HIROSHIMA MATH. J.
15 (1985), 1–25

Convergence of approximate solutions for Kac's model of the Boltzmann equation

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(Received May 19, 1984)

1. Introduction

Kac's model is a one dimensional model of the Boltzmann equation and is written as follows:

$$(1.1)\qquad \begin{cases} \partial_t F = -v\partial_x F + Q(F, F), \\ F(0, x, v) = F_0(x, v), \end{cases} \qquad (t, x, v) \in [0, \infty) \times \boldsymbol{R} \times \boldsymbol{R},$$

where $F = F(t, x, v)$ is a distribution function of particles with velocity v at time t and at position x and $\partial_t F = (\partial/\partial t)F$ etc. Q is a collision operator given by

$$Q(F, G) = (1/2)\int_{-\pi}^{\pi}\int_{\boldsymbol{R}} \{F(v_1')G(v') + F(v')G(v_1') - F(v_1)G(v) - F(v)G(v_1)\} I(\theta)d\theta dv_1,$$

where $v_1' = v\sin\theta + v_1\cos\theta$, $v' = v\cos\theta - v_1\sin\theta$ and $F(v_1') = F(t, x, v_1')$ etc.

Throughout this paper we assume that $I(\theta)$ is a non-negative integrable function on $[-\pi, \pi]$ and satisfies $I(\theta) = I(-\theta)$.

Note that the absolute Maxwellian state $g(v) = \exp(-v^2/2)/\sqrt{2\pi}$ is a stationary solution for (1.1). Putting $F = g + g^{1/2}f$ and substituting it into (1.1), we have the equation for f:

$$(1.2)\qquad \begin{cases} \partial_t f = -v\partial_x f + Lf + \Gamma(f, f) \equiv Bf + \Gamma(f, f), \\ f(0, x, v) = f_0(x, v), \end{cases}$$

where $Lf = 2g^{-1/2}Q(g, g^{1/2}f)$ and $\Gamma(f, f) = g^{-1/2}Q(g^{1/2}f, g^{1/2}f)$. According to [2], the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ and the corresponding eigenvectors $\{e_n\}_{n=0}^{\infty}$ of the linearized collision operator L are given by

$$\lambda_0 = 0, \quad \lambda_n = \int_{-\pi}^{\pi} (\sin^n\theta + \cos^n\theta - 1)I(\theta)d\theta \quad n \geqq 1,$$

$$e_n = e_n(v) = \exp(-v^2/4)H_n(v)/\|\exp(-v^2/4)H_n(v)\|_{L^2(\boldsymbol{R}_v)} \quad n \geqq 0,$$

where $H_n(v)$ are the Hermite polynomials. In particular it should be noted that

$$\lambda_0 = \lambda_2 = 0, \quad \lambda_n < 0\ (n \neq 0, 2),\ \lim_{n\to\infty}\lambda_n = -\nu,$$

where $v=\int_{-\pi}^{\pi} I(\theta)d\theta$. Here we shall suppose that the solution of (1.2) is given by $f(t, x, v)=\sum_{m=0}^{\infty} u_m(t, x)e_m(v)$. Substituting it into (1.2) and using the relation $ve_m(v)=\sqrt{m}e_{m-1}(v)+\sqrt{m+1}e_{m+1}(v)$, we get formally the following system of equations for the unknown functions u_j $j=0, 1,\dots$:

(1.3)

$$\begin{cases} \partial_t \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ \vdots \end{pmatrix} = - \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & \sqrt{2} & & 0 & \\ & \sqrt{2} & 0 & \sqrt{3} & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \sqrt{m} & 0 & \sqrt{m+1} \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \partial_x \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_m \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ \vdots \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ \lambda_1 u_1 u_0 \\ \vdots \\ \lambda_m u_m u_0 + \sum_{n=1}^{m-1} \lambda_{n,m-n} \sqrt{m!/n!(m-n)!}\, u_n u_{m-n} \\ \vdots \end{pmatrix} \\ u_m(0, x) = (f_0(x, v), e_m(v))_{L^2(\mathbf{R}_v)} \quad m \geqq 0, \end{cases}$$

where $\lambda_{n,m}=\int_{-\pi}^{\pi} \cos^n\theta \sin^m\theta I(\theta)d\theta$ $n, m\geqq 1$. If $u_n\equiv 0$ for $n\geqq m+1$, (1.3) is reduced to

(1.4.m)
$$\begin{cases} \partial_t u^{(m)} = - S_m \partial_x u^{(m)} + D_m u^{(m)} + W_m(u^{(m)}, u^{(m)}), \\ u^{(m)}(0, x) = {}^t(u_0(0, x),\dots, u_m(0, x)), \end{cases}$$

where $u^{(m)}=u^{(m)}(t, x)={}^t(u_0(t, x),\dots, u_m(t, x))$,

$$S_m = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \sqrt{2} & & 0 \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \ddots & \ddots & \ddots \\ & 0 & & 0 & \sqrt{m} \\ & & & \sqrt{m} & 0 \end{pmatrix},$$

$$D_m = \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \\ & & & \lambda_m \end{pmatrix}$$

and W_m is a nonlinear operator. See section 4. Throughout this paper we consider (1.4.m) only for $m \geqq 3$.

The purpose of this paper is to show that the solutions of $(1.4.m)_{m=3,4,\dots}$ converge to the solution of the original problem (1.2) for all time $t \geqq 0$ as $m \to \infty$ if the initial value is small enough.

We summarize some results for (1.2) in the appendix without proofs, which will be referred to in the posterior sections. See [6] for details. From Theorem A.8 we see that (1.2) has a unique solution

$$f(t) \in C^0([0, \infty); H_l) \cap C^1([0, \infty); v_{l-1}),$$

where $H_l = H_l(\boldsymbol{R}_x; L^2(\boldsymbol{R}_v)) =$

$$= \{f(x, v) \in L^2(\boldsymbol{R}_x, \boldsymbol{R}_v) |\ |||f|||_l^2 = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} (1+|\xi|)^{2l} |\hat{f}(\xi, v)|^2 dv d\xi < \infty\} \quad l \geqq 0,$$

$V_{l-1} = \{f(x, v) | \{1/(1+|v|)\} f \in H_{l-1}\}$ $l \geqq 1$ and $\hat{f}(\xi, v)$ is the Fourier transform of $f \in L^2(\boldsymbol{R}_x, \boldsymbol{R}_v)$ with respect to x,

$$\hat{f}(\xi, v) = \sqrt{1/2\pi} \int_{\boldsymbol{R}} e^{-i\xi x} f(x, v) dx, \quad i = \sqrt{-1}.$$

In section 2, we discuss the existence and the decay of the solutions for the linearized equations of $(1.4.m)_{m=3,4,\dots}$.

In section 3, we deduce that the solutions for the linearized equations of $(1.4.m)_{m=3,4,\dots}$ converge to the solution for the linearized equation of (1.2) as $m \to \infty$ in the norm

$$\sup_{0 \leqq t < \infty} (1+t)^\alpha ||| \cdot |||_l,$$

for any $\alpha \in [0, \infty)$, $l \geqq 0$.

In section 4, we show the existence and the decay of the solutions for $(1.4.m)_{m=3,4,\dots}$ by estimating the operators W_m and then using an iteration scheme.

Finally in section 5, combining the above results, we deduce that the solutions for $(1.4.m)_{m=3,4,\dots}$ converge to the solution for (1.2) as $m \to \infty$ in the norm

$$\sup_{0 \leqq t < \infty} (1+t)^\alpha ||| \cdot |||_l,$$

for any $\alpha \in [0, 1/2)$, $l \geqq 1$.

Acknowledgement. The author is grateful to Professor K. Asano for his valuable suggestions on this work and to Professor M. Mimura for his encouragement. Also she would like to express her gratitude to Professor Y. Shizuta, Professor S. Ukai, Professor R. Sakamoto and Professor S. Kawashima.

2. Existence and decay of solutions for the linearized equation of (1.4. *m*)

In this section we discuss the linearized equation:

$$(2.1.m)\qquad \begin{cases} \partial_t u^{(m)} = -S_m \partial_x u^{(m)} + D_m u^{(m)}, \\ u^{(m)}(0, x) = u_0^{(m)}(x). \end{cases}$$

By the Fourier transform with respect to x we have

$$\widehat{(2.1.m)}\qquad \begin{cases} \partial_t \hat{u}^{(m)} = (-i\xi S_m + D_m)\hat{u}^{(m)} = T_m(\xi)\hat{u}^{(m)}, \\ \hat{u}^{(m)}(0, \xi) = \hat{u}_0^{(m)}(\xi). \end{cases}$$

Let $\xi \in \boldsymbol{R}$ be a parameter. We consider $\widehat{(2.1.m)}$ in $\boldsymbol{C}^{m+1}$ with the norm $\|x\| = \|x\|_m' = (\sum_{i=0}^m |x_i|^2)^{1/2}$, where $x = {}^t(x_0, x_1, \dots, x_m)$. The following lemmas are easily shown.

LEMMA 2.1 (i) $\sigma(T_m(\xi)) \subset \{z \mid \mathrm{Re}\, z \leqq 0\}$,
(ii) $\sigma(T_m(\xi)) \cap \{z \mid \mathrm{Re}\, z = 0\} = \emptyset$, *if* $\xi \neq 0$,
where $\sigma(T_m(\xi))$ *is the spectrum of* $T_m(\xi)$.

LEMMA 2.2 $T_m(\xi)$ *is a generator of a contraction semi-group* $\{e^{tT_m(\xi)} : t \geqq 0\}$ *in* $\boldsymbol{C}^{m+1}$.

The following proposition gives us an information about the resolvent set of $T_m(\xi)$.

PROPOSITION 2.3
(i) *For any* $\beta_1 \in (0, \kappa/2]$ $(\kappa = -\max_{j \neq 0,2} \lambda_j > 0)$, *there exist constants* $\delta > 0$ *and* $c > 0$ *which are independent of m such that*

(a) $\quad \inf_{|\lambda| \geqq \beta_1, \mathrm{Re}\lambda \geqq -3\kappa/4, |\xi| \leqq \delta} \|(\lambda - T_m(\xi))y\| \geqq c\|y\|$, *for* $y \in \boldsymbol{C}^{m+1}$,

(b) $\quad \sigma(T_m(\xi)) \cap \{\lambda \mid |\lambda| < \beta_1\} = \{\lambda_{m,j}(\xi)\}_{j=0,2}$ for $|\xi| \leqq \delta$,

where $\lambda_{m,j}(\xi)$ *are the perturbed eigenvalues of* λ_j *with respect to* ξ.
(ii) *For any* $\delta' > 0$, *there exist constants* $\beta_2 > 0$ *and* $c' > 0$ *which are independent of m such that*

$$\inf_{\mathrm{Re}\lambda \geqq -\beta_2, |\xi| \geqq \delta'} \|(\lambda - T_m(\xi))y\| \geqq c'\|y\|, \qquad \textit{for} \quad y \in \boldsymbol{C}^{m+1}.$$

REMARK. It is very important that δ, c, β_2 and c' are independent of m. By this fact we can deduce the uniform decay of the solutions for $(2.1.m)_{m=3,4,\dots}$. See Theorem 2.6.

