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Spherical hyperfunctions on the tangent space of symmetric spaces

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Introduction

Let G be connected semisimple Lie group, σ an involutive automorphism of G and H an open subgroup of fixed points of σ. Then G/H is called a semisimple symmetric space and the tangent space at the origin of G/H is identified with a complement $\mathfrak{q}$ of $\mathfrak{h}$ in $\mathfrak{g}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras coresponding to G and H, respectively.

In this paper, we consider spherical hyperfunctions on $\mathfrak{q}$ that are H-invariant and simultaneously eigen hyperfunctions on $\mathfrak{q}$. There have appeared several papers dealing with spherical functions on $\mathfrak{q}$ ([1], [2], [3], [5], [9], [10]). In his paper [2], van Dijk listed up spherical distributions for the rank 1 case. On the other hand, in his paper [1], Cerezo determined the dimension of $O(p, q)$ (or $SO_0(p, q)$) invariant spherical hyperfunctions on $\boldsymbol{R}^{p+q}$, where $\boldsymbol{R}^{p+q}$ can be regarded as the tangent space of the semisimple symmetric space; $SO_0(p+1, q)/SO_0(p, q)$. However, studying spherical hyperfunctions, the author found interesting phenomenon. That is; if f is an H-invariant eigen hyperfunction then f is $\tilde{H}$-invariant, where $\tilde{H}$ is the connected component of the Lie group of all non-singular transformations T on $\mathfrak{q}$ such that $p(Tx) = p(x)$ for any H-invariant polynomial p and $x \in \mathfrak{q}$. In fact, $\tilde{H}$ is "large" (if $G = SL(m+1, \boldsymbol{R})$ and $H = GL^+(m, \boldsymbol{R})$, then $\dim H = m^2$ and $\dim \tilde{H} = 2m^2 - m$). It seems that this phenomenon is independent of the category of functions but is dependent on H or $\tilde{H}$ orbits structure on $\mathfrak{q}$. In his paper [8], Ochiai deals with this problem as $\mathscr{D}$-module structure generated by the Lie algebra $\mathfrak{h}$ or $\tilde{\mathfrak{h}}$ which is the Lie algebra corresponding to $\tilde{H}$.

In this paper, we prove that for "generic" eigen values if f is an H-invariant eigen hyperfunction then f is $\tilde{H}$-invariant (see Theorem 5.1 in §5). From Cerezo's result and Theorem 5.1, we can determine the dimension of spherical hyperfunctions on $\mathfrak{q}$ when rank $\mathfrak{q} = 1$ and eigen value $\mu \neq 0$ (see §5).

§ 0. Notations and preliminaries

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Killing form B and σ an involutive automorphism of $\mathfrak{g}$. Denote $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ the corresponding decomposition on $\mathfrak{g}$ into $+1$ and -1 eigenspaces of σ. In this paper, we denote by V_C the complexification of V, for any $\boldsymbol{R}$-vector space V. Then σ can be extended uniquely to the involutive automorphism (over $\boldsymbol{C}$) of $\mathfrak{g}_C$ and $\mathfrak{g}_C=\mathfrak{h}_C+\mathfrak{q}_C$ the corresponding decomposition on $\mathfrak{g}_C$ into $+1$ and -1 eigenspaces of extended σ. Let G be the connected adjoint group of $\mathfrak{g}$ and H the connected Lie subgroup of G with the Lie algebra $ad\mathfrak{h}$. Then H acts on $\mathfrak{q}$ by the adjoint action. This action is analytic and can be extended uniquely to the holomorphic action on $\mathfrak{q}_C$. Let $P(\mathfrak{q}_C)$ and $S(\mathfrak{q}_C)$ be the polynomial ring and the symmetric algebra on $\mathfrak{q}_C$, respectively. Denote by $P_H(\mathfrak{q}_C)$ and $S_H(\mathfrak{q}_C)$ the subalgebras of all H-invariant polynomials on $\mathfrak{q}_C$ and H-invariant elements in $S(\mathfrak{q}_C)$, respectively.

We denote by $\mathscr{B}(\mathfrak{q})$ the vector space of all hyperfunctions on $\mathfrak{q}$. Let $GL(\mathfrak{q})$ be a Lie group of all non-singular linear transformations on $\mathfrak{q}$. Then $GL(\mathfrak{q})$ acts on $\mathfrak{q}$ naturally. Let A be a subgroup of $GL(\mathfrak{q})$. We denote by $\mathscr{B}^A(\mathfrak{q})$ the subspace (of $\mathscr{B}(\mathfrak{q})$) of all A-invariant hyperfunctions. For each $\lambda\in\mathfrak{q}_C$, put $\chi_\lambda(e)=\nu(e)(\lambda)$ (for the definition ν, see § 2), for $e\in S_H(\mathfrak{q}_C)$. Conversely, for any character χ of $S_H(\mathfrak{q}_C)$, there exists $\lambda\in\mathfrak{q}_C$ such that $\chi_\lambda=\chi$. Indeed, the map; $\lambda\mapsto(p_1(\lambda),\cdots,p_l(\lambda))$ is of $\mathfrak{q}_C$ onto $\boldsymbol{C}^l$, where $p_1,\cdots,p_l$ are homogeneous H-invariant polynomials on q_C and $P_H(\mathfrak{q}_C)=\boldsymbol{C}[p_1,\cdots,p_l]$ (that is a polynomial ring and see [7]).

For each $\lambda\in\mathfrak{q}_C$, We denote by $\mathscr{B}_\lambda(\mathfrak{q})$ the subspace (of $\mathscr{B}(\mathfrak{q})$) of all hyperfunctions f such that $(\partial e)f=\nu(e)(\lambda)f$ for any $e\in S_H(\mathfrak{q}_C)$ (for the definition of ∂, see § 2). For each subgroup A of $GL(\mathfrak{q})$ and $\lambda\in\mathfrak{q}_C$, denote $\mathscr{B}_\lambda^A(\mathfrak{q})=\mathscr{B}_\lambda(\mathfrak{q})\cap\mathscr{B}^A(\mathfrak{q})$. An element f in $\mathscr{B}_\lambda^A(\mathfrak{q})$ is called an A-invariant eigen hyperfunction.

§ 1. Regular elements

In this section, we give two definitions of regular elements in two different ways and consider about their relations.

Let $\mathfrak{g}$ be complex semisimple Lie algebra. Let t be an indeterminate and consider the polynomial;

$$\det(t-adX)=t^N+\Delta_1(X)t^{N-1}+\cdots+\Delta_N(X),$$

where $N=\dim\mathfrak{g}$ and $\det A$ is the determinant of A. Then Δ_k is a homogeneous polynomial function on $\mathfrak{g}$ with degree k. Let m be the smallest integer such that Δ_m is not identically zero. It is well known that $N-m$ coincides with the dimension L of a Cartan subalgebra of $\mathfrak{g}$. Put $\Delta=\Delta_m=\Delta_{N-L}$. Let $\tilde{\mathscr{R}}_{\mathfrak{g}}$ be the set of all elements $X\in\mathfrak{g}$ such that $\Delta(X)\neq 0$.

