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On homology of the double covering over the exterior of a surface in 4-sphere

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Introduction.

We consider a closed connected surface F embedded in a homology 4-sphere M^4 with normal bundle $N(F)$. Of course $N(F)$ always exists as a regular neighborhood of F in the smooth or PL category. The exterior X of F is defined by $X = M^4 - \operatorname{Int} N(F)$. If F is non-orientable (resp. orientable), then $H_1(X) \cong H^2(F) \cong \boldsymbol{Z}_2$ (resp. $\boldsymbol{Z}$) by the Alexander duality, and we have the double covering space X_2 over X associated with the kernel of the non-trivial homomorphism $\pi_1(X) \to \boldsymbol{Z}_2$ through the Hurewicz homomorphism $\pi_1(X) \to H_1(X)$. In this paper, we determine the finitely generated Λ_2-modules $H_*(X_2)$ and $H_*(X_2, \partial X_2)$. Here Λ_2 denotes the integral group ring of $\boldsymbol{Z}_2$ which is generated by t, and t acts on these homology groups by the induced isomorphism of the covering transformation.

THEOREM 1. *If F is non-orientable, we have the following.*

(1) $H_1(X_2) \cong H_1(X_2, \partial X_2) \cong \bigoplus_{i=1}^{n} \Lambda_2/(t+1, c_i)$, *where* c_i $(1 \le i \le n)$ *are odd integers.*

(2) $H_2(X_2) \cong H_2(X_2, \partial X_2) \cong \Lambda_2^{g-1} \oplus \Lambda_2/(t+1) \oplus H_1(X_2)$, *where g is the genus of F.*

(3) $H_i(X_2) = 0$ $(i \ge 3)$, $H_i(X_2, \partial X_2) = 0$ $(i = 0, 3$ *or* $i \ge 5)$, *and* $H_0(X_2) \cong H_4(X_2, \partial X_2) \cong \Lambda_2/(t-1)$.

THEOREM 1′. *If F is orientable, we have the following.*

(1′) $H_1(X_2, \partial X_2) \cong \bigoplus_{i=1}^{n} \Lambda_2/(t+1, c_i)$ *and* $H_1(X_2) \cong \Lambda_2/(t-1) \oplus H_1(X_2, \partial X_2)$, *where* c_i $(1 \le i \le n)$ *are odd integers.*

(2′) $H_2(X_2) \cong H_2(X_2, \partial X_2) \cong \Lambda_2^{2g} \oplus H_1(X_2, \partial X_2)$, *where g is the genus of F.*

(3′) $H_i(X_2) = 0$ $(i \ge 3)$, $H_i(X_2, \partial X_2) = 0$ $(i = 0$ *or* $i \ge 5)$, *and* $H_0(X_2) \cong H_3(X_2, \partial X_2) \cong H_4(X_2, \partial X_2) \cong \Lambda_2/(t-1)$.

REMARK. In the case that $\pi_1(X)$ is an abelian group, the above theorems are well known because F is stably unknotted (cf. [2]).

As for the realization problem of homology modules, we first prove the following theorem.

THEOREM 2. *For any odd integers $c_1, c_2, \dots, c_n$ and positive integer g, there exists a closed connected non-orientable* (*resp. orientable*) *surface of genus g embedded in S^4 such that $H_1(X_2) \cong \bigoplus_{i=1}^{n} \Lambda_2/(t+1, c_i)$* (*resp.* $\bigoplus_{i=1}^{n} \Lambda_2/(t+1, c_i) \oplus \Lambda_2/(t-1)$).