PROOF OF (i) Put

(2.2) $$(\lambda - T_m(\xi))y = x,$$

where $y={}^t(y_0, y_1,\dots, y_m)$, $x={}^t(x_0, x_1,\dots, x_m)$ and $\lambda=-\beta+i\gamma$. Taking the inner product of (2.2) with y and taking the real part of it, we have for $\mathrm{Re}\,\lambda \geqq -3\kappa/4$

(2.3) $$\begin{aligned}(1/\varepsilon)\|x\|^2 + \varepsilon\|y\|^2 &\geqq \|x\|\,\|y\| \\ &\geqq \mathrm{Re}\,((\lambda - T_m(\xi))y,\ y) \\ &\geqq (-3\kappa/4)\textstyle\sum_{j=0,2}|y_j|^2 + (\kappa/4)\sum_{j\neq 0,2}|y_j|^2.\end{aligned}$$

The constant $\varepsilon>0$ is determined later. Considering the first and the third components of (2.2) for $|\lambda|\geqq\beta_1$ and $|\xi|\leqq\delta$, we get

(2.4) $$\begin{cases}(2/\beta_1^2)(|x_0|^2+\delta^2|y_1|^2) \geqq |y_0|^2, \\ (3/\beta_1^2)(|x_2|^2+2\delta^2|y_1|^2+3\delta^2|y_3|^2) \geqq |y_2|^2.\end{cases}$$

The constant $\delta>0$ is determined later. Substitution of (2.4) into the right hand side of (2.3) yields

(2.5) $$\begin{aligned}(1/\varepsilon)\|x\|^2+\varepsilon\|y\|^2 \geqq &- c_1(|x_0|^2+|x_2|^2) - \\ &- \delta^2 c_2(|y_1|^2+|y_3|^2) + c_3\textstyle\sum_{j\neq 0,2}|y_j|^2,\end{aligned}$$

where $c_1=9\kappa/4\beta_1^2$, $c_2=27\kappa/4\beta_1^2$ and $c_3=\kappa/4$. Calculating $2\varepsilon(2.4)+(2.5)$, we have

$$(6\varepsilon/\beta_1^2+1/\varepsilon+c_1)\|x\|^2 \geqq \varepsilon\|y\|^2 + (-\delta^2c_2-\varepsilon\delta^2c_4-2\varepsilon+c_3)\textstyle\sum_{j\neq 0,2}|y_j|^2,$$

where $c_4=18/\beta_1^2$. Consequently, the estimate (a) holds if we choose ε and δ small enough so that

$$-\delta^2c_2 - \varepsilon\delta^2c_4 - 2\varepsilon + c_3 \geqq 0.$$

By the estimate (a), we can set

$$P'_m(\xi) = (1/2\pi i)\int_{S^*}(\lambda - T_m(\xi))^{-1}d\lambda \qquad \text{for} \quad |\xi| \leqq \delta,$$

where $S^*=\{\lambda \mid |\lambda|=\beta_1\}$ and it is positively oriented. Since $(\lambda-T_m(\xi))^{-1}\to(\lambda-T_m(0))^{-1}$ as $|\xi|\to 0$ uniformly on S^* and since $\dim P'_m(0)=2$, $\dim P'_m(\xi)=2$ for $|\xi|\leqq\delta$. This completes the proof of (i).

PROOF OF (ii) Taking the inner product of (2.2) with y and taking the real part of it, we have for $\mathrm{Re}\,\lambda\geqq-\beta$,

$$(1/\varepsilon)\|x\|^2+\varepsilon\|y\|^2 \geqq -\beta\textstyle\sum_{j=0,2}|y_j|^2 + \sum_{j\neq 0,2}(-\beta-\lambda_j)|y_j|^2,$$

where the constants β and ε are determined later. In the case where $|\lambda|\geqq|\xi|\geqq\delta'$, considering the first and the third components of (2.2), we get

$$c_5(\delta')(|x_0|^2+|y_1|^2) \geqq |y_0|^2,$$
$$c_6(\delta')(|x_2|^2+|y_1|^2+|y_3|^2) \geqq |y_2|^2.$$

If $|\lambda| \leqq |\xi|$, it follows from the second and the fourth components of (2.2) that

$$c_7(\delta')(|x_1|^2+|x_3|^2+\sum_{j=1,3,4}|y_j|^2) \geqq |y_0|^2,$$
$$c_8(\delta')(|x_3|^2+|y_3|^2+|y_4|^2) \geqq |y_2|^2.$$

Putting $c \equiv \max_{j=5,6,7,8} c_j(\delta')$, we have

$$c(\sum_{j=0,1,3}|x_j|^2+\sum_{j=1,3,4}|y_j|^2) \geqq |y_0|^2,$$
$$c(|x_2|^2+|x_3|^2+\sum_{j=1,3,4}|y_j|^2) \geqq |y_2|^2.$$

By the calculations similar to those in the proof of (a), we have

$$(1/\varepsilon+4\varepsilon c+2\beta c)\|x\|^2 \geqq \varepsilon\|y\|^2 + \sum_{j\neq 0,2}(-\beta-\lambda_j-2\varepsilon-2\beta c-4\varepsilon c)|y_j|^2.$$

And the proof of (ii) is complete if β and ε are chosen small enough so that

$$-\beta - \lambda_j - 2\varepsilon - 2\beta c - 4\varepsilon c \geqq 0, \quad j \neq 0, 2.$$

PROPOSITION 2.4. *Let $\lambda_{m,j}(\xi)_{j=0,2}$ be the eigenvalues given in Proposition 2.3 and $e_{m,j}(\xi)_{j=0,2}$ be the corresponding eigenvectors. Then there exists a constant $\delta_1>0$, which is independent of m, such that the following properties are satisfied in $|\xi| \leqq \delta_1$:*

(i.a) $\lambda_{m,j}(\xi)=\xi^2 z_{m,j}(\xi)$,
where $z_{m,j}(\xi)$ belong to $C^\infty([-\delta_1, \delta_1])$ and $z_{m,j}(0)\neq 0$.

(i.b) *For any integer $n \geqq 0$, there exists a constant $c>0$ such that*

$$\sup_{m\geqq 3}\sup_{|\xi|\leqq\delta_1}|\partial_\xi^n z_{m,j}(\xi)| \leqq c.$$

(i.c) *There is a constant $\mu_1>0$ such that*

$$\sup_{m\geqq 3}\sup_{|\xi|\leqq\delta_1}\operatorname{Re} z_{m,j}(\xi) < -\mu_1 < 0.$$

(ii.a) $e_{m,j}(\xi)\in C^\infty([-\delta_1, \delta_1]; \boldsymbol{C}^{m+1})$, $(e_{m,i}(\xi), e_{m,j}(-\xi))=\delta_{ij}$,
where δ_{ij} is Kronecker's delta.

(ii.b) *For any integer $n \geqq 0$, there exists a constant $c'>0$ such that*

$$\sup_{m\geqq 3}\sup_{|\xi|\leqq\delta_1}\|\partial_\xi^n e_{m,j}(\xi)\| \leqq c'.$$

PROOF. In this proof, the indices i and j are 0 or 2. Let $\lambda=\lambda_m(\xi)$ be an eigenvalue of $T_m(\xi)$ and let $q=q_m(\xi)$ be the corresponding eigenvector:

(2.6) $$T_m(\xi)q = \lambda q.$$

Put $c^* = \min\{1, -\lambda_n;\ n=1, 3, 4, \dots\} > 0$. If $\operatorname{Re}\lambda > -c^*/2$, then we have

(2.7) $$q = (P-D+i\xi S+\lambda)^{-1}Pq,$$

where $D=D_m$, $S=S_m$ and P is the orthogonal projection onto the null space of D:

$$P = P_{m,0} = \begin{pmatrix} 1 & & & & & \\ & 0 & & 0 & & \\ & & 1 & & & \\ & & & 0 & & \\ & 0 & & & \ddots \ 0 & \\ & & & & & 0 \end{pmatrix}.$$

From the definition of P we can write $Pq = c_0 v_0 + c_2 v_2$, where $c_0 = c_{m,0}(\xi)$ and $c_2 = c_{m,2}(\xi)$ are scalars and $v_0 = {}^t(1, 0, \dots, 0)$ and $v_2 = {}^t(0, 0, 1, 0, \dots, 0)$ form a basis of the null space of D. Taking the inner product of (2.7) with v_0 and v_2, we get

$$\begin{cases} c_0 = c_0(R_1(\xi, \lambda)v_0, v_0) + c_2(R_1(\xi, \lambda)v_2, v_0), \\ c_2 = c_0(R_1(\xi, \lambda)v_0, v_2) + c_2(R_1(\xi, \lambda)v_2, v_2), \end{cases}$$

where $R_1(\xi, \lambda) = R_{m,1}(\xi, \lambda) = (P-D+i\xi S+\lambda)^{-1}$. Since $(c_0, c_2) \neq (0, 0)$, we have

$$\begin{vmatrix} (R_1(\xi, \lambda)v_0 - v_0, v_0) & (R_1(\xi, \lambda)v_2 - v_2, v_0) \\ (R_1(\xi, \lambda)v_0 - v_0, v_2) & (R_1(\xi, \lambda)v_2 - v_2, v_2) \end{vmatrix} = 0.$$

Set $\lambda = z\xi^2$. Noting that $(P-D)^{-1}v_i = v_i$ and $(Sv_i, v_j) = 0$, we have from the resolvent equation

$$(R_2(\xi, z)v_i - v_i, v_j) = \xi^2 M_{i,j}(\xi, z),$$

where $R_2(\xi, z) = R_1(\xi, z\xi^2)$ and

(2.8) $$M_{i,j}(\xi, z) = M_{m,i,j}(\xi, z) = -z(R_2(\xi, z)v_i, v_j) + \\ + (R_2(\xi, z)(iS + z\xi)(P-D)^{-1}iSv_i, v_j).$$