On the other hand, for any $X\in\mathfrak{g}$, let $\mathfrak{g}^X$ be the centralizer of X in $\mathfrak{g}$ and $\mathscr{R}_{\mathfrak{g}}$ the set of all elements $X\in\mathfrak{g}$ such that $\dim\mathfrak{g}^X\leq\dim\mathfrak{g}^Y$ for all $Y\in\mathfrak{g}$. That is sim $\mathfrak{g}^X=L$. Then we have the following assertion.

PROPOSITION 1.1. $\tilde{\mathscr{R}}_{\mathfrak{g}}\subset\mathscr{R}_{\mathfrak{g}}$.

PROOF. For each $X\in\mathfrak{g}$, set $\tilde{\mathfrak{g}}^X=\{Y\in\mathfrak{g};(adX)^kY=0 \text{ for some } k\}$. It is well known that for any $X\in\tilde{\mathscr{R}}_{\mathfrak{g}}$, $\tilde{\mathfrak{g}}^X$ is a Cartan subalgebra of $\mathfrak{g}$. Furthermore, for any $X\in\mathfrak{g}$, $\mathfrak{g}^X\subset\tilde{\mathfrak{g}}^X$. Hence $\dim\tilde{\mathfrak{g}}^X=\dim\mathfrak{g}^X=L$ and $X\in\mathscr{R}_{\mathfrak{g}}$. Therefore $\tilde{\mathscr{R}}_{\mathfrak{g}}\subset\mathscr{R}_{\mathfrak{g}}$.

REMARK. It is not always true that $\tilde{\mathscr{R}}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}$. If $\mathfrak{g}=\mathfrak{sl}(2,\boldsymbol{C})$ then $\Delta(X)=x^2+yz$, where $X=\begin{bmatrix}x & y\\ z & -x\end{bmatrix}$. Let $e=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$. It is easily seen that $\Delta(e)=0$ and $\dim\mathfrak{g}^e=1$. Hence $e\notin\tilde{\mathscr{R}}_{\mathfrak{g}}$, but $e\in\mathscr{R}_{\mathfrak{g}}$.

Let σ be an involutive automorphism of $\mathfrak{g}$ such that $\sigma\neq 1$ and let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ be the decomposition as in §0. Put $\tilde{\mathscr{R}}_{\mathfrak{q}}=\tilde{\mathscr{R}}_{\mathfrak{g}}\cap\mathfrak{q}$. For each $Z\in\mathfrak{q}$, let $\mathfrak{q}^Z$ be the centralizer of Z in $\mathfrak{q}$ and $\mathscr{R}_{\mathfrak{q}}$ the set of all elements that $\dim\mathfrak{q}^Z\leq\dim\mathfrak{q}^Y$ for all $Y\in\mathfrak{q}$. That is; $\dim\mathfrak{q}^Z=\operatorname{rank}\mathfrak{q}=l$ if and only if $Z\in\mathscr{R}_{\mathfrak{q}}$.

PROPOSITION 1.2. $\tilde{\mathscr{R}}_{\mathfrak{q}}\subset\mathscr{R}_{\mathfrak{q}}$.

PROOF. For any $Z\in\mathfrak{q}$, we can prove that

$$\dim\mathfrak{h}-\dim\mathfrak{h}^Z=\dim\mathfrak{q}-\dim\mathfrak{q}^Z$$

by the similar way in Kostant-Rallis [7], where $\mathfrak{h}^Z$ is the centralizer of Z in $\mathfrak{h}$. On the other hand, for any $Z\in\mathfrak{q}$, $\dim\mathfrak{g}^Z=\dim\mathfrak{h}^Z+\dim\mathfrak{q}^Z$, since $\mathfrak{g}^Z=\mathfrak{h}^Z+\mathfrak{q}^Z$. Hence $\dim\mathfrak{g}^Z=\dim\mathfrak{h}-\dim\mathfrak{q}+2\dim\mathfrak{q}^Z$ for any $Z\in\mathfrak{q}$. It implies that $\dim\mathfrak{g}^Z=L$ if and only if $\dim\mathfrak{q}^Z=l$. It follows that $\mathscr{R}_{\mathfrak{q}}=\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{q}$. Therefore $\tilde{\mathscr{R}}_{\mathfrak{q}}\subset\mathscr{R}_{\mathfrak{q}}$ from Proposition 1.1.

§2. Polynomial differential operators

Let V be a vector space over $\boldsymbol{R}$ of finite dimension n. We consider the symmetric algebra $S(V_C)$ over the complexification V_C of V. For any $X\in V$, let $\partial(e)$ denote the differential operator on V given by

$$(\partial(e)f)(x)=\frac{d}{dt}\bigg|_{t=0}f(x+te)\qquad(x\in V, f\in C^\infty(V), t\in\boldsymbol{R}).$$

Then it is well known that the mapping $e \to \partial(e)$ can be extended uniquely to the algebraic isomorphism of $S(V_C)$ (over $\boldsymbol{C}$) into the algebra of differential operators on V. Now suppose there is given a real non-degenerate symmetric bilinear form $B(u, v)(u, v \in V)$ on V. We extend this form B on V_C by its linearity. Let $P(V_C)$ be the algebra of all polynomial functions on V_C and ν denote the linear isomorphism of V_C into $P(V_C)$ given by $\nu(e)(z) = B(e, z)(z, e \in V_C)$. Then it is obvious that the mapping $e \to \nu(e)$ can be extended uniquely to the algebraic isomorphism of $S(V_C)$ onto $P(V_C)$. For each non-negative integer m, we denote $P^m(V_C)$ the subalgebra of all homogeneous polynomial functions of the degree m on V_C and $S^m(V_C)$ the inverse image of $P^m(V_C)$ by ν.

Let $\mathscr{D}(V)$ be the algebra of all differential operators on V. Then $\mathscr{D}(V) \supset C^\infty(V)$ and therefore $P(V_C)$ and $\partial(S(V_C))$ are both subalgebras of $\mathscr{D}(V)$. Let $\mathscr{D}_P(V)$ denote the subalgebra of $\mathscr{D}(V)$ generated by $P(V_C) \cup \partial(S(V_C))$. The elements of $\mathscr{D}_P(V)$ will be called polynomial differential operators on V.

