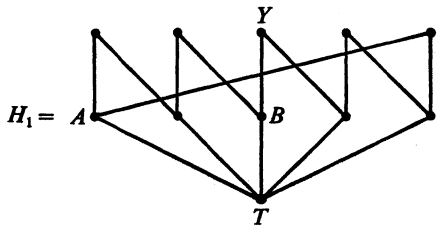


Fig. 30.

and

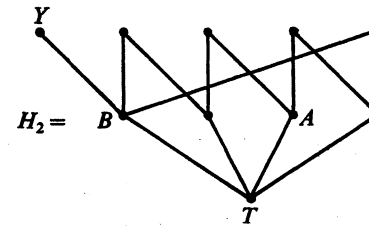


Fig. 31.

Then, by Prop. A, there exists no ASL domain on H_1 or H_2, although $H_i-\{T\}$ $(i=1, 2)$ satisfy the conditions of Prop. C.

REMARK. Let H be a poset of rank 3 with a unique minimal element T. If $H-\{T\}$ is a cycle of length ≥ 10 with branches of height 2, then there exists no ASL domain on H.

SUMMARY. *If R is a homogeneous Gorenstein ASL domain on a poset H of rank 3, then (H has a unique minimal element T and) $H-\{T\}$ is one of the followings*:

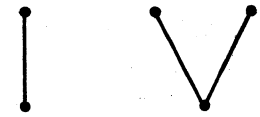


Fig. 32.

Fig. 33.

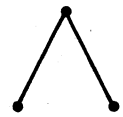


Fig. 34.

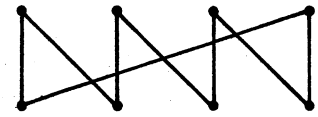
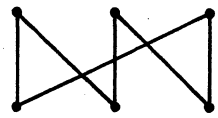
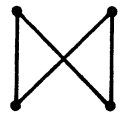


Fig. 35.



with branches of height 2 *or*

Fig. 36.



with branches of height 2.

Fig. 37.

4. The fundamental transformations of a homogeneous ASL

In this section we will define the fundamental transformations of a homogeneous ASL, which are indispensable in the following sections.

Let k be a field and R be a homogeneous ASL on an arbitrary poset H over k. Fix an arbitrary element $\alpha \in H$ and define an embedding $\varphi_\alpha: H \to R$ by

$$\varphi_\alpha(x) = \begin{cases} c_\alpha \cdot \alpha + \sum_{\beta<\alpha} d_\beta \cdot \beta & \text{if } x = \alpha \\ x & \text{if } x \neq \alpha, \end{cases}$$

where $0 \neq c_\alpha \in k$ and $d_\beta \in k$.

PROPOSITION D. *The k-algebra R is a homogeneous ASL with respect to* $\varphi_\alpha(H)$.

PROOF. We have

$$\alpha = c_\alpha^{-1}\{\varphi_\alpha(\alpha) - \textstyle\sum_{\beta<\alpha} d_\beta \cdot \beta\}.$$

If $\gamma, \delta \in H$ and $\gamma \sim \delta$, $\varphi_\alpha(\gamma)\varphi_\alpha(\delta)$ can be expressed as a linear combination of standard monomials of $\varphi_\alpha(H)$ which satisfies the axiom (ASL–2). Hence any non-standard monomial is a linear combination of standard monomials; in other words, the set of standard monomials with respect to $\varphi_\alpha(H)$ spans the k-vector space R. Now the linear independence of the standard monomials follows immediately from the consideration of Poincaré series. Q. E. D.

We call φ_α as above a *fundamental transformation* (or *linear change*) by the element $\alpha \in H$.

DEFINITION. Let R_1 and R_2 be two homogeneous ASL on a poset H with the embeddings $i_m: H \to R_m$ $(m=1, 2)$. We say that R_1 and R_2 are *equivalent* if there exists a k-algebra isomorphism $f: R_1 \to R_2$ such that $f \circ i_1 = i_2$. We say that R_1 and R_2 are *isomorphic* (as ASL) if there exists an embedding $\varphi \circ \psi: H \to R_1$, where φ is a composition of fundamental transformations and ψ is a poset automorphism of H, such that $(R_1, \varphi \circ \psi)$ and (R_2, i_2) are equivalent.

It is easy to see that this isomorphism is an equivalence relation since the inverse of a fundamental transformation is again a fundamental transformation.

EXAMPLE b) Any homogeneous ASL domain on the poset

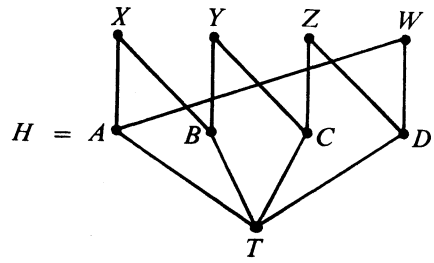


Fig. 38.

is unique up to isomorphism and isomorphic to the *Segre product* $k[s^2, st, t^2] \sharp k[a^2, ab, b^2]$. Here s, t (resp. a, b) are indeterminates over k and the ASL structure of $k[s^2, st, t^2]$ (resp. $k[a^2, ab, b^2]$) is given by

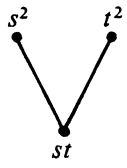
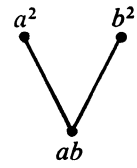


Fig. 39.