This implies

(2.9) $$M(\xi, z) = M_m(\xi, z) = \begin{vmatrix} M_{0,0}(\xi, z) & M_{0,2}(\xi, z) \\ M_{2,0}(\xi, z) & M_{2,2}(\xi. z) \end{vmatrix} = 0.$$

In (2.9) we put $z=\sigma+i\tau$ $\sigma, \tau \in \boldsymbol{R}, f_m(\xi, \sigma, \tau)=\operatorname{Re} M(\xi, \sigma+i\tau)$ and $g_m(\xi, \sigma, \tau)=\operatorname{Im} M(\xi, \sigma+i\tau)$. Then (2.9) is equivalent to

$$
\text{(2.10)} \qquad \begin{cases} f(\xi, \sigma, \tau) = 0, \\ g(\xi, \sigma, \tau) = 0, \end{cases}
$$

where $f=f_m$ and $g=g_m$. Since $M(\xi, z) \in C^\infty(\{(\xi, z)|\operatorname{Re} z\xi^2 > -c^*/2\})$, it follows that

$$
\text{(2.11)} \qquad f, g \in C^\infty(\{(\xi, \sigma, \tau)||\xi|<\delta, \ -c^*/2\delta^2<\sigma, \tau \in \boldsymbol{R}\}),
$$

where δ is any positive real constant. The roots of $M(0, z)=0$ are

$$
\text{(2.12)} \qquad \begin{cases} z_0 = \{3a/b+\sqrt{9a^2/b^2-12/b}\}/2, \\ z_2 = \{3a/b-\sqrt{9a^2/b^2-12/b}\}/2, \end{cases}
$$

where $a=\lambda_1+\lambda_3$, $b=\lambda_1\lambda_3$. It should be noted that $z_2<z_0<0$. By the Cauchy-Riemann differential equation, there holds

$$
\text{(2.13)} \qquad \begin{vmatrix} \partial_\sigma f & \partial_\tau f \\ \partial_\sigma g & \partial_\tau g \end{vmatrix} \neq 0 \quad \text{at} \quad (\xi, \sigma, \tau) = (0, z_j, 0).
$$

By virtue of (2.11), (2.12) and (2.13), we can apply the real implicit function theorem to (2.10) in a δ_1-neighbourhood of $\xi=0$. Moreover δ_1 is independent of m, because the constants $M_m(0, z_j)$ and $\partial_z M_m(0, z_j)$ are independent of m and $\{\partial_z^l M_m(\xi, z)\}_{m=3}^\infty$ $(l=0, 1)$ are equicontinuous families at $(\xi, z)=(0, z_j)$. This completes the proof of (i.a).

Let k and l be non-negative fixed integers. We show that the constants $\partial_\xi^k \partial_z^l M_m(0, z_j)_{m=3,4,\cdots}$ are uniformly bounded and $\{\partial_\xi^k \partial_z^l M_m(\xi, z)\}_{m=3}^\infty$ is an equicontinuous family at $(\xi, z)=(0, z_j)$, which assure (i.b) from the following well-known fact:

$$
\partial_\xi \sigma_{m,j}(\xi) = \left(\frac{\partial(f_m, g_m)}{\partial(\tau, \xi)} \Big/ \frac{\partial(f_m, g_m)}{\partial(\sigma, \tau)}\right)(\xi, \sigma_{m,j}(\xi), \tau_{m,j}(\xi)),
$$

$$
\partial_\xi \tau_{m,j}(\xi) = \left(\frac{\partial(f_m, g_m)}{\partial(\xi, \sigma)} \Big/ \frac{\partial(f_m, g_m)}{\partial(\sigma, \tau)}\right)(\xi, \sigma_{m,j}(\xi), \tau_{m,j}(\xi)).
$$

We shall show only the case of $k=0$, $l=0$. In view of (2.8) and (2.9) it is enough to show that the constants $M_{m,i,j}(0, z_k)_{m=3,4,\cdots}$ are uniformly bounded and $\{M_{m,i,j}(\xi, z)\}_{m=3}^\infty$ is an equicontinuous family at $(\xi, z)=(0, z_k)$, where $k=0, 2$. Note that

$$
\text{(2.i)} \qquad R_{m,2}(0, z_k)v_j = v_j,
$$

$$
\text{(2.ii)} \qquad Sv_j = \sqrt{j}v_{j-1} + \sqrt{j+1}v_{j+1},
$$

and $R_{m,2}(0, z_k)=(P-D)^{-1}$ and S are symmetric operators, from which, it follows that the constants $(R_{m,2}(0, z_k)v_i, v_j)$ and $(R_{m,2}(0, z_k)S(P-D)^{-1}Sv_i, v_j)$ are independent of m, where $k=0, 2$. Therefore the constants $M_{m,i,j}(0, z_k)$ are independent of m. Next, let $|\xi|\leqq 1$ and $|z|\leqq c^*/2$. From (2.8) we have

$$\begin{aligned}&|z(R_{m,2}(\xi, z)v_i, v_j)-z_k(R_{m,2}(0, z_k)v_i, v_j)|\\ &\quad\leqq |z-z_k|\,|(R_{m,2}(\xi, z)v_i, v_j)| + |z_k|\,|(\{R_{m,2}(\xi, z)-R_{m,2}(0, z_k)\}v_i, v_j)|\\ &\quad\leqq |z-z_k|\,\|R_{m,2}(\xi, z)\|\,\|v_i\|\,\|v_j\| +\\ &\qquad + |z_k|\,\|R_{m,2}(\xi, z)\|\,\|(i\xi S+z\xi^2)R_{m,2}(0, z_k)v_i\|\,\|v_j\|,\end{aligned}$$

where $k=0, 2$. Since $\|R_{m,2}(\xi, z)\|\leqq 2/c^*$, (2.i) and (2.ii) yield

$$|z(R_{m,2}(\xi, z)v_i, v_j)-z_k(R_{m,2}(0, z_k)v_i, v_j)| \leqq c(|z-z_k|+|\xi|),$$

where the constant c is independent of m. Similarly we have

$$|(R_{m,2}(\xi, z)(iS+z\xi)(P-D)^{-1}iSv_i, v_j)-(R_{m,2}(0, z_k)iS(P-D)^{-1}iSv_i, v_j)| \leqq c|\xi|,$$

where the constant c is independent of m. Therefore

$$\sup_{m\geqq 3}\sup_{|\xi|\leqq 1, |z|\leqq c^*/2}|M_{m,i,j}(\xi, z)-M_{m,i,j}(0, z_k)| \leqq c(|z-z_k|+|\xi|),$$

where the constant c is independent of m.

To see (ii.a) we substitute $q_{m,j}(\xi)=\sum_{n=0}^{m} c_n v_n$ into (2.6), where $c_n=c_{m,j,n}(\xi)$. Taking the coefficients of v_0 and v_1, we have

$$\begin{aligned}-i\xi c_1 &= \lambda_j(\xi)c_0,\\ -i\xi c_0 + \lambda_1 c_1 - \sqrt{2}i\xi c_2 &= \lambda_j(\xi)c_1,\end{aligned}$$

from which it follows that $k_{m,j}(\xi)c_0=c_2$,
where $k_{m,j}(\xi)=\{-1+\lambda_1 z_{m,j}(\xi)+\xi^2 z_{m,j}(\xi)\}/\sqrt{2}$. Recalling (2.7), we get

$$q_{m,j}(\xi) = R_1(\xi, \lambda_{m,j}(\xi))(v_0+k_{m,j}(\xi)v_2). \tag{2.14}$$

Since $q_{m,j}(\xi)$ belong to $C^\infty([-\delta_1, \delta_1]; \boldsymbol{C}^{m+1})$ and $\|q_{m,j}(0)\|\neq 0$, (ii.a) follows. (ii.b) holds owing to (i.b), (ii.a) and (2.14).

PROPOSITION 2.5. *There are constants* $\delta>0$, $\beta_1>0$ and $\beta_2>0$, *which are independent of m and ξ, such that the semi-group* $\{e^{tT_m(\xi)}: t\geqq 0\}$ *is expressed as follows*:

(i) *For any ξ with* $|\xi|<\delta$,

$$\begin{aligned}e^{tT_m(\xi)}x &= (1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_1-i\gamma}^{-\beta_1+i\gamma} e^{\lambda t}(\lambda-T_m(\xi))^{-1}x\,d\lambda +\\ &\quad + \sum_{j=0,2} e^{t\lambda_{m,j}(\xi)}(x, e_{m,j}(-\xi))_{C^{m+1}}e_{m,j}(\xi).\end{aligned} \tag{2.15}$$

(ii) *For any* ξ *with* $|\xi| \geqq \delta$,

$$e^{tT_m(\xi)}x = (1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_2-i\gamma}^{-\beta_2+i\gamma} e^{\lambda t}(\lambda - T_m(\xi))^{-1}xd\lambda. \tag{2.16}$$

In the above, the first terms on the right hand side of (2.15) *and* (2.16) *have the following estimates*:

$$\|(1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t}(\lambda - T_m(\xi))^{-1}xd\lambda\| \leqq ce^{-\beta_j t}\|x\| \qquad j = 1, 2, \tag{2.17}$$

where the constant c is independent of m and ξ.

PROOF. We give an outline of the proof. Let $\beta > 0$. Then the semi-group is represented by the inverse Laplace transform

$$e^{tT_m(\xi)}x = (1/2\pi i)\lim_{\gamma\to\infty}\int_{\beta-i\gamma}^{\beta+i\gamma} e^{\lambda t}(\lambda - T_m(\xi))^{-1}xd\lambda \quad \text{for any} \quad \xi.$$

By virtue of Proposition 2.3 (i) and Cauchy's integral theorem, we can change the path $\{z|z=\beta+i\gamma \ \gamma\in \boldsymbol{R}\}$ to $\{z|z=-\beta_1+i\gamma \ \gamma\in\boldsymbol{R}\}\cup\{z||z|=\beta_1\}$. Hence we obtain (2.15). The expression (2.16) follows from Proposition 2.3 (ii).