Now, we consider differential operators on V_C. We define the differential operator ∂' on V_C such that $(\partial'(e)f)(z) = \left.\frac{d}{dt}\right|_{t=0} f(z + te)$ for any $e \in V_C$, $z \in V_C$, $f \in C^\infty(V_C)$ and $t \in \boldsymbol{R}$. Then, for $e \in V_C$, $\partial'(e)$ is a first order C^∞-differential operator on V_C. So we can define a C^∞-differential operator $\tilde{\partial}(e)$ on V_C such that $\tilde{\partial}(e) = \frac{1}{2}(\partial'(e) - \partial'(ie))$ for $e \in V$, where $i = \sqrt{-1}$. Then $\tilde{\partial}(e)$ is a holomorphic differential operator on V_C for each $e \in V$. Indeed, for each $z \in V_C$, let $Hol_z(V_C)$ be a subspace (of $T_z^{\boldsymbol{C}}(V_C)$) of all elements v such that $J_z(v) = iv$, where $T_z^{\boldsymbol{C}}(V_C)$ is the complexification of the tangent space $T_z(V_C)$ of V_C at z in V_C and J_z the canonical complex structure. It is easily seen that $(\tilde{\partial}(e))_z \in Hol_z(V_C)$, for any $z \in V_C$. Then it is obvious that the mapping $e \to \tilde{\partial}(e)$ can be extended uniquely to the algebraic isomorphism of $S(V_C)$ (over $\boldsymbol{C}$) into the algebra of holomorphic differential operators on V_C. Let $\tilde{\mathscr{D}}_P(V_C)$ denote the subalgebra of the algebra of holomorphic differential operators on V_C generated by $P(V_C) \cup \tilde{\partial}(S(V_C))$. Then we can identify $\mathscr{D}_P(V)$ with $\tilde{\mathscr{D}}_P(V_C)$ by the algebraic isomorphism defined by $p\partial(e) \to p\tilde{\partial}(e)$, for $p \in P(V_C)$ and $e \in S(V_C)$. In this paper, under the above identification, we use the same notation ∂. That is if f is a C^∞-function on V_C, we write $(\partial(e))f$ instead of $(\tilde{\partial}(e))f$.

Let $\mathfrak{X}(V)$ be the Lie algebra of all C^∞-vector fields on V. Then $\mathscr{D}(V) \supset \mathfrak{X}(V)$. We put $\mathfrak{X}_P(V) = \mathscr{D}_P(V) \cap \mathfrak{X}(V)$. Then $\mathfrak{X}_P(V)$ is a Lie subalgebra of $\mathfrak{X}(V)$. Let E denote the Euler's vector field over V, that is,

$$Ef(x) = \left.\frac{d}{dt}\right|_{t=0} f(x + tx) \qquad (f \in C^\infty(V),\ x \in V,\ t \in \boldsymbol{R}).$$

We denote by $\mathfrak{X}_P^0(V)$ the Lie algebra of all vector fields X $(\in \mathfrak{X}_P(V))$ such that $[E, X]=0$, where $[D_1, D_2]=D_1D_2-D_2D_1$. Indeed, if $X, Y\in\mathscr{X}_P^0(V)$ then

$$[E, [X, Y]]=[X, [E, Y]]+[[E, X], Y]=0 \qquad \text{(Jacobi's identity)}.$$

Hence $[X, Y]\in\mathfrak{X}_P^0(V)$.

Let $\mathfrak{X}_P^0(V; \boldsymbol{R})$ denote the Lie subalgebra (over $\boldsymbol{R}$) (of $\mathfrak{X}_P^0(V)$) of all vector fields $X\in\mathfrak{X}_P^0(V)$ such that Xf is a real-valued function for any real-valued function f. Then it is clear that $\mathfrak{X}_P^0(V; \boldsymbol{R})$ is a real form of $\mathfrak{X}_P^0(V)$.

Let $\mathfrak{gl}(V)$ be the Lie algebra of all linear transformations of V into itself. We define a mapping φ; $\mathfrak{gl}(V)\to\mathscr{D}(V)$ by

$$(\varphi(T)f)(x)=\frac{d}{dt}\bigg|_{t=0} f(x-tT(x)) \qquad (T\in\mathfrak{gl}(V),\ x\in V, f\in C^\infty(V)).$$

PROPOSITION 2.1. *φ is a Lie algebra isomorphism of* $\mathfrak{gl}(V)$ *onto* $\mathfrak{X}_P^0(V; \boldsymbol{R})$.

PROOF. Choose a basis $v_1,\cdots,v_n$ of V. For each $T\in\mathfrak{gl}(V)$, let $M(T)$ be a matrix representation of T with respect to this basis $\{v_1,\cdots,v_n\}$; That is $M(T)=(a_{ij}(T))$, where $Tv_i=\sum a_{ji}(T)v_j$. We identify V with $\boldsymbol{R}^n$ by the mapping; $x=x_1v_1+\cdots+x_nv_n\mapsto(x_1,\cdots,x_n)$. Under this identification, we have the following expression;

$$\varphi(T)=-(x_1,\cdots,x_n)\begin{bmatrix} a_{11}(T)\cdots a_{n1}(T) \\ \vdots \qquad \vdots \\ a_{1n}(T)\cdots a_{nn}(T)\end{bmatrix}\begin{bmatrix}\partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n\end{bmatrix}.$$

The above expression may be written simply

$$\varphi(T)=-x\,{}^tM(T)\frac{\partial}{\partial x}.$$

It is easily seen that $\varphi(T)\in\mathfrak{X}_P^0(V; \boldsymbol{R})$ and φ is a linear map. Moreover if $\varphi(T)=0$ it is obvious that $T=0$. Hence φ is injective. Since $\dim\mathfrak{gl}(V)=n^2$ and $\dim\mathfrak{X}_P^0(V; \boldsymbol{R})=n^2$, it follows that φ is bijective. Finally, we shall show that φ is a Lie algebra homomorphism. Indeed, for $S, T\in\mathfrak{gl}(V)$,

$$\varphi([S, T])=-x\,{}^tM([S, T])\frac{\partial}{\partial x}=-x\,{}^t(M(S)M(T)-M(T)M(S))\frac{\partial}{\partial x}$$

$$=x[{}^tM(S), {}^tM(T)]\frac{\partial}{\partial x}=\left[-x\,{}^tM(S)\frac{\partial}{\partial x}, -x\,{}^tM(T)\frac{\partial}{\partial x}\right]=[\varphi(S), \varphi(T)].$$

Since the above proof is independent of the choice of a basis, the proposition is proved.

REMARK. Let $\mathfrak{gl}(V)_C$ be the complexification of $\mathfrak{gl}(V)$. But, whenever convenient, we can regard an element of $\mathfrak{gl}(V)_C$ also as a C-linear transformation on V_C. We recall that $\mathfrak{X}_P^0(V)$ is the complexification of $\mathfrak{X}_P^0(V; \boldsymbol{R})$. Thus φ can be extended uniquely to a Lie algebra isomorphism (over $\boldsymbol{C}$) of $\mathfrak{gl}(V)_C$ onto $\mathfrak{X}_P^0(V)$. Under the identification of $\mathscr{D}_P(V)$ with $\tilde{\mathscr{D}}_P(V_C)$, we regard $X \in \mathfrak{X}_P^0(V)$ as a holomorphic vector field on V_C.