(resp.



).

Fig. 40.

The k-algebra $k[s^2, st, t^2] \sharp k[a^2, ab, b^2]$ is a homogeneous ASL domain on H by means of

$$X = s^2a^2, \quad Y = t^2a^2, \quad Z = t^2b^2, \quad W = s^2b^2,$$
$$A = s^2ab, \quad B = sta^2, \quad C = t^2ab, \quad D = stb^2,$$
$$T = stab$$

(see [2, III. 10)]).

PROOF OF THE UNIQUENESS. Let R be a homogeneous ASL domain on the poset H over k. By lemma 1, we have

$$AC = T(t_1T + a_1A + c_1C), \quad BD = T(t_2T + b_2B + d_2D).$$

Applying the linear changes

$$\varphi_A(A) = A - c_1T, \quad \varphi_B(B) = B - d_2T,$$
$$\varphi_C(C) = C - a_1T, \quad \varphi_D(D) = D - b_2T,$$

we may assume $AC = T^2$, $BD = T^2$. On the other hand, we have

$$AB = T(t_3T + a_3A + b_3B + x_3X)$$

by Lemma 1. We claim $x_3 \neq 0$. Assume $x_3 = 0$. We have

$$AY = T(t_4T + a_4A + b_4B + x_4X + y_4Y)$$

by Lemma 2. Now if we compare the coefficients of TBY, TBX, TB^2, and T^2B in $(AB)Y = B(AY)$, we get

$$b_3 = y_4, \quad x_4 = 0, \quad b_4 = 0, \quad a_3b_4 = t_4 + a_4b_3.$$

Hence

$$AY = T(t_4T + a_4A + y_4Y), \quad t_4 = -a_4y_4,$$

so we have

$$(A - y_4T)(Y - a_4T) = 0$$

which contradicts our assumption that R is a domain.

Since $x_3 \neq 0$, we may assume $AB = TX$ by the linear change

$$\varphi_X(X) = x_3X + (t_3T + a_3A + b_3B).$$

Similarly we may assume $BC = TY$, $CD = TZ$, $DA = TW$. From these six relations we can get all the straightening relations of R, which coincide with those of $k[s^2, st, t^2] \sharp k[a^2, ab, b^2]$.

5. Branches from the cycle of degree 6

Let k be a field and R be a homogeneous ASL domain on the poset

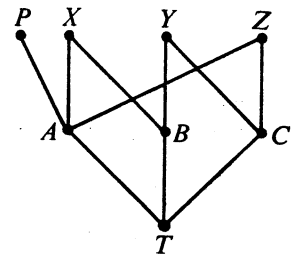


Fig. 4

over k. By Lemma 1 if we put

$$AB = T(t_1T+a_1A+b_1B+x_1X),$$
$$BC = T(t_2T+b_2B+c_2C+y_2Y),$$
$$CA = T(t_3T+c_3C+a_3A+z_3Z),$$

then we have the following

LEMMA 6. $x_1 \neq 0$, $z_3 \neq 0$.

PROOF. Suppose $x_1=0$. Then

$$AB = T(t_1T+a_1A+b_1B).$$

On the other hand,

$$PB = T(tT+aA+bB+pP+xX)$$

by Lemma 2. Comparing the coefficients of TAP and T^2P in $(AB)P=A(PB)$, we have

$$a_1 = p, \quad t_1 + pb_1 = 0,$$

so, $t_1+a_1b_1=0$. Hence $(A-b_1T)(B-a_1T)=0$, a contradiction. Thus, $x_1\neq 0$. Similarly, we can prove $z_3\neq 0$. Q. E. D.

Since $x_1\neq 0$, $z_3\neq 0$, by the linear changes

$$\varphi_X(X) = x_1X + (t_1T+a_1A+b_1B),$$
$$\varphi_Z(Z) = z_3Z + (t_3T+c_3C+a_3A),$$

we may assume $AB=TX$, $CA=TZ$. On the other hand, if $y_2\neq 0$ we may also assume $BC=TY$. If $y_2=0$, by the linear changes $\varphi_B(B)=B-c_2T$, $\varphi_C(C)=C-b_2T$ we may assume $BC=T^2$.

Consequently, we can reduce the straightening relations of AB, BC and CA to the following two types up to isomorphism of ASL:

type [I] $\quad AB = TX, \quad BC = TY, \quad CA = TZ,$

type [II] $\quad AB = TX, \quad BC = T^2, \quad CA = TZ.$

Note that in the case of type [I] we have $CX=AY=BZ$, and in the case of type [II] we have $CX=BZ=TA$, $XZ=A^2$.

Also, after a linear change on P, we may assume

$$PB = T(t_1T+a_1A+p_1P),$$
$$PC = T(t_2T+a_2A+c_2C+p_2P+z_2Z).$$

Case I. Let R be of type [I].

LEMMA 7. $z_2 \neq 0$.

PROOF. Suppose $z_2=0$. Then

$$PC = T(t_2T+a_2A+c_2C+p_2P).$$

Comparing the coefficients of T^2B, T^2X and T^2Y in $(PB)C=(PC)B$, we have $t_2=a_2=c_2=0$, and $PC=p_2TP$, a contradiction. Q. E. D.