To obtain (2.17) we rewrite $(\lambda - T_m(\xi))^{-1}$ by using the resolvent equation as follows:

$$\begin{aligned}(\lambda - T_m(\xi))^{-1} &= (\lambda + a_1 + i\xi S_m)^{-1} + (\lambda + a_1 + i\xi S_m)^{-1}(D_m + a_1)(\lambda + a_1 + i\xi S_m)^{-1} + \\ &\quad + (\lambda + a_1 + i\xi S_m)^{-1}(D_m + a_1)(\lambda - T_m(\xi))^{-1}(D_m + a_1)(\lambda + a_1 + i\xi S_m)^{-1} \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where $a_1 > \max\{\beta_j \ j=1, 2, \ |\lambda_j| \ j=0, 1, 2,\ldots\}$. Hence we get easily

$$\|(1/2\pi i)s-\lim_{\gamma\to\infty}\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t} I_1 d\lambda\| \leqq e^{-t\beta_j}.$$

Since

$$\left|\left(\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t} I_2 x d\lambda, x'\right)_{C^{m+1}}\right| \leqq e^{-\beta_j t}\pi\|D_m + a_1\| \, \|x\| \, \|x'\|/(-\beta_j + a_1)$$

and

$$\begin{aligned}&\left|\left(\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t} I_3 x d\lambda, x'\right)_{C^{m+1}}\right| \\ &\leqq e^{-\beta_j t}\pi \sup_{\gamma\in\boldsymbol{R}, \xi\in S_j}\|(-\beta_j + i\gamma - T_m(\xi))^{-1}\| \, \|D_m + a_1\|^2\|x\| \, \|x'\|/(-\beta_j + a_1)\end{aligned}$$

where $S_1 = \{\xi||\xi| < \delta\}$ and $S_2 = \{\xi||\xi| \geqq \delta\}$, we have

$$\|(1/2\pi i)s-\lim_{\gamma\to\infty}\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t} I_2 d\lambda\| \leqq e^{-\beta_j t}\|D_m + a_1\|/2(-\beta_j + a_1),$$

and

$$\|(1/2\pi i)s-\lim_{\gamma\to\infty}\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t} I_3 d\lambda\|$$
$$\leqq e^{-\beta_j t}\sup_{\gamma\in \boldsymbol{R},\xi\in S_j}\|(-\beta_j+i\gamma-T_m(\xi))^{-1}\|\,\|D_m+a_1\|^2/2(-\beta_j+a_1).$$

The proof is complete by Proposition 2.3.

To state the main theorem in this section we need some definitions.

DEFINITION. Let $l\geqq 0$.

$$H_l(\boldsymbol{R}_x)=\{u(x)\in L^2(\boldsymbol{R}_x)|\,\|u\|_l^2=\int_{\boldsymbol{R}}(1+|\xi|)^{2l}|\hat{u}(\xi)|^2d\xi<\infty\},$$

$$H_{l,m}=\{u(x)={}^t(u_0,u_1,\dots,u_m)|u_j\in H_l(\boldsymbol{R}_x)\ j=0,\dots,m\},\quad |||u|||_{l,m}^2=\sum_{j=0}^m\|u_j\|_l^2,$$

$$H_{l,\infty}=\{u(x)={}^t(u_0,u_1,\dots,u_m,\dots)|u_j\in H_l(\boldsymbol{R}_x)\ j=0,1,\dots\ |||u|||_{l,\infty}^2=\sum_{j=0}^\infty\|u_j\|_l^2<\infty\},$$

$\mathscr{P}$: operator from H_l to $H_{l,\infty}$:

$$(\mathscr{P}f)(x)={}^t((f(x,\cdot),e_0),(f(x,\cdot),e_1),\dots,(f(x,\cdot),e_m),\dots)\quad f\in H_l,$$

$$\mathscr{P}^{-1}:(\mathscr{P}^{-1}u)(x,v)=\begin{cases}\sum_{j=0}^\infty u_je_j & \text{if}\quad u(x)={}^t(u_0(x),u_1(x),\dots,u_m(x),\dots),\\ \sum_{j=0}^m u_je_j & \text{if}\quad u(x)={}^t(u_0(x),u_1(x),\dots,u_m(x)),\end{cases}$$

(Here we formally define $\mathscr{P}^{-1}$.)

$$L_m^1=\{u(x)={}^t(u_0(x),u_1(x),\dots,u_m(x))|\ |||\mathscr{P}^{-1}u|||_L=|||\mathscr{P}^{-1}u|||_{L^2(\boldsymbol{R}_v;L^1(\boldsymbol{R}_x))}<\infty\},$$

$$E_m=H_{l,m}\cap L_m^1,\quad |||\cdot|||_{E,m}=|||\cdot|||_{l,m}+|||\mathscr{P}^{-1}\cdot|||_L,$$

$$e^{tT_m}u=\sqrt{1/2\pi}\int_{\boldsymbol{R}}e^{i\xi x}e^{tT_m(\xi)}\hat{u}(\xi)d\xi\quad u\in H_{l,m}.$$

THEOREM 2.6. *Let $l\geqq 0$. Then $\{e^{tT_m}: t\geqq 0\}$ is a contraction semi-group on $H_{l,m}$. Moreover, there exists a constant $c_1>0$ which is independent of m such that e^{tT_m} has the following decay estimates*:

(i) *Let $u\in E_m$. Then*

$$|||e^{tT_m}u|||_{l,m}\leqq c_1(|||u|||_{l,m}+\sup_{|\xi|\leqq\delta}\|\mathscr{P}^{-1}\hat{u}\|_{L^2(\boldsymbol{R}_v)})/(1+t)^{1/4}\leqq c_1|||u|||_{E,m}/(1+t)^{1/4},$$

where δ is the constant given in Proposition 2.5.

(ii) *Let $u\in E_m$ and $u_j(x)=0$ for a.e. x and $j=0,2$. Then*

$$|||e^{tT_m}u|||_{l,m}\leqq c_1|||u|||_{E,m}/(1+t)^{3/4}.$$

PROOF. Nishida and Imai proved the existence and the decay of the solutions for the Boltzmann equation. (See [5].) Referring to [5] we can similarly construct the solutions of (2.1.m). Evidently the constant c_1 is independent of m by virtue of Proposition 2.3, 2.4 and 2.5.

3. Convergence of solutions for linearized equations of (1.4.m)

In the preceding section we obtained the solution $e^{tT_m}u$ for the linearized equation of (1.4.m). In this section we shall show that $\{e^{tT_m}u\}_{m=3}^{\infty}$ converges to the solution for the linearized equation of (1.2). First, we define infinite dimensional vector spaces and infinite dimensional matrix operators.

DEFINITION 3.1.

$$E = H_l \cap L^2(\boldsymbol{R}_v; L^1(\boldsymbol{R}_x)), \quad |||\cdot|||_E = |||\cdot|||_l + |||\cdot|||_L,$$

$$\mathscr{S}_\infty = \{u = {}^t(u_0(x), u_1(x), \dots, u_m(x), \dots) | u_j(x) \in \mathscr{S} \text{ for any } j \text{ and }$$
$$u_j(x) \equiv 0 \text{ if } j \notin M, \text{ where } M \text{ is some finite set} \subset \{0, 1, 2, \dots\}\},$$

$$T_m = -S_m \partial_x + D_m,$$

$$T_m^\infty = \begin{pmatrix} T_m & 0 \cdots \\ \vdots & \vdots \\ 0 & 0 \cdots \\ \vdots & \vdots \end{pmatrix},$$

$$T^\infty = -\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & \ddots & 0 & & \\ & \ddots & \ddots & \ddots & & \\ & 0 & \ddots & 0 & \sqrt{m} & \\ & & & \sqrt{m} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \partial_x + \begin{pmatrix} \lambda_0 & & & & 0 \\ & \lambda_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \lambda_m \\ & & & & & \ddots \end{pmatrix}.$$

The following lemma is easily shown from Lemma 2.2.

LEMMA 3.1. *T_m^∞ is a generator of a contraction semi-group $\{e^{tT_m^\infty}: t \geqq 0\}$ in $H_{l,\infty}$.*

REMARK. $e^{tT_m^\infty}u = {}^t(e^{tT_m}P_m u, u_{m+1}, \dots)$ holds for $u \in H_{l,\infty}$, where P_m is the orthogonal projection from $H_{l,\infty}$ to $H_{l,m}$.

LEMMA 3.2. *T^∞ is a generator of a contraction semi-group $\{e^{tT^\infty}: t \geqq 0\}$ in $H_{l,\infty}$. Moreover*

$$\mathscr{P}^{-1}(\lambda - T^\infty)^{-1}\mathscr{P}f = (\lambda - B)^{-1}f \text{ for any } f \in H_l, \ \mathrm{Re}\,\lambda > 0.$$

REMARK. $\mathscr{P}^{-1}e^{tT^\infty}\mathscr{P}f = e^{tB}f$ holds for any $f \in H_l$.

PROOF. Let $u \in H_{l,\infty}$. Since $(\lambda - B)^{-1}\mathscr{P}^{-1}u \in H_l$, we can set $(\lambda - B)^{-1}\mathscr{P}^{-1}u = \sum_{n=0}^{\infty} w_n(x)e_n = \mathscr{P}^{-1}w$, where $w = {}^t(w_0, w_1,\dots, w_m,\dots) \in H_{l,\infty}$. So we define the operator A from $H_{l,\infty}$ to $H_{l,\infty}$ by $Au = w$. Then A is a bounded operator. Noting this and $A(\lambda - T^\infty)u = u$ for $u \in \mathscr{S}_\infty$, we see $A(\lambda - T^\infty)u = u$ for $u \in \mathscr{D}(\lambda - T^\infty)$, which shows that λ belongs to the resolvent set of T^∞. Since T^∞ is dissipative, the proof is complete.

In order to obtain Proposition 3.4 we shall prepare the following lemma.