For each $e \in S(V_C)$, let μ_e be the derivation of $\mathscr{D}_P(V)$ given by $\mu_e(D) = [\partial(e), D]$ $(D \in \mathscr{D}_P(V))$. On the other hand, for each $q \in P^1(V_C)$, there exists unique derivation δ_q of $S(V_C)$ such that $\delta_q(v) = \langle v, q\rangle$ $(v \in V_C)$, where $\langle v, q\rangle = \nu(v)(q)(0)$. Let m be a positive integer, then we have

PROPOSITION 2.2. *If* $q_j \in P^1(V_C)$ $(1 \le j \le m)$ *then*

$$\mu_e^m(q_1 \cdots q_m) = m!\,\partial(\delta_{q_1}(e) \cdots \delta_{q_m}(e)),$$

for any $e \in S(V_C)$.

PROOF. We shall prove the proposition by induction on m. Let $\mathfrak{Z}$ be a subalgebra (of $\mathscr{D}_P(V)$) of all polynomial differential operators D such that $[\partial(v), D] = 0$ for any $v \in V_C$. It is obvious that $\partial(S(V_C)) \subset \mathfrak{Z}$. Conversely, if $D \in \mathfrak{Z}$ there exist $q_j \in P(V_C)$ and $e_j \in S^j(V_C)$ such that $D = \sum q_j \partial(e_j)$ and $\sum \partial v(q_j)\partial e_j = 0$ for any $v \in V_C$. Hence $\partial v(q_j) = 0$ for any $v \in V_C$ (for any j such that $e_j \neq 0$). Then $q_j \in P^0(V_C)$ $(= \boldsymbol{C})$, for any j such that $e_j \neq 0$. Therefore $\mathfrak{Z} = \partial(S(V_C))$.

Let $v \in V_C$, $q \in P^1(V_C)$ and $e \in S(V_C)$ then

$$[\partial v, [\partial e, q]] = [\partial e, [\partial v, q]] + [[\partial v, \partial e], q] = [\partial e, \langle v, q\rangle] = 0.$$

Hence $[\partial e, q] \in \mathfrak{Z}$. Therefore $[\partial e, q] \in \partial(S(V_C))$ for any $e \in S(V_C)$ and $q \in P^1(V_C)$.

Let $m = 1$. From the above argument, for each $q \in P^1(V_C)$, we can define a linear map τ_q of $S(V_C)$ into itself such that $\tau_q(e) = \partial^{-1}[\partial e, q]$. Moreover τ_q is a derivation of $S(V_C)$. Indeed, since $\partial^{-1}[\partial e_1 e_2, q] = \partial^{-1}\{\partial e_1[\partial e_2, q] + [\partial e_1, q]\partial e_2\} = e_1\partial^{-1}[\partial e_2, q] + e_2\partial^{-1}[\partial e_1, q]$, we have $\tau_q(e_1 e_2) = e_1\tau_q(e_2) + e_2\tau_q(e_1)$ for any $e_1, e_2 \in S(V_C)$.

On the other hand, $\tau_q(v) = \partial^{-1}[\partial v, q] = \langle v, q\rangle$ for any $v \in V_C$. Therefore $\tau_q = \delta_q$ for any $q \in P^1(V_C)$. It follows that $[\partial e, q] = \partial\delta_q(e)$, for any $q \in P^1(V_C)$.

Now, let $q_1, \cdots, q_m \in P^1(V_C)$ and $e \in S^1(V_C)$, we have

$$\mu_e^m(q_1 \cdots q_m) = \sum_{0 \le k \le m} \binom{m}{k} \mu_e{}^k(q_1 \cdots q_{m-1})\, \mu_e{}^{m-k}(q_m),$$

from the Leibniz rule for derivations. But, if $m \ge 2$ then

$$\mu_e^m(q_1 \cdots q_{m-1}) = 0 \quad \text{and} \quad \mu_e^{m-k}(q_m) = 0 \qquad \text{for } 0 \le k \le m-2,$$

because $\mu_e^{m-1}(q_1 \cdots q_{m-1}) = (m-1)!\,\partial(\delta_{q_1}(e) \cdots \delta_{q_{m-1}}(e))$ and $\mu_e(q_m) = \partial\delta_{q_m}(e)$ by induction hypothesis. Hence

$$\mu_e^m(q_1 \cdots q_m) = m\mu_e^{m-1}(q_1 \cdots q_{m-1})\,\mu_e(q_m) = m!\,\partial(\delta_{q_1}(e) \cdots \delta_{q_m}(e)).$$

Therefore the proposition is proved.

Let $v_1, \cdots, v_n$ be a basis of V_C. Since B is a symmetric non-degenerate bilinear form, we can choose a basis $u_1, \cdots, u_n$ such that $B(v_i, u_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker's δ.
Put

$$\omega = \frac{1}{2} \sum_{1 \le i \le n} u_i v_i \in S^2(V_C).$$

This element ω is independent of a choice of a basis and is called the Casimir element. Then we have the following

LEMMA 2.3. *If $q \in P^m(V_C)$ then $\mu_\omega^m(q) = m!\,\partial(v^{-1}(q))$.*

PROOF. First we will show that $\delta_q(\omega) = v^{-1}(q)$ for any $q \in P^1(V_C)$. From the definition of δ_q,

$$\begin{aligned} v\delta_q(\omega) &= \frac{1}{2} v \sum \{\delta_q(u_i) v_i + \delta_q(v_i) u_i\} \\ &= \frac{1}{2} \sum \{\langle u_i, q\rangle v(v_i) + \langle v_i, q\rangle v(u_i)\}. \end{aligned}$$

Hence

$$\begin{aligned} v\delta_q(\omega)(z) &= \frac{1}{2} \sum \{q(u_i) B(v_i, z) + q(v_i) B(u_i, z)\} \\ &= \frac{1}{2} q\left(\sum \{B(v_i, z) u_i + B(u_i, z) v_i\}\right). \end{aligned}$$

But $\sum B(v_i, z) u_i = \sum B(u_i, z) v_i = z$. Therefore $\delta_q(\omega) = v^{-1}(q)$ for any $q \in P^1(V_C)$. Next, from Proposition 2.2, we have

$$\mu_\omega^m(q) = \mu_\omega^m(q_1 \cdots q_m) = m!\,\partial(v^{-1}(q_1) \cdots v^{-1}(q_m)) = m!\,\partial(v^{-1}(q)),$$

for $q = q_1 \cdots q_m (q_i \in P^1(V_C),\ 1 \le i \le m)$.
This shows that if $q \in P^m(V_C)$, then $\mu_\omega^m(q) = m!\,\partial(v^{-1}(q))$.