Moreover, in Section 3, we consider the torsion pairing

$$\ell\colon \mathrm{tor}_{\mathbf{Z}} H_1(X_2) \times \mathrm{tor}_{\mathbf{Z}} H_2(X_2, \partial X_2) \longrightarrow \mathbf{Q}/\mathbf{Z},$$

which is Λ_2-bilinear and nonsingular. Here $\mathrm{tor}_{\mathbf{Z}} H$ denotes the $\mathbf{Z}$-torsion part of H. Let $\mathfrak{N}$ be the monoid of isomorphism classes of odd order finite abelian groups with nonsingular symmetric bilinear form. According to Poincaré duality and Universal coefficient theorem, $\mathrm{tor}_{\mathbf{Z}} H_1(X_2)$ is canonically isomorphic to $\mathrm{tor}_{\mathbf{Z}} H_2(X_2, \partial X_2)$. Since these groups are of odd order by Theorems 1 and 1′, ℓ determines an element of $\mathfrak{N}$. Since the structure of $\mathfrak{N}$ is known (cf. [5]), we can prove the following

THEOREM 3. *Let ℓ be an element of $\mathfrak{N}$ and g be a positive integer. Then there exists a closed connected non-orientable surface of genus g embedded in S^4 such that its torsion pairing corresponds to ℓ. There also exists an orientable one.*

§1. Proof of Theorems 1 and 1′.

We will give a rather detailed proof of Theorem 1 and only an outline of that of Theorem 1′. First we assume that F is non-orientable. Let $C_*(X)$ be the cellular chain complex of X with integral coefficients. Tensoring the chain complex of $\mathbf{Z}$-free modules $C_*(X)$ to the exact sequence $0 \to \Lambda_2/(t+1) \to \Lambda_2 \to \Lambda_2/(t-1) \to 0$, we have a short exact sequence of chain complexes of Λ_2-modules

$$0 \longrightarrow C_*(X) \underset{\mathbf{Z}}{\otimes} (\Lambda_2/(t+1)) \longrightarrow C_*(X) \underset{\mathbf{Z}}{\otimes} \Lambda_2 \longrightarrow C_*(X) \underset{\mathbf{Z}}{\otimes} (\Lambda_2/(t-1)) \longrightarrow 0 \tag{1.1}$$

Note that $C_*(X) \underset{\mathbf{Z}}{\otimes} \Lambda_2$ (resp. $C_*(X) \underset{\mathbf{Z}}{\otimes} (\Lambda_2/(t-1))$) is naturally isomorphic to $C_*(X_2)$ (resp. $C_*(X)$) and we introduce the abbreviation $\hat{C}_* = C_*(X) \underset{\mathbf{Z}}{\otimes} (\Lambda_2/(t+1))$. Since $\hat{C}_* \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_2$ is isomorphic to $C_*(X) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_2$, we have $H_*(\hat{C}; \mathbf{Z}_2) \cong H_*(X; \mathbf{Z}_2)$. In the derived homology exact sequence of (1.1), it is easily seen that $\partial: H_1(X) \to H_0(\hat{C})$ is an isomorphism. Thus $H_1(\hat{C}; \boldsymbol{Z}_2) \cong H_1(X; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2$ implies $H_1(\hat{C}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_2 = 0$. So we see that $H_1(X_2)$ is finite of odd order and so is $H_1(X_2, \partial X_2)$. Now we remark that $(t+1)H_*(\hat{C}) = 0$, therefore we obtain $(t+1)H_1(X_2) = 0$ and $H_1(X_2)$ is isomorphic to $\bigoplus_{i=1}^{n} \Lambda_2/(t+1, c_i)$, where c_i are odd integers.

LEMMA 1.1. *As Λ_2-modules, $H_1(X_2)$, $H_1(X_2, \partial X_2)$, $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2)$ and $\mathrm{tor}_{\boldsymbol{Z}}(X_2, \partial X_2)$ are isomorphic to each other.*

PROOF. Using $H_3(X_2, \partial X_2) \cong H^1(X_2) = 0$, we consider the homology exact sequence of the pair $(X_2, \partial X_2)$:

$$0 \longrightarrow H_2(\partial X_2) \longrightarrow H_2(X_2) \longrightarrow H_2(X_2, \partial X_2) \longrightarrow H_1(\partial X_2) \longrightarrow \cdots.$$