By Lemma 7, applying the linear change $\varphi_P(P)=P/z_2$, we may assume $z_2=1$, that is,

$$PB = T(t_1T+a_1A+p_1P), \tag{2}$$

$$PC = T(t_2T+a_2A+c_2C+p_2P+Z). \tag{3}$$

We put

$$BZ = CX = AY = T(tT+aA+bB+cC+xX+yY+zZ).$$

Substituting these relations to the standard monomial expression of $(PB)C=(PC)B$, we get the following relations:

$$p_1t_2 - p_2t_1 = t, \quad p_1a_2 - p_2a_1 = a, \quad -t_2 = b,$$
$$t_1 + p_1c_2 = c, \quad -a_2 = x, \quad -c_2 = y, \quad a_1 + p_1 = z.$$

From these relations we get the following equations

$$(b+xy)(p_1)^2 + (-bz+cx+t+ay)p_1 + (ac-tz) = 0, \tag{4}$$

$$(c+yz)(p_2)^2 + (bz-cx+t+ay)p_2 + (ab-tx) = 0 \tag{5}$$

on p_1 and p_2.

LEMMA 8. At least one of the coefficients of above equations is not zero.

PROOF. Assume that all the coefficients of (4) and (5) are zero. To begin with, $b+xy=0$, $c+yz=0$, hence $bz-cx=0$. Accordingly $t+ay=0$ since $bz-cx+t+ay=0$. Then

$$(A-yT)(B-zT)(C-xT) = (a+zx)T^2(A-yT).$$

Since $A-yT\neq 0$, we have $(B-zT)(C-xT)=(a+zx)T^2$. Thus $TY-xTB-zTC-aT^2=0$, which contradicts the axiom (ASL-1). Q. E. D.

Case II. Let R be of type [II].

Applying a linear change on Y, if necessary, we may put

$$AY = T(t_0T+b_0B+c_0C+y_0Y),$$
$$PY = T(t'T+a'A+b'B+c'C+x'X+y'Y+z'Z+p'P).$$

From $(PY)B=(PB)Y$, we have the following relations:

$$a' = b' = x' = y' = z' = 0, \quad p' = 0, \quad c' = a_1t_0 + p_1t',$$
$$a_1c_0 + p_1c' = 0, \quad t' = a_1b_0, \quad t_1 + a_1y_0 = 0.$$

Note that $a_1 \neq 0$, $p_1 \neq 0$.

From these relations we get the following equation

$$b_0(p_1)^2 + t_0p_1 + c_0 = 0 \tag{6}$$

on p_1, where at least one of b_0, t_0, c_0 is not zero.

Consequently, summarizing the above calculations, in both type [I] and type [II] we have

PROPOSITION E. *The number of the branches from A is at most two.*

PROOF. Thanks to the equations (4), (5) and (6), we have only to show that there is no branch P' from A, except P, with

$$P'B = T(t_1'T + a_1'A + p_1P') \tag{7}$$

and that there is no branch P' from A, except P, with

$$P'C = T(t_2'T + a_2'A + c_2'C + p_2P' + Z). \tag{8}$$

Firstly, suppose that there exists a branch P' with the relation (7). Then we have $t_1 = a_1 = 0$, and $t_1' = a_1' = 0$, since $(PB)P' = (P'B)P$, a contradiction.

Secondly, suppose that there exists a branch P' with the relation (8). Since $CA = TZ$, we have

$$PZ = (PC)A/T = t_2TA + a_2A^2 + c_2TZ + p_2AP + AZ,$$
$$P'Z = (P'C)A/T = t_2'TA + a_2'A^2 + c_2'TZ + p_2AP' + AZ.$$

As usual, if we compare the coefficients of T^2P, T^2P', TAP and TAP' in $(PC)P' = (P'C)P$, we have

$$a_2 = a_2' = -p_2, \quad t_2 + c_2p_2 = 0, \quad t_2' + c_2'p_2 = 0.$$

Hence

$$(P-P')C = -p_2(c_2-c_2')T^2 + (c_2-c_2')TC + p_2T(P-P').$$

So, we have

$$\{(P-P')-(c_2-c_2')T\}(C-p_2T) = 0,$$

which is a contradiction. Q. E. D.

PROPOSITION F. *If there exists a branch from B (or C), then the number of the branches from A is at most one.*

PROOF. Let Q be a branch from B. By Lemma 3 we have

$$PQ = T(t'T+a'A+b'B+x'X+p'P+q'Q).$$

Note that this case is of type [I] by Lemma 6 applied to the branch Q.

Now $PB=T(t_1T+a_1A+p_1P)$ and, without loss of generality, we may also assume $QA=T(t_2T+b_2B+q_2Q)$. Then, since $(PB)Q=(PQ)B$ and $(QA)P=(PQ)A$, we have $a'=b'=x'=p'=q'=0$. Hence $PQ=t'T^2$.

If there exists another branch P' from A, then $P'Q=t''T^2$ in a similar way. Accordingly we have $t'T^2P'=t''T^2P$, which means $t'=t''=0$ and we have a contradiction. Q.E.D.