REMARK. Under the identification of $\mathscr{D}_P(V)$ with $\tilde{\mathscr{D}}_P(V_C)$, we have

$$\tilde{\mu}_e^m(q_1 \cdots q_m) = m!\,\tilde{\partial}(\delta_{q_1}(e) \cdots \delta_{q_m}(e)) \text{ and } \tilde{\mu}_\omega^m(q) = m!\,\tilde{\partial}(v^{-1}(q)),$$

where $\tilde{\mu}_e$ is the derivation of $\tilde{\mathscr{D}}_P(V_C)$ such that $\tilde{\mu}_e(D) = [\tilde{\partial}e, D]$.

§ 3. Analytic solutions

Let θ be a Cartan involution of $\mathfrak{g}$ such that $\theta\sigma = \sigma\theta$ (see § 0, for the notations $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{q}$, σ). Then $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p}$ (direct sum) and $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{p} + \mathfrak{q} \cap \mathfrak{k}$ (direct sum), where $\mathfrak{k} = \{X \in \mathfrak{g}; \theta X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g}: \theta X = -X\}$. It is clear that

$$\mathfrak{h}_C = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p} + i\mathfrak{h} \cap \mathfrak{k} + i\mathfrak{h} \cap \mathfrak{p} \quad \text{(direct sum as real vector spaces)},$$

$$\mathfrak{q}_C = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p} + i\mathfrak{q} \cap \mathfrak{k} + i\mathfrak{q} \cap \mathfrak{p} \quad \text{(direct sum as real vector spaces)}.$$

Set $\mathfrak{k}^d = \mathfrak{h} \cap \mathfrak{k} + i\mathfrak{h} \cap \mathfrak{p}$, $\mathfrak{p}^d = \mathfrak{q} \cap \mathfrak{p} + i\mathfrak{q} \cap \mathfrak{k}$ and $\mathfrak{g}^d = \mathfrak{k}^d + \mathfrak{p}^d$. Let G^d (or G_C^d) be the connected adjoint group of $\mathfrak{g}^d$ (or $\mathfrak{g}_C^d$) and K^d (or K_C^d) the connected Lie subgroup of G^d (or G_C^d) with Lie algebra $ad\ \mathfrak{k}^d$ (or $ad\ \mathfrak{k}_C^d$), respectively. It is known that the pair (G^d, K^d) is a Riemannian symmetric pair with the Cartan involution σ and the Killing form of $\mathfrak{g}^d$ is the restriction of the Killing form B of $\mathfrak{g}_C$. We define the linear map ξ (over $\boldsymbol{R}$) of $\mathfrak{g}_C$ into $\mathfrak{g}_C^d$ such that

$$\xi(e \otimes a) = e \otimes a \qquad \text{for } e \in \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p},\ a \in \boldsymbol{C}$$

$$\xi(e \otimes a) = (ie) \otimes (-ia) \qquad \text{for } e \in \mathfrak{h} \cap \mathfrak{p} + \mathfrak{q} \cap \mathfrak{k},\ a \in \boldsymbol{C}.$$

Then it is easily seen that ξ is a linear isomorphism (over $\boldsymbol{C}$) of $\mathfrak{g}_C$ onto $\mathfrak{g}_C^d$. By restricting this map ξ, we have the linear isomorphisms (over $\boldsymbol{C}$) of $\mathfrak{h}_C$ onto $\mathfrak{k}_C^d$ and of $\mathfrak{q}_C$ onto $\mathfrak{p}_C^d$. Moreover, it is obvious that this map ξ can be extended uniquely to the algebraic isomorphism (over $\boldsymbol{C}$) of $S(\mathfrak{q}_C)$ onto $S(\mathfrak{p}_C^d)$ and the map ξ of $\mathfrak{h}_C$ onto $\mathfrak{k}_C^d$ induces a Lie group isomorphism of H_C onto K_C^d. One can easily see that for any $h \in \mathfrak{h}_C$ and $e \in S(\mathfrak{q}_C)$ $\xi([h, e]) = [\xi(h), \xi(e)]$. Hence the restriction of ξ to $S_H(\mathfrak{q}_C)$ is an algebraic isomorphism (over $\boldsymbol{C}$) of $S_H(\mathfrak{q}_C)$ onto $S_{K^d}(\mathfrak{p}_C^d)$. Indeed, if $e \in S_H(\mathfrak{q}_C)$ then $\xi(e) \in S_{K^d}(\mathfrak{p}_C^d)$ by the above equality. Conversely, if $e \in S_{K^d}(\mathfrak{p}_C^d)$ then $\xi^{-1}(e) \in S_H(\mathfrak{q}_C)$ by the above equality. Let μ be the algebraic isomorphism of $S_{K^d}(\mathfrak{p}_C^d)$ onto $P_{K^d}(\mathfrak{p}_C^d)$ defined by the same way as the map ν. Then it is easily seen that for any $e \in S_H(\mathfrak{q}_C)$ and $\lambda \in \mathfrak{q}_C$ we have $\nu(e)(\lambda) = \mu(\xi(e))(\xi(\lambda))$, because $B(\xi(e), \xi(\lambda)) = B(e, \lambda)$ for any $e \in \mathfrak{q}_C$ and $\lambda \in \mathfrak{q}_C$.

Let φ (or ψ) be the Lie isomorphism (over $\boldsymbol{R}$) of $\mathfrak{gl}(\mathfrak{q})$ (or $\mathfrak{gl}(\mathfrak{p}^d)$) onto $\mathfrak{X}_P^0(\mathfrak{q}; \boldsymbol{R})$ (or $\mathfrak{X}_P^0(\mathfrak{p}^d; \boldsymbol{R})$) defined in § 2, respectively. Then we have the Lie isomorphism φ (or ψ) (over $(\boldsymbol{C})$ of $ad\ \mathfrak{h}_C$ (or $ad\ \mathfrak{k}_C^d$) onto $\varphi(ad\ \mathfrak{h}_C)$ (or $\psi(ad\ \mathfrak{k}_C^d)$) whose restriction to $ad\ \mathfrak{h}$ (or $ad\ \mathfrak{k}^d$) is a Lie isomorphism (over $\boldsymbol{R}$) of $ad\ \mathfrak{h}$ (or $ad\ \mathfrak{k}^d$) onto φ $(ad\ \mathfrak{h}^d)$ (or $\psi(ad\ \mathfrak{k}^d)$), respectively.

Let V be a real vector space and $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. We denote by $\mathscr{A}(U)$ the vector space of all analytic functions on U which is an open subset of V and $\mathscr{A}^{\mathfrak{a}}(U)$ the vector space of all $\varphi_V(\mathfrak{a})$-invariant analytic functions on U (where φ_V is defined by Proposition 2.1). Let A be a connected Lie subgroup of $GL(V)$ corresponding with the Lie algebra $\mathfrak{a}$. If U is A-invariant (that is, $ax \in U$ for any $a \in A$ and $x \in U$), we denote by $\mathscr{A}^A(U)$ the vector subspace (of $\mathscr{A}(U)$) of all A-invariant analytic functions on U.