Since ∂X_2 is not only an orientable 3-manifold but also the total space of S^1-bundle over the non-oricntable surface F, $H_2(\partial X_2)$ is isomorphic to $\boldsymbol{Z}^{g-1}$ and $H_1(\partial X_2)$ is isomorphic to $\boldsymbol{Z}^{g-1} \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ (resp. $\boldsymbol{Z}^{g-1} \oplus \boldsymbol{Z}_4$) if $e \equiv 0 \pmod 4$ (resp. $e \equiv 2 \pmod 4$), where e is the Euler number of the normal bundle $N(F) \longrightarrow F$ and even in our situation. For the reason that the $\boldsymbol{Z}$-torsion of $H_2(X_2, \partial X_2)$ is odd torsion, the above exact sequence implies that $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2)$ is isomorphic to $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2, \partial X_2)$ as Λ_2-module. On the other hand, $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2)$ (resp. $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2, \partial X_2)$) is isomorphic to $H_1(X_2, \partial X_2)$ (resp. $H_1(X_2)$) as $\boldsymbol{Z}$-module by Poincaré duality and universal coefficient theorem. So, to conclude the proof of the lemma, we have only to show that $(t+1)$ $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2) = 0$. We consider the derived homology exact sequence of (1.1):

$$H_3(X) \longrightarrow H_2(\hat{C}) \xrightarrow{f} H_2(X_2) \xrightarrow{h} H_2(X) \tag{1.2}$$

Note that $H_3(X) = 0$ and $H_2(X) \cong \boldsymbol{Z}^{g-1}$ by Alexander duality. Since $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2)$ is a Λ_2-submodule of $\mathrm{Im}\, f$ and $(t+1)H_2(\hat{C}) = 0$, we obtain the desired result.

This lemma and the fact $H_1(X_2) \cong \bigoplus_{i=1}^{n} \Lambda_2/(t+1, c_i)$ imply (1) of Theorem 1. It is also easy to see that (3) of Theorem 1 holds. So, we shall prove (2) of Theorem 1 hereafter.

For a finitely generated Λ_2-module H, we denote the Λ_2-module $H/\mathrm{tor}_{\boldsymbol{Z}} H$ by $\bar{H}$. Then, the induced short exact sequence $0 \to \overline{H_2(\hat{C})} \to \overline{H_2(X_2)} \to \mathrm{Im}\, h \to 0$ from (1.2) reduces to the following short exact sequence of Λ_2-modules

$$0 \longrightarrow (\Lambda_2/(t+1))^g \longrightarrow \overline{H_2(X_2)} \longrightarrow (\Lambda_2/(t-1))^{g-1} \longrightarrow 0.$$

By the calculation of Euler characteristic, we have $\mathrm{rank}_{\boldsymbol{Z}} H_2(X_2) = 2g-1$ and

$\operatorname{rank}_{\boldsymbol{Z}} H_2(\hat{C}) = g$. Since $\operatorname{Ext}^1_{\Lambda_2}(\Lambda_2/(t-1), \Lambda_2/(t+1)) \cong \boldsymbol{Z}_2$ and the corresponding extended module is Λ_2 or $(\Lambda_2/(t+1)) \oplus (\Lambda_2/(t-1))$, we have

$$\overline{H_2(X_2)} \cong \Lambda_2^k \oplus (\Lambda_2/(t+1))^{\ell+1} \oplus (\Lambda_2/(t-1))^{\ell} \qquad (1.3),$$

for some non negative integers k and ℓ satisfying $k + \ell = g - 1$. To prove (2) of Theorem 1 for $H_2(X_2)$, it is enough to show $\ell = 0$. We first show the following lemma.