EXAMPLE c) Let $k[x, y, z]^{(3)}$ be the Veronese subring of the polynomial ring $k[x, y, z]$ which is generated by all the monomials of degree 3. The k-algebra $k[x, y, z]^{(3)}$ turns out to be a homogeneous ASL domain on the poset

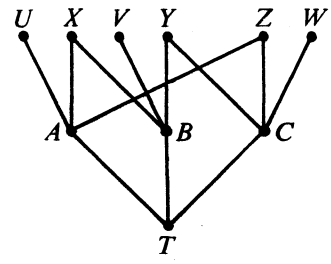


Fig. 42.

by means of

$$T = xyz, \quad A = yz(y-z), \quad B = zx(z-x), \quad C = xy(x-y),$$
$$X = z(z-x)(z-y), \quad Y = x(x-y)(x-z), \quad Z = y(y-z)(y-x),$$
$$U = yz^2, \quad V = zx^2, \quad W = xy^2.$$

EXAMPLE d) Let α and β be non-zero elements of k with $\alpha \neq \beta$, and x, y, z be indeterminates over k. We can construct a homogeneous ASL domain on

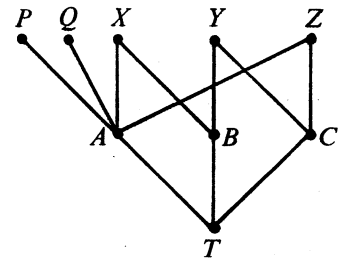


Fig. 43.

over k by means of

$$T = xyz, \quad A = xz(y+z), \quad B = x^2y, \quad C = yz^2,$$
$$X = x^2(y+z), \quad Y = y^2(x-\alpha z)(x-\beta z)/(y+z), \quad Z = z^2(y+z),$$
$$P = xz^2(y+z)/(x-\alpha z), \quad Q = xz^2(y+z)/(x-\beta z),$$

whose straightening relations are

$$AB = TX, \quad BC = T^2, \quad CA = TZ, \quad AY = T\{-(\alpha+\beta)T+B+\alpha\beta C\},$$
$$BZ = TA, \quad CX = TA, \quad XY = -(\alpha+\beta)TB + B^2 + \alpha\beta T^2,$$
$$YZ = -(\alpha+\beta)TC + T^2 + \alpha\beta C^2, \quad ZX = A^2, \quad PB = T(A+\alpha P),$$
$$PC = (1/\alpha)T(P-Z), \quad PX = A(A+\alpha P), \quad PY = T(T-\beta C),$$
$$PZ = (1/\alpha)A(P-Z), \quad QB = T(A+\beta Q), \quad QC = (1/\beta)T(Q-Z),$$
$$QX = A(A+\beta Q), \quad QY = T(T-\alpha C), \quad QZ = (1/\beta)A(Q-Z),$$
$$PQ = (1/(\alpha-\beta))A(P-Q).$$

Note that this example is of type [II], and α, β are the roots of the equation (6).

6. Branches from the cycle of degree 4

Let k be a field and R be a homogeneous ASL domain on a poset H which is of the following type.

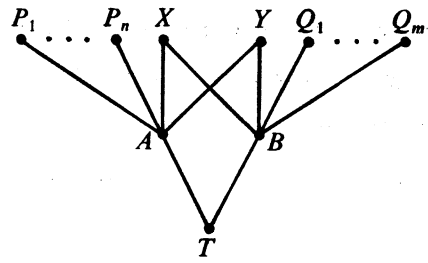


Fig. 44.

Our final results in this section are as follows.

PROPOSITION G. *If there exists a branch from B, then the number of the branches from A is at most four.*

PROPOSITION H. *If there exist two branches from B, then the number of the branches from A is at most two.*

Now let R be a homogeneous ASL domain on the poset

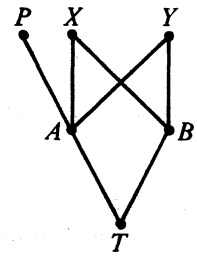


Fig. 45.

over k. After applying linear changes on A, B, X, Y, we may assume

$$AB = T(tT+xX+yY), \tag{9}$$

$$XY = T(t'T+a'A+b'B) + aA^2 + bB^2. \tag{10}$$

On the other hand, we have

$$PB = T(t_1T+a_1A+b_1B+x_1X+y_1Y+p_1P),$$
$$PX = T(t_2T+a_2A+b_2B+x_2X+y_2Y+p_2P) + A(a'_2A+x'_2X+p'_2P),$$
$$PY = T(t_3T+a_3A+b_3B+x_3X+y_3Y+p_3P) + A(a'_3A+y'_3Y+p'_3P).$$

Apply, if necessary,

$$\varphi_P(P) = P - x'_2A - b_1T.$$

Then we may assume $b_1=0$, $x'_2=0$.

LEMMA 9. *The coefficients x_1, x_2, x_3, y_2 and y_3 are all zero.*

PROOF. Comparing the coefficients of TX^2, TBX and TBY in $(PB)X=(PX)B$, we get $x_1=x_2=y_2=0$. Also, comparing the coefficients of TBX and TBY in $(PB)Y=(PY)B$, we get $x_3=y_3=0$. Q.E.D.

LEMMA 10. $a_3 = a'_3 = 0$.

PROOF. Compare the coefficients of TAX and A^2X in $(PX)Y=(PY)X$. Q.E.D.

LEMMA 11. $y'_3 \neq 0$.