LEMMA 3.3. *Let* $\operatorname{Re}\lambda > 0$. *Then*

$$\lim_{m\to\infty} (\lambda - T_m^\infty)^{-1} = (\lambda - T^\infty)^{-1} \text{ strongly in } H_{l,\infty}.$$

PROOF. Let $x \in H_{l,\infty}$ and $\varepsilon > 0$. Since $(\lambda - T^\infty)(\mathscr{S}_\infty)$ is dense in $H_{l,\infty}$ there exists $x' = {}^t(x'_0, x'_1,\dots, x'_m,\dots) \in (\lambda - T^\infty)(\mathscr{S}_\infty)$ such that $|||x - x'|||_{l,\infty} < \varepsilon \operatorname{Re}\lambda/2$. In view of $x' \in (\lambda - T^\infty)(\mathscr{S}_\infty)$ there exists $y = {}^t(y_0, y_1,\dots, y_m,\dots) \in \mathscr{S}_\infty$ such that $x' = (\lambda - T^\infty)y$. Since $y \in \mathscr{S}_\infty$, there is an integer $N > 0$ such that for any $j \geqq N$, $y_j \equiv 0$. If $m \geqq N+1$, we have $T_m^\infty T^\infty y = T^\infty T_m^\infty y$, which implies $(\lambda - T^\infty)^{-1} T_m^\infty (\lambda - T^\infty) y = T_m^\infty y$. Since $x' = (\lambda - T^\infty)y$ we get $(\lambda - T^\infty)^{-1} T_m^\infty x' = T_m^\infty (\lambda - T^\infty)^{-1} x$. Hence we have

$$\begin{aligned}
&|||\{(\lambda - T_m^\infty)^{-1} - (\lambda - T^\infty)^{-1}\}x|||_{l,\infty} \leqq \\
&\leqq |||\{(\lambda - T_m^\infty)^{-1} - (\lambda - T^\infty)^{-1}\}(x - x')|||_{l,\infty} + |||\{(\lambda - T_m^\infty)^{-1} - (\lambda - T^\infty)^{-1}\}x'|||_{l,\infty} \\
&\leqq (2/\operatorname{Re}\lambda)|||x - x'|||_{l,\infty} + |||(\lambda - T_m^\infty)^{-1}(T_m^\infty - T^\infty)(\lambda - T^\infty)^{-1}x'|||_{l,\infty} \\
&< \varepsilon + |||(\lambda - T_m^\infty)^{-1}(\lambda - T^\infty)^{-1}(T_m^\infty - T^\infty)x'|||_{l,\infty} \\
&\qquad (\text{since } (T_m^\infty - T^\infty)x' = 0) \\
&= \varepsilon,
\end{aligned}$$

which completes the proof.

PROPOSITION 3.4. *Let* $T > 0$ *and* $u \in H_{l,\infty}$. *Then*

$$\lim_{m\to\infty} \sup_{0 \leqq t \leqq T} |||e^{tT_m^\infty}u - e^{tT^\infty}u|||_{l,\infty} = 0.$$

See [4] for a complete proof.

PROPOSITION 3.5

(i) $\lambda_j(\xi)_{j=0,2}$ *and* $\lambda_{m,j}(\xi)_{j=0,2}$ *are given in Proposition* A.3 *and* 2.3 *respectively. Then we have*

$$(\partial_\xi^n \lambda_j)(0) = (\partial_\xi^n \lambda_{m,j})(0) \qquad \text{for} \quad n \leqq 2m-3.$$

(ii) $e_j(\xi)_{j=0,2}$ *and* $e_{m,j}(\xi)_{j=0,2}$ *are given in Proposition* A.4 *and* 2.4 *respectively. Then we have*

$$P\{(\partial_\xi^n e_j)(0)\} = \begin{pmatrix} \partial_\xi^n e_{m,j}(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad \textit{for} \quad n \leqq m-2,$$

where P is defined by $Pf={}^t((f, e_0), (f, e_1), \ldots, (f, e_m), \ldots)$.

PROOF. First, we define some notations:
P^*: the orthogonal projection onto the null space of L,

$$R_2(\xi, z) = (P^* - L + i\xi v + z\xi^2)^{-1},$$
$$M_{i,j}(\xi, z) = -z(R_2(\xi, z)e_i, e_j)_{L^2(\boldsymbol{R}_v)} + (R_2(\xi, z)(iv+z\xi)(P^*-L)^{-1}ive_i, e_j)_{L^2(\boldsymbol{R}_v)},$$
$$M(\xi, z) = \begin{vmatrix} M_{0,0}(\xi, z) & M_{0,2}(\xi, z) \\ M_{2,0}(\xi, z) & M_{2,2}(\xi, z) \end{vmatrix},$$
$$z = \sigma + i\tau \qquad \sigma, \tau \in \boldsymbol{R},$$
$$f(\xi, \sigma, \tau) = \operatorname{Re} M(\xi, \sigma + i\tau),$$
$$g(\xi, \sigma, \tau) = \operatorname{Im} M(\xi, \sigma + i\tau).$$

Applying the real implicit function theorem to

$$\begin{cases} f = 0, \\ g = 0, \end{cases}$$

in a neighbourhood of $(\xi, \sigma, \tau) = (0, z_j, 0)$, we obtain the solutions $z_j(\xi) = \sigma_j(\xi) + i\tau_j(\xi)$ $j=0, 2$ of $M(\xi, z)=0$ in the same way as in section 2. See [6]. Moreover we have

$$(3.1) \qquad \begin{cases} \partial_\xi \sigma_j(\xi) = \left(\dfrac{\partial(f, g)}{\partial(\tau, \xi)} \Big/ \dfrac{\partial(f, g)}{\partial(\sigma, \tau)}\right)(\xi, \sigma_j(\xi), \tau_j(\xi)), \\ \partial_\xi \tau_j(\xi) = \left(\dfrac{\partial(f, g)}{\partial(\xi, \sigma)} \Big/ \dfrac{\partial(f, g)}{\partial(\sigma, \tau)}\right)(\xi, \sigma_j(\xi), \tau_j(\xi)). \end{cases}$$

Hence in view of the expressions for $M_{m,i,j}$ and $M_{i,j}$ it is enough to investigate

$$(\partial_\xi^k \partial_z^l R_{m,2}(\xi, z)v_h)(0, z_j) \quad h = 0, 1, 2, 3 \quad j = 0, 2,$$
$$(\partial_\xi^k \partial_z^l R_2(\xi, z)e_h)(0, z_j) \quad h = 0, 1, 2, 3 \quad j = 0, 2,$$
$$(S_m \partial_\xi^k \partial_z^l R_{m,2}(\xi, z)v_h)(0, z_j) \quad h = 0, 2, \quad j = 0, 2,$$

and

$$(v\partial_\xi^k\partial_z^l R_2(\xi, z)e_h)(0, z_j) \quad h = 0, \quad j = 0, 2.$$

Put $\partial_\xi^k\partial_z^l R_{m,2}(\xi, z)v_h = \sum_{n=0}^k Q_n(\xi, z, R_{m,2}(\xi, z), S_m)v_h$, where $Q_n(\xi, z, X, Y)$ is a non-commutative polynomial in ξ, z, X and Y, which is independent of m and whose degree with respetc to Y is just n. Replacing D_m and S_m by L and v respectively, we have

$$\partial_\xi^k\partial_z^l R_2(\xi, z)e_h = \sum_{n=0}^k Q_n(\xi, z, R_2(\xi, z), v)e_h.$$

Note the following facts:

$$(3.2)\quad \begin{cases} R_{m,2}(0, z_j)v_i = v_i, \quad R_2(0, z_j)e_i = e_i \quad i, j = 0, 2. \\ R_{m,2}(0, z_j)v_i = (-1/\lambda_i)v_i, \quad R_2(0, z_j)e_k = (-1/\lambda_k)e_k \\ \qquad\qquad j = 0, 2, \quad i = 1, 3, 4,\dots, m, \quad k = 1, 3, 4,\dots, \\ S_m v_j = \sqrt{j}v_{j-1} + \sqrt{j+1}v_{j+1} \quad 0 \leqq j \leqq m-1, \quad S_m v_m = \sqrt{m}v_{m-1}, \\ ve_j = \sqrt{j}e_{j-1} + \sqrt{j+1}e_{j+1} \quad 0 \leqq j < \infty. \end{cases}$$

It follows from the above that:

(i) if $n+h \leqq m$,

(v.1) $Q_n(0, z_j, R_{m,2}(0, z_j), S_m)v_h = \sum_{r=0}^{n+h} a_{n,h,r}v_r$,

(e.1) $Q_n(0, z_j, R_2(0, z_j), v)e_h = \sum_{r=0}^{n+h} a_{n,h,r}e_r$;

(ii) if $n+h = m+1$,

(v.2) $Q_n(0, z_j, R_{m,2}(0, z_j), S_m)v_h = \sum_{r=0}^{m} a'_{n,h,r}v_r$,

(e.2) $Q_n(0, z_j, R_2(0, z_j), v)e_h = \sum_{r=0}^{m} a'_{n,h,r}e_r + a'_{n,h,m+1}e_{m+1}$,

and (iii) if $2m-1 \geqq n+h > m+1$,

(v.3) $Q_n(0, z_j, R_{m,2}(0, z_j), S_m)v_h = \sum_{r=0}^{2m+1-(n+h)} a''_{n,h,r}v_r +$
$\quad + \sum_{r=2m+2-(n+h)}^{m} c_{n,h,r}v_r$,

(e.3) $Q_n(0, z_j, R_2(0, z_j), v)e_h = \sum_{r=0}^{2m+1-(n+h)} a''_{n,h,r}e_r + \sum_{r=2m+2-(n+h)}^{n+h} d_{n,h,r}e_r$,

where the coefficients of v_r and e_r in the right side hands are the constants and $j=0, 2$. Taking the inner product of $(\text{v}.k)_{k=1,2,3}$ and $(\text{e}.k)_{k=1,2,3}$ with v_s and e_s respectively, we get for $0 \leqq n+h \leqq 2m-1$, in view of (3.2),

$$(Q_n(0, z_j, R_{m,2}(0, z_j), S_m)v_h, v_s) = (Q_n(0, z_j, R_2(0, z_j), v)e_h, e_s)$$

where $j=0,2$ and $s=0,2$. This implies

$$(3.3)\qquad ((\partial_\xi^k\partial_z^l R_{m,2}(\xi, z)v_h)(0, z_j), v_s) = ((\partial_\xi^k\partial_z^l R_2(\xi, z)e_h)(0, z_j), e_s),$$

where $k\leqq 2m-4$, $h=0, 1, 2, 3$, $j=0, 2$, $s=0, 2$. We have similarly

$$(3.4) \qquad ((S_m\partial_\xi^k\partial_z^l R_{m,2}(\xi, z)v_h)(0, z_j), v_s) = ((v\partial_\xi^k\partial_z^l R_2(\xi, z)e_h)(0, z_j), e_s),$$

where $k\leqq 2m-5$, $h=0, 2$, $j=0, 2$, $s=1,3$. (3.3) and (3.4) complete the proof of the statement (i).

Replacing $\lambda_{m,j}$, v_0 and v_2 in (2.14) by λ_j, e_0 and e_2 respectively, we obtain the representation of $e_j(\xi)$. (This is proved in [6].) Using the representations of $e_{m,j}(\xi)$ and $e_j(\xi)$ together with the statements (i), (v.1) and (e.1), we get the statement (ii).