LEMMA 1.2. *$H_2(X_2)$ is isomorphic to $\overline{H_2(X_2)} \oplus \operatorname{tor}_{\boldsymbol{Z}} H_2(X_2)$ as Λ_2-module.*

PROOF. We will show that $\operatorname{Ext}^1_{\Lambda_2}(\overline{H_2(X_2)}, \operatorname{tor}_{\boldsymbol{Z}} H_2(X_2)) = 0$. By the above argument and lemma 1.1, it is sufficient to show that

$$\operatorname{Ext}^1_{\Lambda_2}((\Lambda_2/(t+1)), \Lambda_2/(t+1, c)) = 0 \qquad (1.4)$$

$$\text{and } \operatorname{Ext}^1_{\Lambda_2}((\Lambda_2/(t-1)), \Lambda_2/(t+1, c)) = 0 \qquad (1.5),$$

where c is odd. To calculate the Ext group (1.4), we take a Λ_2-free resolution of $\Lambda_2/(t+1)$:

$$\cdots \longrightarrow \Lambda_2 \xrightarrow{t-1} \Lambda_2 \xrightarrow{t+1} \Lambda_2 \longrightarrow \Lambda_2/(t+1) \longrightarrow 0$$

Applying $\operatorname{Hom}_{\Lambda_2}(-, \Lambda_2/(t+1, c))$ to this, we obtain the following.

$$\Lambda_2/(t+1, c) \xrightarrow{0} \Lambda_2/(t+1, c) \xrightarrow{-2} \Lambda_2/(t+1, c) \longrightarrow \cdots$$

Since c is odd, $-2\colon \Lambda_2/(t+1, c) \to \Lambda_2/(t+1, c)$ is an isomorphim. and hence (1.4) holds. Similarly, (1.5) also holds.

Next we calculate $H^3_{\Lambda_2}(X_2; \Lambda_2/(t-1))$ the third cohomology of $\operatorname{Hom}_{\Lambda_2}(C_*(X_2), \Lambda_2/(t-1))$ by using the universal coefficient spectral sequence (cf. [3]). This spectral sequence induces a filtration

$$H^3_{\Lambda_2}(X_2; \Lambda_2/(t-1)) = J_{3,0} \supset J_{2,1} \supset J_{1,2} \supset J_{0,3} \supset J_{-1,4} = 0$$

with $J_{p,q}/J_{p-1,q+1} \cong E_\infty^{p,q}$ and $E_2^{p,q} = \operatorname{Ext}^q_{\Lambda_2}(H_p(X_2), \Lambda_2/(t-1))$ and differential d_r has degree $(1-r, r)$. To obtain the E_2-term, we need the following lemma.

Lemma 1.3.

(1) $\operatorname{Ext}^i_{\Lambda_2}(\Lambda_2/(t+1), \Lambda_2/(t-1)) \cong \boldsymbol{Z}_2$ *(i = odd) or* 0 *(i = even).*

(2) $\operatorname{Ext}^i_{\Lambda_2}(\Lambda_2/(t-1), \Lambda_2/(t-1)) \cong \boldsymbol{Z}_2$ *(i = even ≥ 2) or* 0 *(i = odd) or* $\Lambda_2/(t-1)$ *($i = 0$).*

(3) $\operatorname{Ext}^i_{\Lambda_2}(\Lambda_2/(t+1, c), \Lambda_2/(t-1)) = 0$ *for all i, where c is an odd integer.*

PROOF. (1) and (2) are easily seen, so we omit the proofs. To calculate (3), take the following Λ_2-free resolution of $\Lambda_2/(t+1, c)$.

$$\cdots \longrightarrow \Lambda_2^2 \xrightarrow{\partial_3} \Lambda_2^2 \xrightarrow{\partial_2} \Lambda_2^2 \xrightarrow{\partial_1} \Lambda_2 \longrightarrow \Lambda_2/(t+1, c) \longrightarrow 0$$

∂_i are represented by the following matrices:

$$\partial_1;(t+1\ c),\ \partial_2;\begin{pmatrix} t-1 & -c \\ 0 & t+1 \end{pmatrix},\ \partial_3;\begin{pmatrix} c & t+1 \\ t-1 & 0 \end{pmatrix},\ \partial_4;\begin{pmatrix} t+1 & 0 \\ -c & t-1 \end{pmatrix},$$

$$\partial_5;\begin{pmatrix} t-1 & 0 \\ c & t+1 \end{pmatrix},\ \text{and}\ \partial_{n+2}=\partial_n\ \text{for}\ n\geq 4,$$

where every element of Λ_2^m is represented by a row vector. Applying $\mathrm{Hom}_{\Lambda_2}(-, \Lambda_2/(t-1))$ to this resolution, we obtain the desired result.