Let U be an open subset of $\mathfrak{q}_C$. Then $\xi(U)$ is an open subset of $\mathfrak{p}_C^d$. Let $\mathcal{O}(U)$ (or $\mathcal{O}(\xi(U))$) be the vector space of all holomorphic functions on U (or $\xi(U)$), respectively. Then it is obvious that ξ^* is a linear isomorphism of $\mathcal{O}(\xi(U))$ onto $\mathcal{O}(U)$, where $(\xi^* F)(z) = F(\xi(z))$ for any $F \in \mathcal{O}(\xi(U))$ and $z \in U$.

LEMMA 3.1. *For any $h \in \mathfrak{h}_C$, $F \in \mathcal{O}(\xi(U))$ and $z \in U$, we have*

$$(\varphi(adh)(\xi^* F))(z) = (\psi(ad\, \xi(h))F)(\xi(z)).$$

PROOF. From the definition of φ (or ψ), we have

$$\begin{aligned}(\varphi(adh)(\xi^* F))(z) &= \frac{d}{dt}\bigg|_{t=0} (F \circ \xi)(z - t[h, z]) \\ &= \frac{d}{dt}\bigg|_{t=0} F(\xi(z) - t[\xi(h), \xi(z)]) \\ &= (\psi(ad\, \xi(h))F)(\xi(z)),\end{aligned}$$

for any $h \in \mathfrak{h}_C$, $F \in \mathcal{O}(\xi(U))$ and $z \in U$, since F is holomorphic. This implies the lemma.

For each $\lambda \in \mathfrak{q}_C$ (or $\lambda' \in \mathfrak{p}_C^d$) and an open subset U of $\mathfrak{q}$ (or $\mathfrak{p}^d$), we denote by $\mathscr{A}_\lambda(U)$ (or $\mathscr{A}_{\lambda'}(U)$) the vector space of all analytic functions f such that for any $e \in S_H(\mathfrak{q}_C)$ (or $e \in S_{K^d}(\mathfrak{p}_C^d)$) $(\partial e)f = \nu(e)(\lambda)f$ (or $(\partial e)f = \mu(e)(\lambda)f$), respectively. Set $\mathscr{A}_\lambda{}^H(U) = \mathscr{A}_\lambda(U) \cap \mathscr{A}^H(U)$, $\mathscr{A}_{\lambda'}^{K^d}(U') = \mathscr{A}_{\lambda'}(U') \cap \mathscr{A}^{K^d}(U')$, $\mathscr{A}_\lambda^{\mathfrak{h}}(U) = \mathscr{A}_\lambda(U) \cap \mathscr{A}^{\mathfrak{h}}(U)$ and $\mathscr{A}_{\lambda'}^{\mathfrak{k}^d}(U') = \mathscr{A}_{\lambda'}(U') \cap \mathscr{A}^{\mathfrak{k}^d}(U')$, for each open subset U of $\mathfrak{q}$ and U' of $\mathfrak{p}^d$.

It is well known that if $f \in \mathscr{A}(\mathfrak{q})$ then there exist a domain U of $\mathfrak{q}_C$ and unique holomorphic function $F \in \mathcal{O}(U)$ such that $U \cap \mathfrak{q} = \mathfrak{q}$ and f is the restriction of F to $\mathfrak{q}$. Set $\tilde{F} = (\xi^{-1})^* F$. Then $\tilde{F}$ is a holomorphic function on $\xi(U)$. Set $W = \xi(U) \cap \mathfrak{p}^d$. Then W is an open subset of $\mathfrak{p}^d$ and $0 \in W$. Let g be the restriction of $\tilde{F}$ to W. Then g is an analytic function on W. In this section we call that g is a pure imaginary analytic continuation of f.

LEMMA 3.2. *If $f \in \mathscr{A}_\lambda^H(\mathfrak{q})$ then $g \in \mathscr{A}_{\xi(\lambda)}^{\mathfrak{k}^d}(W)$.*

PROOF. Let $f \in \mathscr{A}^H(\mathfrak{q})$. Then $\varphi(adh)f = 0$ on $\mathfrak{q}$, for any $h \in \mathfrak{h}$. It is obvious that $\varphi(adh)F = 0$ on U for any $h \in \mathfrak{h}_C$. Here $\varphi(adh)$ is regarded as a holomorphic vector field (see Remark of Proposition 2.1). From Lemma 3.1, we have $\psi(ad\, \xi(h))\tilde{F} = 0$ on $\xi(U)$ for any $h \in \mathfrak{h}_C$, where $\tilde{F} = (\xi^{-1})^* F$. Hence

$\psi(adk)\tilde{F} = 0$ on $\xi(U)$ for any $k \in \mathfrak{k}^d$, since ξ is bijective. It implies that $\psi(adk)g = 0$ on W for any $k \in \mathfrak{k}^d$. Therefore $g \in \mathscr{A}^{\mathfrak{k}^d}(W)$.

Let $f \in \mathscr{A}_\lambda(\mathfrak{q})$. Then $(\partial e)F = v(e)(\lambda)F$ on U for any $e \in S_H(\mathfrak{q}_C)$. Here ∂e is regarded as a holomorphic differential operator (see §2). Indeed, the restricted function of $\partial(e)F - v(e)(\lambda)F$ to $\mathfrak{q}$ is zero on $\mathfrak{q}$, since $(\partial e)f = v(e)(\lambda)f$ on $\mathfrak{q}$. But $\partial(e)F - v(e)(\lambda)F$ is holomorphic on U. Hence $(\partial e)F - v(e)(\lambda)F = 0$ on U from the identity theorem for an analytic function. On the other hand, it is easily seen that for any $e \in S(\mathfrak{q}_C)$ and $z \in U$ we have $(\partial e)(\xi^* \tilde{F})(z) = \partial(\xi e)\tilde{F}(\xi(z))$. Hence, for any $e \in S_H(\mathfrak{q}_C)$ and $z \in U$, we have $\partial(\xi e)\tilde{F}(\xi(z)) = v(e)(\lambda)\tilde{F}(\xi(z))$, since $\tilde{F} = (\xi^{-1})^* F$. Therefore, by restricting the above equality to $\xi(U) \cap \mathfrak{p}^d$, we have $\partial(\xi e)g = v(e)(\lambda)g$ on W for any $e \in S_H(\mathfrak{q}_C)$. This implies that $g \in \mathscr{A}_{\xi(\lambda)}(W)$, because $v(e)(\lambda) = \mu(\xi e)(\xi\lambda)$ and ξ is bijective. Therefore the lemma is proved.