PROPOSITION 3.6. *Let $M\geqq 3$ be an integer. Then there exists a constant $c(M)>0$ such that for any $m\geqq M$ and any $f\in E$*

$$|||\mathscr{P}^{-1} e^{tT_m} P_m\mathscr{P}f-e^{tB}f|||_l \leqq c(M)|||f|||_E/(1+t)^{(2M-1)/4}.$$

PROOF. We estimate $|||\mathscr{P}^{-1} e^{tT_m} P_m\mathscr{P}f-e^{tB}f|||_l^2$ as follows:

$$\int_{\boldsymbol{R}}\int_{\boldsymbol{R}}(1+|\xi|)^{2l}|\mathscr{P}^{-1} e^{tT_m(\xi)} P_m\mathscr{P}\hat{f}(\xi, v) - e^{tB(\xi)}\hat{f}(\xi, v)|^2 dv d\xi \leqq$$

$$\leqq \int_{\boldsymbol{R}}\int_{|\xi|\geqq\delta}(1+|\xi|)^{2l}|\mathscr{P}^{-1}(1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_2-i\gamma}^{-\beta_2+i\gamma} e^{\lambda t}(\lambda-T_m(\xi))^{-1}P_m\mathscr{P}\hat{f}(\xi, v)d\lambda -$$
$$- (1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_2-i\gamma}^{-\beta_2+i\gamma} e^{\lambda t}(\lambda-B(\xi))^{-1}\hat{f}(\xi, v)d\lambda|^2 dv d\xi +$$
$$+ 2\int_{\boldsymbol{R}}\int_{|\xi|<\delta}(1+|\xi|)^{2l}|\mathscr{P}^{-1}(1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_1-i\gamma}^{-\beta_1+i\gamma} e^{\lambda t}(\lambda-T_m(\xi))^{-1}P_m\mathscr{P}\hat{f}(\xi, v)d\lambda -$$
$$- (1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_1-i\gamma}^{-\beta_1+i\gamma} e^{\lambda t}(\lambda-B(\xi))^{-1}\hat{f}(\xi, v)d\lambda|^2 dv d\xi +$$
$$+ 2\int_{\boldsymbol{R}}\int_{|\xi|<\delta}(1+|\xi|)^{2l}|\mathscr{P}^{-1}\textstyle\sum_{j=0,2} e^{t\lambda_{m,j}(\xi)}(P_m\mathscr{P}\hat{f}(\xi, v), e_{m,j}(-\xi))_{\boldsymbol{C}^{m+1}}e_{m,j}(\xi) -$$
$$- \textstyle\sum_{j=0,2} e^{t\lambda_j(\xi)}(\hat{f}(\xi, v), e_j(-\xi))_{L^2(\boldsymbol{R}_v)}e_j(\xi)|^2 dv d\xi$$
$$= I_1 + I_2 + I_3.$$

By the estimates in Proposition 2.5 and A.5

$$(3.5) \qquad I_j \leqq c\, e^{-2t\beta}|||f|||_l^2 \quad j=1, 2,$$

where the constant $c>0$ is independent of m and $\beta=\min_{j=1,2}\beta_j$.

To estimate I_3 we shall first give some estimates:

$$\|e_j(\xi)-\mathscr{P}^{-1}e_{m,j}(\xi)\|_{L^2(\boldsymbol{R}_v)} \leqq c(M)|\xi|^{M-1},$$
$$|e^{t\lambda_j(\xi)}-e^{t\lambda_{m,j}(\xi)}| \leqq c(M)e^{-t\mu_1\xi^2/2}t|\xi|^{M+1} \quad j=0, 2,$$

where the constant $c(M)$ is independent of m, but it is dependent on M, which are deduced from Proposition 2.4 and 3.5. Using the decomposition:

$$e^{t\lambda_j(\xi)}(\hat{f}, e_j(-\xi))_{L^2(\mathbf{R}_v)}e_j(\xi) - \mathscr{P}^{-1} e^{t\lambda_{m,j}(\xi)}(P_m\mathscr{P}\hat{f}, e_{m,j}(-\xi))_{\mathbf{C}^{m+1}}e_{m,j}(\xi) =$$
$$= e^{t\lambda_j(\xi)}\{(\hat{f}, e_j(-\xi)-\mathscr{P}^{-1}e_{m,j}(-\xi))_{L^2(\mathbf{R}_v)}e_j(\xi)\} +$$
$$+ e^{t\lambda_j(\xi)}\{(\hat{f}, \mathscr{P}^{-1}e_{m,j}(-\xi))_{L^2(\mathbf{R}_v)}e_j(\xi)-\mathscr{P}^{-1}(P_m\mathscr{P}\hat{f}, e_{m,j}(-\xi))_{\mathbf{C}^{m+1}}e_{m,j}(\xi)\} +$$
$$+ (e^{t\lambda_j(\xi)}-e^{t\lambda_{m,j}(\xi)})\mathscr{P}^{-1}(P_m\mathscr{P}\hat{f}, e_{m,j}(-\xi))_{\mathbf{C}^{m+1}}e_{m,j}(\xi)$$
$$= K_4 + K_5 + K_6 \quad j = 0, 2,$$

we have from Proposition 2.4

$$(3.6) \quad \int_{\mathbf{R}}\int_{-\delta}^{\delta} |K_4|^2 dv d\xi$$
$$\leqq \int_{-\delta}^{\delta} |e^{2t\lambda_j(\xi)}| \|\hat{f}(\xi,\cdot)\|_{L^2(\mathbf{R}_v)}^2 \|e_j(-\xi)-\mathscr{P}^{-1}e_{m,j}(-\xi)\|_{L^2(\mathbf{R}_v)}^2 \|e_j(\xi)\|_{L^2(\mathbf{R}_v)}^2 d\xi$$
$$\leqq c(M)\int_{-\delta}^{\delta} e^{-t\xi^2\mu_1}|\xi|^{2M-2}\|\hat{f}(\xi,\cdot)\|_{L^2(\mathbf{R}_v)}^2 d\xi$$
$$\leqq c(M)\sup_{|\xi|\leqq\delta}\|\hat{f}(\xi,\cdot)\|_{L^2(\mathbf{R}_v)}^2 \int_{-\delta}^{\delta} e^{-t\xi^2\mu_1}|\xi|^{2M-2}d\xi$$
$$\leqq c(M)|||f|||_L^2/(1+t)^{(2M-1)/2},$$

and

$$(3.7) \qquad \int_{\mathbf{R}}\int_{-\delta}^{\delta} |K_k|^2 dv d\xi \leqq c(M)|||f|||_L^2/(1+t)^{(2M-1)/2} \quad k = 5, 6.$$

The summation of (3.5) (3.6) and (3.7) completes the proof.

THEOREM 3.7. *Suppose* $\alpha \geqq 0$ *and* $f \in E$. *Then*

$$\lim_{m\to\infty}\sup_{0\leqq t<\infty}(1+t)^\alpha |||\mathscr{P}^{-1}e^{tT_m}P_m\mathscr{P}f - e^{tB}f|||_l = 0.$$

PROOF. Let $\varepsilon>0$. Choose an integer $N\geqq 3$ with $\alpha<(2N-1)/4$. Owing to Proposition 3.6 there is a constant $T>0$ such that for any $t\geqq T$ and $m\geqq N$

$$(3.8) \qquad |||\mathscr{P}^{-1}e^{tT_m}P_m\mathscr{P}f - e^{tB}f|||_l < \varepsilon/(1+t)^\alpha.$$

In view of the remarks in Lemma 3.1 and 3.2 and Proposition 3.4, there exists an integer $M(\geqq N)$ such that for any $m\geqq M$

$$(3.9) \qquad |||\mathscr{P}^{-1}e^{tT_m}P_m\mathscr{P}f - e^{tB}f|||_l < \varepsilon/(1+T)^{(2M-1)/4} \quad \text{on } [0, T].$$

Therefore (3.8) and (3.9) complete the proof.

4. Existence and decay of solutions for (1.4.*m*)

We define W_m as follows:

$$W_m(u, v) = (1/2)\begin{pmatrix} 0 \\ \lambda_1(u_0v_1+u_1v_0) \\ \vdots \\ \lambda_m(u_0v_m+u_mv_0)+\sum_{n=1}^{m-1}\lambda_{n,m-n}\sqrt{m!/n!(m-n)!}(u_nv_{m-n}+u_{m-n}v_n) \end{pmatrix}$$

where $u={}^t(u_0, u_1,\dots, u_m)$ and $v={}^t(v_0, v_1,\dots, v_m)$. Then W_m can be regarded as a bilinear operator from $H_{l,m}\times H_{l,m}$ to $H_{l,m}$.

LEMMA 4.1. *Let* $l\geqq 1$. *Suppose* $u, v\in H_{l,m}$. *Then*

(i) $|||W_m(u, v)|||_{l,m}\leqq c^{**}|||u|||_{l,m}|||v|||_{l,m}$,

(ii) $|||\mathscr{P}^{-1}W_m(u, v)|||_L\leqq 2\nu|||u|||_{l,m}|||v|||_{l,m}$,

where $c^{**}=2\nu d$ *and the constant* d *depends only on* l. *Therefore the constant* c^{**} *is independent of* m.