Now we are in a position to prove that $\ell=0$ in (1.3). First by Lemma 1.3 and (1), (3) of Theorem 1, we have $E_2^{p,q}=0$ for $p\neq 0, 2$. Thus d_r is the zero map for $r\neq 3$. Hence we obtain

$$H^3_{\Lambda_2}(X_2; \Lambda_2/(t-1)) \cong E_\infty^{2,1} \cong E_4^{2,1} = \mathrm{Ker}\,[d_3^{2,1}: E_3^{2,1} \longrightarrow E_3^{0,4}].$$

Substituting the right hand side of (1.3) for $\overline{H_2(X_2)}$ in Lemma 1.2, we obtain

$$E_3^{2,1} \cong \boldsymbol{Z}_2^{\ell+1} \quad \text{and} \quad E_3^{0,4} \cong \boldsymbol{Z}_2.$$

On the other hand, $\mathrm{Hom}_{\Lambda_2}(C_*(X_2), \Lambda_2/(t-1))$ is naturally isomorphic to $\mathrm{Hom}_{\boldsymbol{Z}}(C_*(X), \boldsymbol{Z})$ with the trivial action of t. So $H^*_{\Lambda_2}(X_2; \Lambda_2/(t-1))$ is isomorphic to $H^*(X)$. Since $H^3(X)=0$ by the Alexander duality, $E_\infty^{2,1}=0$ and $d_3^{2,1}$ is injective. We have proved that $d_3^{2,1}: \boldsymbol{Z}_2^{\ell+1} \to \boldsymbol{Z}_2$ in the above. Thus we obtain $\ell=0$ and determine the structure of $H_2(X_2)$. The relative homology group $H_2(X_2, \partial X_2)$ can be similarly determined and isomorphic to $H_2(X_2)$ but not canonically. This ends the proof of (2) of Theorem 1 and also that of Theorem 1.

To prove Theorem 1′ we assume that F is orientable. In this case, we note that $H_1(X_2)$ is isomorphic to $(H_1(\tilde{X})/(t+1)H_1(\tilde{X})) \oplus \Lambda_2/(t-1)$ as Λ_2-module, where $\tilde{X}$ is the infinite cyclic covering. (See (2.1) in the next section.) Since it is known that $t-1$ induces an automorphism on the first summand, $H_1(\tilde{X})/(t+1)H_1(\tilde{X})$ is finite of odd order. Thus we obtain (1′) of Theorem 1′. Moreover, the structure of the second homology can be determined by using the spectral sequence as is the non-orientable case. So we omit the proof.

§ 2. Proof of Theorem 2.

In the rest of this paper, we consider a *knotted surface* (S^4, F), that is, an embedded closed connected surface F in S^4 and use the following notation; $\Phi_i(F) = H_i(X_2)$. If F is orientable, then we denote $\tilde{\Phi}_i(F) = H_i(\tilde{X})$. Here $\tilde{X}$ is the infinite cyclic universal abelian covering of X. For knotted surfaces (S^4, F) and (S^4, F'), we consider the connected sum

$$(S^4, F)\#(S^4, F') = (S^4, F\# F').$$

Then it is easy to see that $\bar{X}_2 \approx X_2 \cup X_2'$ and $X_2 \cap X_2' \approx D^2 \times S^1$, where $\bar{X}_2$ is the double covering of the exterior of $F\# F'$ and $\approx$ means a homeomorphism. Using this splitting, we obtain the following

LEMMA 2.1. $\mathrm{tor}_{\boldsymbol{Z}}\Phi_2(F) \oplus \mathrm{tor}_{\boldsymbol{Z}}\Phi_2(F')$ *is isomorphic to* $\mathrm{tor}_{\boldsymbol{Z}}\Phi_2(F\# F')$ *as* Λ_2*-module.*

PROOF. Consider the Mayer-Vietoris exact sequence of the splitting $(\bar{X}_2, X_2, X_2')$.