Let B be the restricted Killing form of $\mathfrak{p}^d$. It is easily seen that B is a positive definite symmetric bilinear form on $\mathfrak{p}^d$. Since $0 \in W$ and W is an open subset of $\mathfrak{p}^d$, there exists a positive number r such that if $B(x, x) < r$ and $x \in \mathfrak{p}^d$ then $x \in W$. We fix r. But r is dependent on a given analytic function f, since W is so. Let W_0 be a (connected open) subset (of W) of all elements $x \in \mathfrak{p}^d$ such that $B(x, x) < r$. Then W_0 is a K^d-invariant open subset, since B is K^d-invariant. We have the following lemma by the usual way in the analysis of Lie groups (see [6] or [11]).

LEMMA 3.3. *For any $\eta \in \mathfrak{p}^d$, we have*

$$\mathscr{A}_\eta^{\mathfrak{k}^d}(W_0) = \mathscr{A}_\eta^{K^d}(W_0) \quad \textit{and} \quad \dim \mathscr{A}_\eta^{k^d} = 1.$$

PROOF. For each $e \in S(\mathfrak{p}_C^d)$, set $\rho(e) = \int_{K^d} ke\, dk$, where dk is the normalized Haar measure of K^d such that $\int_{K^d} dk = 1$. Then ρ is the projection of $S(\mathfrak{p}_C^d)$ onto $S_{K^d}(\mathfrak{p}_C^d)$. Let $u \in \mathscr{A}_\eta^{K^d}(W_0)$. Then for any $e \in S(\mathfrak{p}_C^d)$,

$$\mu(\rho(e))u(0) = (\partial(\rho(e))u)(0) = \int_{K^d} (L_k \circ \partial e \circ L_{k^{-1}})u(0)\, dk$$

$$= \int_{K^d} ((\partial e)u)(0)\, dk = (\partial e)u(0),$$

where $(L_k u)(x) = u(k^{-1}x)(x \in \mathfrak{p}^d)$. This implies that if $u(0) = 0$ then $u = 0$ on W_0, since W_0 is connected. Therefore $\dim \mathscr{A}_\eta^{K^d}(W_0) \leq 1$ for any $\eta \in \mathfrak{p}_C^d$. It is obvious that $\mathscr{A}_\eta^{K^d}(W_0) \subset \mathscr{A}_\eta^{\mathfrak{k}^d}(W_0)$. But if $u \in \mathscr{A}_\eta^{\mathfrak{k}^d}(W_0)$ then $u \in \mathscr{A}_\eta^{K^d}(W_0)$. Indeed, for any $X \in \mathfrak{k}^d$ and $x \in W_0$,

$$\frac{d}{dt}u(e^{tX}x) = \frac{d}{ds}\bigg|_{s=0} u(e^{sX}e^{tX}x) = (\psi(adX)u)(e^{tX}x) = 0.$$

Hence $u(e^X x) - u(x) = \int_0^1 \frac{d}{dt}u(Ad(e^{tX})x)\,dt = 0$ for any $X \in \mathfrak{k}^d$ and $x \in W_0$. This implies that u is K^d-invariant, since K^d is connected. Thus we have $\mathscr{A}_\eta^{\mathfrak{k}^d}(W_0) = \mathscr{A}_\eta^{K^d}(W_0)$ for any $\eta \in \mathfrak{p}_C^d$.

For any $\eta \in \mathfrak{p}_C^d$ and $w \in \mathfrak{p}_C^d$, set

$$\Psi_\eta(w) = \int_{K^d} e^{B(kw,\eta)}\,dk.$$

Then it is clear that Ψ_η is an entire holomorphic function of $\mathfrak{p}_C^d$ such that $\Psi_\eta(0) = 1$.Moreover Ψ_η is K_C^d-invariant. Indeed it is trivial that Ψ_η is K^d-invariant. But, for each $w \in \mathfrak{p}_C^d$, it is obvious that the function $\Psi_\eta(kw) - \Psi_\eta(w)$ of K_C^d is an entire holomorphic function on K_C^d, since the adjoint action of K_C^d on $\mathfrak{p}_C^d$ is holomorphic. Hence $\Psi_\eta(kw) - \Psi_\eta(w) = 0$ for any $w \in \mathfrak{p}_C^d$ and $k \in K_C^d$ from the identity theorem for an analytic function. Therefore Ψ_η is K_C^d-invariant. Moreover, it is easily seen that $(\partial e)e^{B(kw,\eta))} = B(ke,\eta)e^{B(kw,\eta)}$ for any $e \in \mathfrak{p}_C^d$ and $k \in K^d$. Thus if $e \in S_{K^d}(\mathfrak{p}_C^d)$ then $(\partial e)e^{B(kw,\eta)} = \mu(e)(\eta)e^{B(kw,\eta)}$. Therefore $(\partial e)\Psi_\eta = \mu(e)(\eta)\Psi_\eta$ for any $e \in S_{K^d}(\mathfrak{p}_C^d)$. Let g_η be the restriction of Ψ_η to W_0. Then it is obvious that $g_\eta \in \mathscr{A}_\eta^{K^d}(W_0)$ and $g_\eta(0) = 1$. Hence the lemma is proved.

Now we have the following.

THEOREM 3.4. $\dim \mathscr{A}_\lambda^H(\mathfrak{q}) = 1$ *for any* $\lambda \in \mathfrak{q}_C$.

PROOF. Let $f_i \in \mathscr{A}_\lambda^H(\mathfrak{q})(i = 1, 2)$. Then there exist K^d-invariant open connected subset W_i $(i = 1, 2)$ of $\mathfrak{p}^d$ and analytic functions $g_i \in \mathscr{A}_{\xi(\lambda)}^{\mathfrak{k}^d}(W_i)$ such that $0 \in W_i$ and g_i is the pure imaginary analytic continuation of f_i $(i = 1, 2)$. Put $c_i = f_i(0)$ $(= g_i(0))$ $(i = 1, 2)$, $f = c_2 f_1 - c_1 f_2$, $g = c_2 g_1 - c_1 g_2$ and $W = W_1 \cap W_2$. Then it is obvious that $f \in \mathscr{A}_\lambda^H(\mathfrak{q})$, g is the pure imaginary analytic continuation of f and $g \in \mathscr{A}_{\xi(\lambda)}^{\mathfrak{k}^d}(W)$. But $g = 0$ on W, since $g(0) = 0$. From the identity theorem for an analytic function, we have $f = 0$ on $\mathfrak{q}$. It implies that $\dim \mathscr{A}_\lambda^H(\mathfrak{q}) \leq 1$ for any $\lambda \in \mathfrak{q}_C$.