PROOF. We first evaluate the k-th component of $W_m(u, v)$. Owing to Schwarz's inequality, we get

$$(1/4)\Big|\sum_{n=0}^{k}\sqrt{k!/n!(k-n)!}\int_{-\pi}^{\pi}\cos^n\theta\sin^{k-n}\theta I(\theta)d\theta(u_nv_{k-n}+u_{k-n}v_n)-$$
$$-\nu(u_0v_k+u_kv_0)\Big|^2$$
$$\leqq \sum_{n=0}^{k}\{k!/n!(k-n)!\}\Big|\int_{-\pi}^{\pi}\cos^n\theta\sin^{k-n}\theta I(\theta)d\theta\Big|^2\sum_{n=0}^{k}|u_nv_{k-n}|^2+$$
$$+\sum_{n=0}^{k}\{k!/n!(k-n)!\}\Big|\int_{-\pi}^{\pi}\cos^n\theta\sin^{k-n}\theta I(\theta)d\theta\Big|^2\sum_{n=0}^{k}|u_{k-n}v_n|^2+$$
$$+\nu^2(|u_0v_k|^2+|u_kv_0|^2) = I_k.$$

Noting that $\sum_{n=0}^{k}\{k!/n!(k-n)!\}\left(\int_{-\pi}^{\pi}\cos^n\theta\sin^{k-n}\theta I(\theta)d\theta\right)^2\leqq\nu^2$ (see [2]), we have

$$I_k\leqq 4\nu^2\sum_{n=0}^{k}|u_{k-n}v_n|^2.$$

By Sobolev's inequality:

$$\|fg\|_l\leqq d\|f\|_l\|g\|_l \quad f, g\in H_l(\boldsymbol{R}_x)$$

we have

$$|||W_m(u, v)|||_{l,m}^2=\int_{\boldsymbol{R}}(1+|\xi|)^{2l}\sum_{k=0}^{m}|\widehat{\text{the } k\text{-th component of } W_m(u, v)}|^2d\xi$$

$$\leqq 4\nu^2 \textstyle\sum_{k=0}^m \int_{\boldsymbol{R}} (1+|\xi|)^{2l} \sum_{n=0}^k |\widehat{u_{k-n}v_n}|^2 d\xi$$

$$\leqq 4\nu^2 d^2 \textstyle\sum_{k=0}^m \sum_{n=0}^k \|u_{k-n}\|_l^2 \|v_n\|_l^2$$

$$= (c^{**})^2 |||u|||_{l,m}^2 |||v|||_{l,m}^2.$$

This shows (i).

Next, summing up $I_{k,k=0,\cdots,m}$, we have

$$\|W_m(u, v)\| \leqq 2\nu \|u\| \, \|v\| . \tag{4.1}$$

From the definition of L_m^1,

$$|||\mathscr{P}^{-1} W_m(u, v)|||_L^2 = \left\| \int_{\boldsymbol{R}} |\mathscr{P}^{-1} W_m(u, v)| dx \right\|_{L^2(\boldsymbol{R}_v)}^2$$

$$\leqq \left(\int_{\boldsymbol{R}} \|\mathscr{P}^{-1} W_m(u, v)\|_{L^2(\boldsymbol{R}_v)} dx \right)^2$$

$$= \left(\int_{\boldsymbol{R}} \|W_m(u, v)\| dx \right)^2 .$$

Applying (4.1) and Schwarz's inequality, we obtain the estimate (ii), and so the proof is complete.

REMARK 4.2. It is easily seen that

$$W_m(u, u) - W_m(v, v) = W_m(u+v, u-v) .$$

Making use of Theorem 2.6, Lemma 4.1 and Remark 4.2, we obtain the following theorem.

THEOREM 4.3. *Let $l \geqq 1$. There exist constants $c_E > 0$ and $c_2 > 0$, which are independent of m, such that for any initial value $u_0 \in E_m$ with $|||u_0|||_{E,m} < c_E$, (1.4.m) has a unique solution $u(t) \in C^0([0, \infty); H_{l,m}) \cap C^1([0, \infty); H_{l-1,m})$. Moreover*

$$|||u(t)|||_{l,m} \leqq c_2(|||u_0|||_{l,m} + \sup_{|\xi| \leqq \delta} \|\mathscr{P}^{-1} \hat{u}_0\|_{L^2(\boldsymbol{R}_v)})/(1+t)^{1/4} \leqq c_2 |||u_0|||_{E,m}/(1+t)^{1/4},$$

where δ is the constant given in Proposition 2.5.

We can prove this theorem by the usual technique. So we omit the proof. See [5] for a complete proof.

5. Convergence of solutions for (1.4.m)

In this section we show that the solutions constructed in section 4 converge to the solution for (1.2).

We consider the following equations:

$$f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} \Gamma(f(s), f(s)) ds, \tag{5.1}$$

$$u^{(m)}(t) = e^{tT_m} P_m \mathscr{P} f_0 + \int_0^t e^{(t-s)T_m} W_m(u^{(m)}(s), u^{(m)}(s)) ds. \tag{5.2.m}$$

PROPOSITION 5.1. *There exists a constant $c_E > 0$ such that for any $m \geqq 3$ and $f_0 \in E$ with $|||f_0|||_E < c_E$, the equations (5.1) and (5.2.m) have unique solutions $f(t)$ and $u^{(m)}(t)$, respectively. Moreover there is a constant $c > 0$ such that for any $m \geqq 3$,*

$$\sup_{0 \leqq t < \infty} (1+t)^{1/2} |||f(t) - \mathscr{P}^{-1} u^{(m)}(t)|||_l \leqq c |||f_0|||_E. \tag{5.3}$$

PROOF. It is clear from Theorem 4.3 and A.8 that the solutions for (5.1) and (5.2.m) exist. In order to prove (5.3) we directly evaluate $X(0, t)$, where $X(\alpha, t) = X(\alpha, t, m) = (1+t)^\alpha |||f(t) - \mathscr{P}^{-1} u^{(m)}(t)|||_l$:

$$\begin{aligned} X(0, t) \leqq\ & |||e^{tB} f_0 - \mathscr{P}^{-1} e^{tT_m} P_m \mathscr{P} f_0|||_l + \\ & + \left|\left|\left| \int_0^t \{e^{(t-s)B} \Gamma(f(s), f(s)) - \mathscr{P}^{-1} e^{(t-s)T_m} P_m \mathscr{P} \Gamma(f(s), f(s))\} ds \right|\right|\right|_l + \\ & + \left|\left|\left| \int_0^t \mathscr{P}^{-1} e^{(t-s)T_m} \{P_m \mathscr{P} \Gamma(f(s), f(s)) - W_m(u^{(m)}(s), u^{(m)}(s))\} ds \right|\right|\right|_l \\ =\ & I + II_1 + II_2. \end{aligned}$$

By Proposition 3.6 we see

$$I \leqq c |||f_0|||_E / (1+t)^{5/4}, \tag{5.4}$$

and

$$\begin{aligned} II_1 &\leqq \int_0^t c |||\Gamma(f(s), f(s))|||_E / (1+t-s)^{5/4} ds \\ &\leqq c c(\Gamma) \{\sup_{0 \leqq s \leqq t} (1+s)^{1/4} |||f(s)|||_l\}^2 \int_0^t 1/\{(1+t-s)^{5/4} (1+s)^{1/2}\} ds \\ &= 6\sqrt{2} c c(\Gamma) \{\sup_{0 \leqq s \leqq t} (1+s)^{1/4} |||f(s)|||_l\}^2 / (1+t)^{1/2}, \end{aligned} \tag{5.5}$$

where $c(\Gamma) = 2\nu(1+d)$. Next, noting that

$$P_m \mathscr{P} \Gamma(f(s), f(s)) - W_m(u^{(m)}(s), u^{(m)}(s)) = W_m(P_m \mathscr{P} f(s) + u^{(m)}(s), P_m \mathscr{P} f(s) - u^{(m)}(s))$$

we get

$$\begin{aligned} II_2 &\leqq \int_0^t |||e^{(t-s)T_m} W_m(P_m \mathscr{P} f(s) + u^{(m)}(s), P_m \mathscr{P} f(s) - u^{(m)}(s))|||_{l,\infty} ds \\ &\leqq c_1 c(\Gamma) \int_0^t |||P_m \mathscr{P} f(s) + u^{(m)}(s)|||_{l,\infty} X_m(0, s) / (1+t-s)^{3/4} ds \end{aligned}$$

$$\leqq d^* \sup_{0\leqq s\leqq t}(1+s)^{1/4}(|||P_m\mathscr{P}f(s)|||_{l,\infty}+ \\ +|||u^{(m)}(s)|||_{l,\infty})\sup_{0\leqq s\leqq t}X_m(1/2, s)/(1+t)^{1/2}$$

where $d^*=8\sqrt{2}c_1c(\Gamma)$, c_1 is the constant given in Theorem 2.6 and $X_m(\alpha, t)=(1+t)^\alpha|||P_m\mathscr{P}f(t)-u^{(m)}(t)|||_{l,m}$. Next, we use the following estimates:

$$\sup_{0\leqq s<\infty}(1+s)^{1/4}|||f(s)|||_l,\ \sup_{0\leqq s<\infty}(1+s)^{1/4}|||u^{(m)}(s)|||_{l,m}\leqq c_2|||f_0|||_E,$$

where the constant c_2 is independent of m. These show that

$$II_2\leqq 2c_2d^*|||f_0|||_E\sup_{0\leqq s\leqq t}X_m(1/2, s)/(1+t)^{1/2}. \tag{5.6}$$

Summing up (5.4), (5.5) and (5.6) yields

$$\sup_{0\leqq t<\infty}X(1/2, t)\leqq c|||f_0|||_E(1+6\sqrt{2}c(\Gamma)c_2^2|||f_0|||_E)/(1-2c_2d^*|||f_0|||_E).$$

The proof is complete.

LEMMA 5.2. *Let $T\geqq 0$. Suppose $g(t)\in C^0([0, T]; H_l)$. Then*

$$\lim_{m\to\infty}\sup_{0\leqq t\leqq T}\left|\left|\left|\int_0^t\{e^{(t-s)B}g(s)-\mathscr{P}^{-1}e^{(t-s)T_m}P_m\mathscr{P}g(s))ds\right|\right|\right|_l=0.$$

PROOF. Let $\varepsilon>0$ and put $c=\max_{0\leqq t\leqq T}|||g(t)|||_l$. Here we may assume $c\neq 0$. Since $g(t)$ is uniformly continuous on $[0, T]$, there exists a partition $0=s_0<s_1<\cdots<s_k=T$ such that for any i $0\leqq i\leqq k$,

$$s_i-s_{i-1}<\varepsilon/6c,\ |||g(s)-g(s_{i-1})|||_l<\varepsilon/6T,\quad\text{for any}\quad s\in[s_{i-1}, s_i].$$

By Proposition 3.4 there is an integer $M\geqq 3$ such that for any $m\geqq M$

$$\max_{0\leqq i\leqq k}\sup_{0\leqq t\leqq T}|||G(t, g(s_i), m)|||_l<\varepsilon/3T,$$

where $G(t, g(s), m)=e^{tB}g(s)-\mathscr{P}^{-1}e^{tT_m}P_m\mathscr{P}g(s)$. Let $m\geqq M$ and $s_{h-1}\leqq t\leqq s_h$. we have

$$\begin{aligned}&\left|\left|\left|\int_0^t G(t-s, g(s), m)ds\right|\right|\right|_l\\ &\leqq\left|\left|\left|\sum_{i=1}^{h-1}\int_{s_{i-1}}^{s_i}\{G(t-s, g(s), m)-G(t-s, g(s_i), m)\}ds\right|\right|\right|_l+\\ &\quad+\left|\left|\left|\sum_{i=1}^{h-1}\int_{s_{i-1}}^{s_i}G(t-s, g(s_i), m)ds\right|\right|\right|_l+\left|\left|\left|\int_{s_{h-1}}^t G(t-s, g(s), m)ds\right|\right|\right|_l\\ &\leqq\sum_{i=1}^{h-1}\int_{s_{i-1}}^{s_i}2\varepsilon/6T ds+\sum_{i=1}^{h-1}\int_{s_{i-1}}^{s_i}\varepsilon/3T ds+\int_{s_{h-1}}^t 2c ds\\ &=\varepsilon.\end{aligned}$$

The proof is complete.