If F is non-orientable, then $\mathrm{tor}_{\boldsymbol{Z}} H_2(X_2) \cong H_1(X_2)$ by Theorem 1. So we have the following as a corollary of Lemma 2.1.

COROLLARY 2.2. *If F and F' are non-orientable, then* $\Phi_1(F) \oplus \Phi_1(F')$ *is isomorphic to* $\Phi_1(F\# F')$ *as* Λ_2*-module.*

LEMMA 2.3. *If F is orientable and F' is non-orientable, then* $\Phi_1(F\# F')$ *is isomorphic to* $(\tilde{\Phi}_1(F)/(t+1)\tilde{\Phi}_1(F)) \oplus \Phi_1(F')$ *as* Λ_2*-module.*

PROOF. Consider the exact sequence

$$\longrightarrow H_1(\tilde{X}) \xrightarrow{t^2-1} H_1(\tilde{X}) \longrightarrow H_1(X_2) \xrightarrow{\partial_*} H_0(\tilde{X}) \xrightarrow{t^2-1} H_0(\tilde{X}) \longrightarrow$$

which is derived from the short exact sequence

$$0 \longrightarrow C_*(\tilde{X}) \xrightarrow{t^2-1} C_*(\tilde{X}) \xrightarrow{p_\#} C_*(X_2) \longrightarrow 0.$$

Here, p is the projection map $\tilde{X} \to X_2$ and $H_0(\tilde{X}) \cong \boldsymbol{Z}$. This induces an isomorphism of $\boldsymbol{Z}$-module

$$H_1(X_2) \cong (H_1(\tilde{X})/(t^2-1)H_1(\tilde{X})) \oplus H_0(\tilde{X}) \qquad (2.1).$$

Now, it is well known that $H_1(\tilde{X})$ is of type K, that is, $t-1$ is an automorphism. Hence $(t^2-1)H_1(\tilde{X}) \cong (t+1)H_1(\tilde{X})$. Moreover, we remark that the second direct summand is the image of the infinite cyclic group generated by the meridian element, which is a generator of

$H_1(X_2 \cap X_2')$. Finally notice that the direct sum decomposition (2.1) induces an isomorphism of Λ_2-module, because $t+1$ is the zero map. So this completes the proof.

The above argument also shows the following lemma.

LEMMA 2.4. *If F and F' are orientable, then* $\mathrm{tor}_{\mathbf{Z}}\Phi_1(F \# F')$ *is isomorphic to* $\mathrm{tor}_{\mathbf{Z}}\Phi_1(F) \oplus \mathrm{tor}_{\mathbf{Z}}\Phi_1(F')$ *as Λ_2-module.*

Let (S^4, S_c) be the 2-sphere in S^4 which is called the 2-twist spun of the (2, c)-torus knot (cf. [6]), where c is an odd integer, and (S^4, P) (resp. (S^4, T)) be unknotted real projective plane (resp. unknotted torus). It is easy to see that $\tilde{\Phi}_1(S_c) \cong \Lambda/(t+1, c)$, $\Phi_1(P) = 0$ and $\Phi_1(T) \cong \Lambda_2/(t-1)$. Here Λ is the integral group ring of the infinite cyclic group generated by t. We denote $(S^4, F_c) = (S^4, S_c) \# (S^4, P)$ and $(S^4, F_c') = (S^4, S_c) \# (S^4, T)$. Then $\Phi_1(F_c) \cong \Lambda_2/(t+1, c)$ and $\Phi_1(F_c') \cong \Lambda_2/(t+1, c) \oplus \Lambda_2/(t-1)$ by Lemmas 2.3 and 2.4. Thus we can prove Theorem 2 by taking $\overset{n}{\underset{i=1}{\#}} (S^4, S_{c_i}) \# (\overset{g}{\#}(S^4, P))$ (non-orientable case) or $\overset{n}{\underset{i=1}{\#}} (S^4, S_{c_i}) \# (\overset{g}{\#}(S^4, T))$ (orientable case).