PROOF. Comparing the coefficients of A^2Y and TAY in $(PX)Y=(PY)X$, we have

$$a'_2 + p'_2y'_3 = 0, \quad a_2 + p_2y'_3 = 0.$$

On the other hand, comparing the coefficients of T^2A, TA^2, TAY and TAP in $(AB)P=(PB)A$, we have

$$t_1 = xa_2 = -p_2y'_3x, \quad a_1 = xa'_2 = -p'_2y'_3x, \quad y_1 = yy'_3, \quad p_1 = xp'_2 + yp'_3.$$

Hence, if $y'_3=0$ then $PB=p_1TP$, which contradicts our assumption that R is a domain. Q.E.D.

Since $y'_3 \neq 0$, we consider P/y'_3 instead of P, and we may assume $y_3=1$.

Consequently, we may start from the following relations:

$$PB = T(t_1T+a_1A+y_1Y+p_1P); \tag{11}$$

$$PX = T(t_2T+a_2A+b_2B+p_2P) + A(a'_2A+p'_2P); \tag{12}$$

$$PY = T(t_3T+b_3B+p_3P) + A(Y+p'_3P). \tag{13}$$

LEMMA 12. $t_1 = -xp_2$, $a_1 = -xp'_2$, $y_1 = y$, $p_1 = xp'_2 + yp'_3$,
$t_2 = y(b'+xp'_2b+yp'_3b)$, $a_2 = -p_2$, $b_2 = yb$, $a'_2 = -p'_2$,
$t_3 = -x(b'+xp'_2b+yp'_3b)$, $b_3 = -xb$.

PROOF. In the proof of Lemma 11, if we put $y'_3=1$, then we get

$$t_1 = -xp_2, \quad a_1 = -xp'_2, \quad y_1 = y, \quad a_2 = -p_2, \quad a'_2 = -p'_2,$$
$$p_1 = xp'_2 + yp'_3.$$

On the other hand, comparing the coefficients of TB^2 in $(PB)X=(PX)B$ and $(PB)Y=(PY)B$, we get

$$b_2 = yb, \quad b_3 = -xb.$$

Also, comparing the coefficients of T^2Y and T^2X in $(PX)Y=(PY)X$, we get

$$t_2 = y(b'+xp'_2b+yp'_3b), \quad t_3 = -x(b'+xp'_2b+yp'_3b)$$

as desired. Q.E.D.

LEMMA 13. $t + xp_2 + yp_3 = 0$, $a = p'_2p'_3$, $a' = p_2p'_3 + p'_2p_3$,
$t' = p_2p_3 - (xp'_2+yp'_3)(b'+xp'_2b+yp'_3b)$.

PROOF. Compare the coefficients of T^2P in $(AB)P=(PB)A$, and A^2P, TAP, T^2P in $P(XY)=(PX)Y$. Q.E.D.

Next, we consider a homogeneous ASL domain on the poset

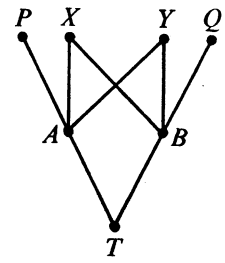


Fig. 46.

over a field k. By the same argument as above, we may assume that the straightening relations of QA, QY and QX are

$$QA = T(\bar{t}_1 T + \bar{b}_1 B + \bar{x}_1 X + q_1 Q),$$
$$QY = T(\bar{t}_2 T + \bar{b}_2 B + \bar{a}_2 A + q_2 Q) + B(\bar{b}'_2 B + q'_2 Q),$$
$$QX = T(\bar{t}_3 T + \bar{a}_3 A + q_3 Q) + B(X + q'_3 Q).$$

If we interchange A with B and X with Y in Lemma 12 and Lemma 13, we have

$$\bar{t}_1 = -yq_2, \quad \bar{b}_1 = -yq'_2, \quad \bar{x}_1 = x, \quad q_1 = yq'_2 + xq'_3,$$
$$\bar{t}_2 = x(a' + yq'_2 a + xq'_3 a), \quad \bar{b}_2 = -q_2, \quad \bar{a}_2 = xa, \quad \bar{b}'_2 = -q'_2,$$
$$\bar{t}_3 = -y(a' + yq'_2 a + xq'_3 a), \quad \bar{a}_3 = -ya,$$
$$t + yq_2 + xq_3 = 0, \quad b = q'_2 q'_3, \quad b' = q_2 q'_3 + q'_2 q_3,$$
$$t' = q_2 q_3 - (yq'_2 + xq'_3)(a' + yq'_2 a + xq'_3 a).$$

If we put the straightening relation of PQ to be

$$PQ = T(t_0 T + a_0 A + b_0 B + x_0 X + y_0 Y + p_0 P + q_0 Q),$$

then we have

LEMMA 14. $a_0 = -xp'_2$, $b_0 = -yq'_2$, $x_0 = 0$, $y_0 = 0$,

$p_0 = xp'_2$, $q_0 = yq'_2$.

PROOF. Comparing the coefficients of TAX, TAY, TA^2 and TAP in $(PQ)A = P(QA)$, we get

$$x_0 = 0, \quad y_0 = 0, \quad a_0 = \bar{x}_1 a'_2 = -xp'_2, \quad p_0 = \bar{x}_1 p'_2 = xp'_2.$$

Also, comparing the coefficients of TB^2 and TBQ in $(PQ)B = Q(PB)$, we get

$$b_0 = y_1 \bar{b}'_2 = -yq'_2, \quad q_0 = y_1 q'_2 = yq'_2$$

as desired. Q.E.D.