THEOREM 5.3. *Let* $0 \leqq \alpha < 1/2$ *and* $f_0 \in E$ *with* $|||f_0|||_E < c_E$, *where the constant* c_E *is given in Proposition* 5.1. *Suppose that* $f(t)$ *and* $u^{(m)}(t)$ *are solutions of* (5.1) *and* (5.2.*m*) *with the initial value* f_0 *and* $P_m \mathscr{P} f_0$ *respectively. Then we have*

$$\lim_{m\to\infty} \sup_{0 \leqq t < \infty} (1+t)^{\alpha} |||f(t) - \mathscr{P}^{-1} u^{(m)}(t)|||_l = 0.$$

PROOF. In order to evaluate directly we use the same decomposition in the proof of Proposition 5.1. According to the proof of Proposition 5.1 we have

$$(5.7) \qquad II_2 \leqq a_2 \sup_{0 \leqq s < \infty} X_m(\alpha, s)/(1+t)^{\alpha} \quad \text{for any} \quad m \geqq 3,$$

where the constant $a_2 < 1$ is independent of m. Let $\varepsilon > 0$. By Theorem 3.7 there is an integer $M_1 \geqq 3$ such that for any $m \geqq M_1$

$$(5.8) \qquad I < (1-a_2)\varepsilon/2(1+t)^{\alpha}.$$

In view of (5.5) for the estimate of II_1, there exists a constant $T > 0$ such that for any $m \geqq M_1$, $t \geqq T$

$$(5.9) \qquad II_1 \leqq (1-a_2)\varepsilon/2(1+t)^{\alpha}.$$

Hence, summing up (5.7), (5.8) and (5.9), we get

$$(5.10) \quad \sup_{T \leqq t} X(\alpha, t, m) < (1-a_2)\varepsilon + a_2 \sup_{0 \leqq t < \infty} X_m(\alpha, t) \qquad \text{for} \quad m \geqq M_1.$$

To obtain the estimate on $[0, T]$ we note that $f(s)$ is uniformly continuous on $[0, T]$. It follows from Lemma 5.2 that there is an integer $M_2 (\geqq M_1)$ such that for any $m \geqq M_2$

$$II_1 < (1-a_2)\varepsilon/2(1+T)^{\alpha} \qquad \text{for} \quad 0 \leqq t \leqq T.$$

Consequently, we have

$$(5.11) \quad \sup_{0 \leqq t \leqq T} X(\alpha, t, m) < (1-a_2)\varepsilon + a_2 \sup_{0 \leqq t < \infty} X_m(\alpha, t) \quad \text{for } m \geqq M_2.$$

The estimates (5.10) and 5.11) imply the result.

Appendix

We first consider the linearized equation of (1.2):

$$(\text{A.1}) \qquad \begin{cases} \partial_t f = Bf, \\ f(0, x, v) = f_0(x, v). \end{cases}$$

(A.1) is rewritten, by the Fourier transform with respect to x, into

$$(\text{A.2}) \qquad \begin{cases} \partial_t \hat{f} = (-i\xi v + L)\hat{f} \equiv B(\xi)\hat{f}, \\ \hat{f}(0, \xi, v) = \hat{f}_0(\xi, v). \end{cases}$$

Regarding $\xi \in \boldsymbol{R}$ as a parameter, we consider (A.2) in $L^2(\boldsymbol{R}_v)$. In this appendix we use the short notation $\|\cdot\| = \|\cdot\|_{L^2(\boldsymbol{R}_v)}$.

LEMMA A.1. (i) $\sigma(B(\xi)) \subset \{z | \mathrm{Re}\, z \leqq 0\}$,
(ii) $\sigma(B(\xi)) \cap \{z | \mathrm{Re}\, z = 0\} = \emptyset$, *if* $\xi \neq 0$,
(iii) $\sigma(B(\xi)) = \sigma_e(B(\xi)) \cup \sigma_d(B(\xi))$, $\sigma_e(B(\xi)) = \{z | z = -i\gamma - v,\ \gamma \in \boldsymbol{R}\}$,
where $\sigma(B(\xi))$, $\sigma_e(B(\xi))$ *and* $\sigma_d(B(\xi))$ *are the spectrum, the essential spectrum and the set of the isolated eigenvalues with finite multiplicity of* $B(\xi)$ *respectively.* (*See* [7].)

LEMMA A.2. $B(\xi)$ *is a generator of a contraction semi-group* $\{e^{tB(\xi)}: t \geqq 0\}$ *in* $L^2(\boldsymbol{R}_v)$.

PROPOSITION A.3.
(i) *For any* $\beta_1 \in (0, \kappa/2]$, *there exist constants* $\delta > 0$ *and* $c > 0$ *such that*

(a) $\inf_{|\lambda| \geqq \beta_1, \mathrm{Re}\lambda \geqq -3\kappa/4, |\xi| \leqq \delta} \|(\lambda - B(\xi))f\| \geqq c\|f\|$, *for* $f \in L^2(\boldsymbol{R}_v)$,

(b) $\sigma(\hat{B}(\xi)) \cap \{\lambda | |\lambda| < \beta_1\} = \{\lambda_j(\xi)\}_{j=0,2}$ *for* $|\xi| \leqq \delta$,

where $\lambda_j(\xi)$ *are the perturbed eigenvalues of* λ_j *with respect to* ξ.
(ii) *For any* $\delta' > 0$, *there exist constants* $\beta_2 > 0$ *and* $c' > 0$ *such that*

$$\inf_{\mathrm{Re}\lambda \geqq -\beta_2, |\xi| \geqq \delta'} \|(\lambda - B(\xi))f\| \geqq c'\|f\|, \quad \text{for } f \in L^2(\boldsymbol{R}_v).$$

PROPOSITION A.4. *Let* $\lambda_j(\xi)_{j=0,2}$ *be the eigenvalues given in Proposition* A.3 *and* $e_j(\xi)_{j=0,2}$ *be the corresponding eigenvectors. Then there exists a constant* $\delta_1 > 0$ *such that for* $|\xi| \leqq \delta_1$ *we have the following results*:

(i.a) $\lambda_j(\xi) = \xi^2 z_j(\xi)$, $\sup_{|\xi| \leqq \delta_1} \mathrm{Re}\, z_j(\xi) \leqq -\mu_1 < 0$,

where $z_j(\xi)$ *belong to* $C^\infty([-\delta_1, \delta_1])$ *and* μ_1 *is a positive constant.*

(ii.a) $e_j(\xi) \in C^\infty([-\delta_1, \delta_1]; L^2(\boldsymbol{R}_v))$, $(e_i(\xi), e_j(\xi)) = \delta_{ij}$,

where δ_{ij} *is Kronecker's delta.*

PROPOSITION A.5. *There are constants* $\delta > 0$, $\beta_1 > 0$ *and* $\beta_2 > 0$ *such that the semi-group* $\{e^{tB(\xi)}: t \geqq 0\}$ *is expressed as follows*:
(i) *For any* ξ *with* $|\xi| < \delta$,

$$e^{tB(\xi)}f = (1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_1-i\gamma}^{-\beta_1+i\gamma} e^{\lambda t}(\lambda - B(\xi))^{-1} f d\lambda + \sum_{j=0,2} e^{t\lambda_j(\xi)}(f, e_j(-\xi))_{L^2(\boldsymbol{R}_v)} e_j(\xi). \tag{A.3}$$

(ii) *For any* ξ *with* $|\xi| \geqq \delta$,

$$(\text{A.4}) \qquad e^{tB(\xi)}f = (1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_2-i\gamma}^{-\beta_2+i\gamma} e^{\lambda t}(\lambda - B(\xi))^{-1}fd\lambda.$$

In the above, the first terms on the right hand side of (A.3) *and* (A.4) *have the following estimates*:

$$\|(1/2\pi i)\lim_{\gamma\to\infty}\int_{-\beta_j-i\gamma}^{-\beta_j+i\gamma} e^{\lambda t}(\lambda - B(\xi))^{-1}fd\lambda\| \leqq c\, e^{-\beta_j t}\|f\|, \quad j = 1, 2,$$

where the constant c is independent of ξ.

From the above results we obtain the existence and the decay of the solutions for (A.1) in H_l.

THEOREM A.6. *Let* $l\geqq 0$. *Then B is a generator of a contraction semigroup* $\{e^{tB}: t\geqq 0\}$ *in* H_l. *Moreover there exists a constant* $c_1>0$ *such that* e^{tB} *has the following decay estimates*:

(i) *Let* $f\in E$. *Then*

$$|||e^{tB}f|||_l \leqq c_1|||f|||_E/(1+t)^{1/4}.$$

(ii) *Let* $f\in E$ *and* $\int_{\boldsymbol{R}} e_j(v)f(x, v)dv=0$, *a.e.x*, $j=0, 2$. *Then*

$$|||e^{tB}f|||_l \leqq c_1|||f|||_E/(1+t)^{3/4}.$$

LEMMA A.7. *Let* $l\geqq 1$. *Suppose* $f, g\in H_l$. *Then*

(i) $|||\Gamma(f, g)|||_l \leqq c^{**}|||f|||_l|||g|||_l$.

(ii) $|||\Gamma(f, g)|||_L \leqq 2\nu|||f|||_l|||g|||_l$.

Theorem A.6 and Lemma A.7 together imply the following theorem, which is our main result in this section.

THEOREM A.8. *Let* $l\geqq 1$. *There exist constants* $c_E>0$ *and* $c_2>0$ *such that for any initial value* $f_0\in E$ *with* $|||f_0|||_E<c_E$, (1.2) *has a unique solution* $f(t)\in C^0([0, \infty); H_l)\cap C^1([0, \infty); V_{l-1})$, *satisfying the estimate*

$$|||f(t)|||_l \leqq c_2|||f_0|||_E/(1+t)^{1/4}.$$

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