§3. Proof of Theorem 3.

First we present the following proposition. The first isomorphism is easily obtained by the direct calculation. The other isomorphisms can be proved by the same method of Levine [3, p.12] and we omit the proofs.

PROPOSITION 3.1. *Let A be a finitely generated Λ_2-module of odd order and assume that* $(t+1)A = 0$. *Then*

$$A \cong \mathrm{Ext}^1_\Lambda(A, \Lambda_2) \cong \mathrm{Hom}_\Lambda(A, \mathbf{Q}/\mathbf{Z} \underset{\mathbf{Z}}{\otimes} \Lambda_2) \cong \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z}),$$

where $\cong$ means a Λ_2-isomorphism.

REMARK. For a finite Λ-module A, A is always isomorphic to $\mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})$ as $\mathbf{Z}$-module, but, in general, not isomorphic as Λ-module (cf. [4]).

Poincaré duality and universal coefficient theorem induce a canonical Λ_2-isomorphism $\mathrm{tor}_{\mathbf{Z}}H_1(X_2) \cong \mathrm{tor}_{\mathbf{Z}}H_2(X_2, \partial X_2)$. Using Proposition 3.1 and this isomorphism, we have the pairing $\ell\colon \mathrm{tor}_{\mathbf{Z}}H_1(X_2) \times \mathrm{tor}_{\mathbf{Z}}H_2(X_2, \partial X_2) \to \mathbf{Q}/\mathbf{Z}$, which is stated in Introduction.

The monoid $\mathfrak{N}$, as stated in Introduction, is decomposed into direct sum of monoids $\mathfrak{N}_p$ corresponding to p-primary groups for odd primes p. $\mathfrak{N}_p$ is generated by $A(p^k)$ and $B(p^k)$ ($k \geq 1$). Here $A(p^k)$ (resp. $B(p^k)$) denotes the form

ℓ over the cyclic group of order p^k, generated by x, with $\ell(x, x) = a/p^k$, where a is a residue (resp. a non-residue) (cf. [5]). We consider the 2-twist spun of the 2-bridge knot of type n/m with G.C.D. $(m, n) = 1$ and $m =$ odd. We denote it by $(S^4, S_{m,n})$. It is a fibered knot and its fiber is the punctured lens space $L(m, n)$. Farber [1] and Levine [3] showed that the torsion pairing on $\tilde{\Phi}_1(S_{m,n})$, which we denote by $\tilde{\ell}$, is isomorphic to the linking pairing on $H_1(L(m, n))$. Note that $\tilde{\Phi}_1(S_{m,n}) \cong \Lambda/(t+1, m)$. Therefore when $m = p^k$ this pairing is $A(p^k)$ or $B(p^k)$, if n is a residue or a non-residue respectively. Moreover, $\tilde{\Phi}_1(S_{m,n})$ is isomorphic to $\Phi_1(S_{m,n})$ as Λ_2-module and $\tilde{\ell}$ is also isomorphic to ℓ by the natural projection $\tilde{X} \to X_2$. Since it is easy to see that the connected sum of knotted surfaces induces a direct sum decomposition of the corresponding linking pairing, Theorem 3 holds by taking an appropriate connected sum of $(S^4, S_{m,n})$ with $m = p^k$, (S^4, P) and (S^4, T).

References

[1] M. Š. Farber, Duality in an infinite cyclic covering and even-dimensional knots, Math. USSR-Izv., **11** (1977), 749–781.

[2] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-spaces, Osaka J. Math., **16** (1979), 233–248.

[3] J. Levine, Knot modules. I, Trans. Amer. Math. Soc., **229** (1977), 1–50.

[4] M. Sekine, On the minimum number of generators of a finite module over $\boldsymbol{Z}[t, t^{-1}]$, Kobe J. Math., **6** (1989), 159–162.

[5] C. T. C. Wall, Quadratic forms on finite groups, and related topics, Topology, **2** (1964), 281–298.

[6] E. C. Zeeman, Twisting spun knots, Trans. Amer. Math. Soc., **115** (1965), 471–495.

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