LEMMA 15. $t_0 = -xp_2 - x^2p'_2q'_3 = -yq_2 - y^2p'_3q'_2$.

PROOF. Comparing the coefficients of T^2A in $(PQ)A=P(QA)$ and T^2B in $(PQ)B=Q(PB)$, we get

$$t_0 = \bar{b}_1a_1 + \bar{x}_1a_2 + q_1a_0 = a_1\bar{b}_1 + y_1\bar{b}_2 + p_1b_0.$$

Express this relation by x, y, p_2, p'_2, p'_3, q_2, q'_2 and q'_3, and we get the desired result. Q. E. D.

Moreover, we consider a homogeneous ASL domain on the poset

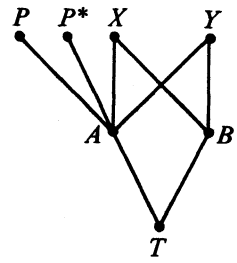


Fig. 47.

over a field k. Suppose that the straightening relations concerning P^* are

$$P^*B = T(t_1^*T + a_1^*A + y_1^*Y + p_1^*P^*)$$
$$P^*X = T(t_2^*T + a_2^*A + b_2^*B + p_2^*P^*) + A(a_2'^*A + p_2'^*P^*)$$
$$P^*Y = T(t_3^*T + b_3^*B + p_3^*P^*) + A(Y + p_3'^*P^*).$$

Then we have

LEMMA 16. *If* $p'_2 = p'^*_2$ *then* $p'_2 = p'^*_2 = 0$ *and* $a = 0$.

PROOF. Comparing the coefficients of A^2P and A^2P^* in $(PX)P^*=P(P^*X)$, we get $a'_2=a'^*_2=0$ since $p'_2=p'^*_2$. Hence $p'_2=0$ since $a'_2=-p'_2$. The equality $a=0$ follows from Lemma 13. Q. E. D.

We have now finished the preliminary steps for the proofs of Proposition G and Proposition H.

PROOF OF PROPOSITION G. By Lemma 15, we have

$$xp_2 + x^2p'_2q'_3 = yq_2 + y^2p'_3q'_2. \tag{14}$$

If $(x, y)=(0, 0)$, by Lemma 12, $PB=0$ and R is not an integral domain. As x and y are in symmetric situation, we may assume $x\neq 0$. Then we may assume $x=1$. So, it is sufficient to consider the following two cases, (i) $x=y=1$, (ii) $x=1$, $y=0$.

Case (i) $x=y=1$.

Substituting $x=y=1$ into (14), we have

$$p_2 + p'_2q'_3 = q_2 + p'_3q'_2, \tag{15}$$

here we consider q_2, q'_2 and q'_3 to be given. Multiplying p'_2 to the both sides of the relation (15), we get

$$p_2p'_2 + (p'_2)^2q'_3 = q_2p'_2 + aq'_2, \tag{16}$$

since $p'_2p'_3=a$ by Lemma 13. Moreover, since we get

$$\{a-(p'_2)^2\}p_2 = t(p'_2)^2 + a'p'_2$$

from Lemma 13, we can eliminate p_2 from (16) and we have

$$q'_3(p'_2)^4 - (t+q_2)(p'_2)^3 - \{a'+a(q'_2+q'_3)\}(p'_2)^2 + aq_2p'_2 + a^2q'_2 = 0. \tag{17}$$

[1] The case $a\neq 0$.

As an equation on p'_2, (17) is non-trivial. In fact, if all the coefficients of the equation (17) are zoro, then $t=0$, $t'=0$, $b=0$, $b'=0$ and $a'=0$. Thus we get

$$AB = T(X+Y), \quad XY = aA^2,$$
$$QA = TX, \quad QY = aTA, \quad QX = -aTA + BX.$$

Multiplying Q to the equality $AB=T(X+Y)$, we have $ABQ=Q^2A+TQY=Q^2A+aT^2A$, hence $BQ-Q^2-aT^2=0$, which contradicts the linear independence of the standard monomials.

Consequently, if $a\neq 0$ then, by the equation (17), p'_2 can take at most four different values. Thanks to Lemma 16, we get the desired result.

[2] The case $a=0$.

In this case, the equation (17) on p'_2 becomes

$$q'_3(p'_2)^4 + q_3(p'_2)^3 - a'(p'_2)^2 = 0, \tag{18}$$

which is non-trivial since $q'_3=q_3=a'=0$ implies $QX=BX$.

If $a'\neq 0$, by (18) the number of possible values of p'_2 except zero is at most two. If $p'_2=0$, from (15) we get

$$p_2p'_3 = q_2p'_3 + (p'_3)^2q'_2.$$

Substitute $a'=p_2p'_3 + p'_2p_3(=p_2p'_3)$, and we have

$$q'_2(p'_3)^2 + q_2p'_3 - a' = 0,$$

which is a non-trivial equation on p'_3 since $a'\neq 0$. Hence the number of possible

values of p_3' is at most two. If p_3' is determined, then p_2 (hence p_3) is also determined by (15).

If $a'=0$, the equation (18) on p_2' turns out to be

$$(p_2')^3(q_3'p_2'+q_3) = 0,$$

and the number of possible values of p_2' except zero is at most one. If $p_2'=0$ then $p_2p_3'=0$ since $a'=0$. When $p_3'=0$ we get $p_2=q_2$ by (15), so $p_3=-t-q_2$. On the other hand, if $p_2=0$ (hence $p_3=-t$), we have

$$b(p_3')^2 + b'p_3' + t' = 0$$

from Lemma 13, which is a non-trivial equation on p_3'. Hence the number of possible values of p_3' is at most two. Thus the number of the branches from A is at most four.

Case (ii) $x=1$, $y=0$.

Substituting $x=1$, $y=0$ to the equations in Lemma 13, and eliminating p_2, p_3, p_3', we get

$$b(p_2')^4 + b'(p_2')^3 + t'(p_2')^2 + a'tp_2' + t^2a = 0. \tag{19}$$

The equation (19) on p_2' is non-trivial. In fact, assume that all the coefficients of (19) are zero. Then, to begin with, $b=b'=t'=0$. By (10), $(a, a')=(0, 0)$ is not possible, hence $t=0$. Accordingly, we have

$$AB = TX, \quad XY = a'TA + aA^2.$$

So, $ABY=TXY=a'T^2A+aTA^2$, that is, $BY-a'T^2-aTA=0$, which contradicts the linear independence of the standard monomials. Consequently, by the equation (19) on p_2', the number of possible values of p_2' is at most four. From Lemma 13, (12), and $y=0$, we get

$$PX = tT(A-P) - p_2'A(A-P).$$

Hence P is determined if a value of p_2' is determined. Q. E. D.

REMARK. (1) If $x=1$, $y=0$, then the number of the branches from A is at most four, regardless of whether there is a branch from B or not.

(2) If $x=1$, $y=0$ and if there exist branches from both A and B, then both the numbers of the branches from A and from B are at most one. In fact, by Lemma 14 we get

$$PQ = - p_2'T(A-P).$$

This means that the branch Q from B is uniquely determined by the branch P from A. Also, the converse follows immediately from the symmetry of the poset

and the straightening relations of AB and XY.

PROOF OF PROPOSITION H. Let Q and Q^* be branches from B and r_2, r'_2 (resp. r_3, r'_3) be the coefficients of TQ^*, BQ^* in the straightening relations of Q^*Y (resp. Q^*X).

Thanks to the Remark after the proof of Prop. G, we have only to consider the case $(x, y)=(1, 1)$. By Lemma 13, we get

$$p_2 + p'_2q'_3 = q_2 + p'_3q'_2, \tag{20}$$

$$p_2 + p'_2r'_3 = r_2 + p'_3r'_2. \tag{21}$$

Subtracting (21) from (20), we have

$$(q'_3-r'_3)p'_2 = (q_2-r_2) + (q'_2-r'_2)p'_3. \tag{22}$$

Moreover, multiplying p'_2 we have

$$(q'_3-r'_3)(p'_2)^2 - (q_2-r_2)p'_2 - a(q'_2-r'_2) = 0 \tag{23}$$

since $a=p'_2p'_3$.

[1] The case $a\neq 0$.

The equation (23) on p'_2 is non-trivial, since $q'_3=r'_3$, $q_2=r_2$ and $q'_2=r'_2$ (hence $q_3=r_3$) imply $Q=Q^*$. So, we have the desired result from Lemma 16.

[2] The case $a=0$.

In this case the equation (23) turns out to be

$$(q'_3-r'_3)(p'_2)^2 - (q_2-r_2)p'_2 = 0.$$

Suppose $q'_3\neq r'_3$ or $q_2\neq r_2$. Then the number of possible value of p'_2 except zero is at most one. Let $p'_2=0$. Firstly, if $q'_2\neq r'_2$ then p'_3 is uniquely determined by (22). Accordingly, p_2 (hence p_3) is uniquely determined by (20). Secondly, if $q'_2=r'_2$ then $q'_2=r'_2=0$ by Lemma 16, so p_2 is uniquely determined by (20) and (21), that is, $p_2=q_2=r_2$. Note that $p_2\neq 0$, since $p_2=0$ implies $q_2=r_2=0$, thus $QY=Q^*Y=a'T^2$. Since $p_2\neq 0$, p'_3 is also determined by $a'=p_2p'_3+p'_2p_3$.

Suppose $q'_3=r'_3$ and $q_2=r_2$. Then $q'_2\neq r'_2$ since $Q\neq Q^*$. Hence $q'_3=r'_3=0$ and $b=0$ since $b=q'_2q'_3=r'_2r'_3$. In this case we get $p'_3=0$ by (22). So, we have $p_2=q_2$ by (20). Accordingly,

$$PY = -T\{b'T+(t+q_2)P\} + AY$$

and the branch P from A is uniquely determined if it exists. Q.E.D.

EXAMPLE e) The Veronese subring $k[x, y, z]^{(3)}$ of $k[x, y, z]$, treated in example c), also turns out to be a homogeneous ASL on the poset

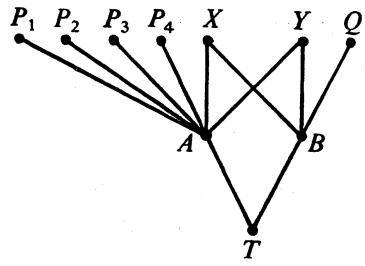


Fig. 48.

over k by means of

$$T = x(x^2-yz), \quad A = x\ell_{\alpha,\beta}\ell_{\gamma,\delta}, \quad B = (x^2-yz)\ell_{\alpha,\beta}, \quad X = z\ell_{\alpha,\beta}\ell_{\gamma,\delta},$$
$$Y = y\ell_{\alpha,\beta}\ell_{\gamma,\delta}, \quad P_1 = x\ell_{\alpha,\beta}\ell_{\beta,\gamma}, \quad P_2 = x\ell_{\alpha,\beta}\ell_{\beta,\delta}, \quad P_3 = x\ell_{\alpha,\gamma}\ell_{\gamma,\delta},$$
$$P_4 = x\ell_{\beta,\gamma}\ell_{\gamma,\delta}, \quad Q = y(x^2-yz),$$

where α, β, γ, δ are distinct non-zero elements of k and $\ell_{a,b}=(a+b)x-aby-z$ is a linear form in $k[x, y, z]$.

EXAMPLE f) As usual, let x, y, z be indeterminates over k. We can construct a homogeneous ASL domain on

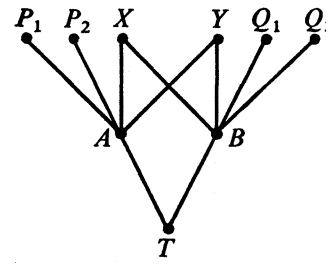


Fig. 49.

over k by means of

$$T = x(x^2-yz), \quad A = x\{x^2+(y-z)x-y^2\}, \quad B = z(x^2-yz), \quad X = y(x^2-yz),$$
$$Y = x\{x^2+(z-y)x-z^2\}, \quad P_1 = x^2(y-z), \quad P_2 = -x^2(y-z)(2x-y-z)/(z-x),$$
$$Q_1 = -(x^2-yz)^2/(y-z), \quad Q_2 = (x^2-yz)^2/(x-y),$$

whose straightening relations are

$$AB=T(-T+X+Y), \quad XY=T(-T+A+B), \quad P_1B=T(-T+Y+P_1),$$
$$P_2B=T(-A+Y+P_2), \quad P_1X=T(T-A+P_1), \quad P_2X=T^2+A(-A+P_2),$$
$$P_1Y=-T^2+A(Y+P_1), \quad P_2Y=T(-T+P_2)+AY, \quad Q_1A=T(-T+X+Q_1),$$

$Q_2A=T(-B+X+Q_2)$, $Q_1Y=T(T-B+Q_1)$, $Q_2Y=T^2+B(-B+Q_2)$,
$Q_1X=-T^2+B(X+Q_1)$, $Q_2X=T(-T+Q_2)+BX$, $P_1Q_1=-T^2$,
$P_1Q_2=T(-T-B+Q_2)$, $P_2Q_1=T(-T-A+P_2)$, $P_2Q_2=T(-A-B+P_2+Q_2)$,
$P_1P_2=T(P_1-P_2)+AP_1$, $Q_1Q_2=T(Q_1-Q_2)+BQ_1$.

EXAMPLE g) Let n be an arbitrary positive integer. We can construct a homogeneous ASL domain R on

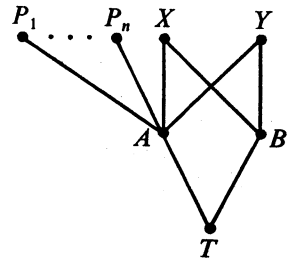


Fig. 50.

over k. In fact, put

$$T=xz^2,\quad A=xyz,\quad B=(x^2+z^2)z,\quad X=x^2y,\quad Y=yz^2,$$
$$P_i=xyz^2/(z-x/p_i)\qquad (i=1,2,\ldots,n),$$

where $0\neq p_i\in k$, $p_i\neq p_j$ if $i\neq j$, and we have the straightening relations as follows:

$$AB=T(X+Y),\quad XY=A^2,\quad P_iB=T\{-p_iA+Y+(p_i+1/p_i)P_i\},$$
$$P_iX=-p_iA(A-P_i),\quad P_iY=A(Y+P_i/p_i),$$
$$P_iP_j=A(p_iP_j-p_jP_i)/(p_i-p_j)\qquad (i\neq j).$$

Note that this example is not normal and the normalization of R is $R[TY/A]=R[z^3]$.

This example was discovered in the following way. Suppose $x=1$, $y=1$, $a\neq 0$. Then by Lemma 13 we get

$$(24)\qquad b\xi^4+b'\xi^3+(t'-4ab)\xi^2+(ta'-4ab')\xi+\{t^2a-4at'+(a')^2\}=0,$$

where $\xi=p_2'+a/p_2'$. If this equation (24) on ξ is non-trivial, then the number of the branches from A is at most eight. Accordingly, only when all the coefficients of (24) are zero, there is the possibility of having an arbitrary number of the branches from A. In this case

$$AB=T(X+Y),\quad XY=aA^2,$$

and we get this example g) if we put $a=1$.

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