Set $\Phi_\lambda = \xi^* \Psi_\eta$, where $\lambda = \xi^{-1}(\eta)$ (see Lemma 3.3, for the notations η, Ψ_η). Then Φ_λ is an H_C-invariant entire holomorphic function of $\mathfrak{q}_C$ and $(\partial e)\Phi_\lambda = \nu(e)(\lambda)\Phi_\lambda$ for any $e \in S_H(\mathfrak{q}_C)$. Indeed, for any $z \in \mathfrak{q}_C$, we have

$$\Phi_\lambda(z) = \int_{K^d} e^{B(k\xi(z),\xi(\lambda))}\,dk.$$

Since Ψ_η is K_C^d-invariant and $\xi(hz) = \xi(h)\xi(z)$ for any $h \in H_C$ and $z \in \mathfrak{q}_C$, it is clear that Φ_λ is H_C-invariant. By the same way as Lemma 3.3, we have $(\partial e)\Phi_\lambda$

$= \nu(e)(\lambda)\Phi_\lambda$ on $\mathfrak{q}_C$, for any $e \in S_H(\mathfrak{q}_C)$. Let f_λ be the restriction of Φ_λ to $\mathfrak{q}$. Then it is obvious that $f_\lambda \in \mathscr{A}_\lambda^H(\mathfrak{q})$ and $f_\lambda(0) = 1$. Therefore the theorem is proved.

Note that the technique described in this section is based on Flested-Jensen's idea in [4].

§4. The definition of $\tilde{H}$ and $\tilde{\mathfrak{h}}$

We consider a real semi-simple symmetric pair (G, H). We recall that $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and H is acting on $\mathfrak{q}$ by the adjoint action. Let $P_H(\mathfrak{q}_C)$ (or $S_H(\mathfrak{q}_C)$) be a subalgebra of $P(\mathfrak{q}_C)$ (or $S(\mathfrak{q}_C)$) of all H-invariant polynomials (or H-invariant elements) on $\mathfrak{q}_C$ as the above H-action. Then from Chevalley's theorem, $P_H(\mathfrak{q}_C) = C[p_1, \cdots, p_l]$, where p_j is a homogeneous polynomial and $C[p_1, \cdots, p_l]$ is the polynomial ring ($l = \text{rank } \mathfrak{q}$). Put $e_i = \nu^{-1}(p_i)$ $(1 \le i \le l)$. Then $S_H(\mathfrak{q}_C)$ is generated by $1, e_1, \cdots, e_l$.

Let $GL(\mathfrak{q})$ be the Lie group of all non-singular linear transformations on $\mathfrak{q}$. Then the Lie algebra of $GL(\mathfrak{q})$ is $\mathfrak{gl}(\mathfrak{q})$. Let H' be the subgroup of $GL(\mathfrak{q})$ of all non-singular linear transformations T of $\mathfrak{q}$ such that $P(Tx) = P(x)$ for any $x \in \mathfrak{q}$ and $P \in P_H(\mathfrak{q}_C)$. It is obvious that H' is a closed subgroup of $GL(\mathfrak{q})$. Thus H' is a Lie group. We denote by $\tilde{H}$ the connected component of the Lie group H'. Let $Ad(H)$ be the Lie subgroup of $GL(\mathfrak{q})$ of all non-singular transformations $Ad(h)$ $(h \in H)$. Then the Lie algebra of $Ad(H)$ is $ad\, \mathfrak{h}$ which is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{q})$ of all linear transformations adx $(x \in \mathfrak{h})$. We assume H is connected. Then the definition of $\tilde{H}$ implies that $Ad(H)$ is a connected subgroup of $\tilde{H}$. Let $\tilde{\mathfrak{h}}$ be the Lie subalgebra of $\mathfrak{gl}(\mathfrak{q})$ of all elements X such that $\varphi(X)p = 0$ for any $p \in P_H(\mathfrak{q}_C)$, where φ is defined in §2. Then it is clear that $\tilde{\mathfrak{h}}$ is the Lie algebra corresponding to $\tilde{H}$ (or H') and $\tilde{\mathfrak{h}} \supset ad\, \mathfrak{h}$.

Under the identification of $\mathscr{D}_P(\mathfrak{q})$ with $\tilde{\mathscr{D}}_P(\mathfrak{q}_C)$ (see §2), the mapping i_z; $e \mapsto (\partial e)_z$ is a linear isomorphism (over C) of $\mathfrak{q}_C$ onto $Hol_z(\mathfrak{q}_C)$ for any $z \in \mathfrak{q}_C$. Let $[z, \mathfrak{h}_C]$ be the subspace of $\mathfrak{q}_C$ of all elements $[z, w]$ $(w \in \mathfrak{h}_C)$ for each $z \in \mathfrak{q}_C$ and $Hol_z(\mathfrak{q}_C; I)$ the subspace of $Hol_z(q_C)$ of all elements v such that $(dp)_z v = 0$ for any $p \in P_H(\mathfrak{q}_C)$. Then we have the following.

PROPOSITION 4.1. *If $z \in \mathscr{R}_{\mathfrak{q}_C}$ then i_z gives a linear isomorphism of $[z, \mathfrak{h}_C]$ onto $Hol_z(\mathfrak{q}_C; I)$.*

PROOF. It is trivial that the map i_z is linear and injective. But, it is obvious that $\dim_C [z, \mathfrak{h}_C] \le n - l$ for any $z \in \mathfrak{q}_C$ and $\dim_C [z, \mathfrak{h}_C] = n - l$ if and only if $z \in \mathscr{R}_{\mathfrak{q}_C}$, where $n = \dim_C \mathfrak{q}$, $l = \text{rank } \mathfrak{q}$. Indeed, for each $z \in \mathfrak{q}_C$ the map;

$$\mathfrak{h}_C / \mathfrak{h}_C^z \ni w + \mathfrak{h}_C^z \longmapsto [z, w] \in [z, \mathfrak{h}_C]$$

is well defined and a linear isomorphism of $\mathfrak{h}_{C}/\mathfrak{h}_{C}^{z}$ onto $[z, \mathfrak{h}_{C}]$ (for the notation $\mathfrak{h}_{C}^{z}$, see §1). By the similar proof of Proposition 5 in [7], we have $\dim_{C}\mathfrak{h}_{C}/\mathfrak{h}_{C}^{z} = \dim_{C}\mathfrak{q}_{C}/\mathfrak{q}_{C}^{z}$ for any $z\in\mathfrak{q}_{C}$. Hence $\dim_{C}[z, \mathfrak{h}_{C}] = n - \dim_{C}\mathfrak{q}_{C}^{z}$ for any $z\in\mathfrak{q}_{C}$. Thus we have the assertion from the definition of $\mathscr{R}_{\mathfrak{q}_{C}}$ (see §1). On the other hand, $\dim_{C} Hol_{z}(\mathfrak{q}_{C}; I) \geq n - l$ for any $z\in\mathfrak{q}_{C}$ and if $z\in\mathscr{R}_{\mathfrak{q}_{C}}$ then $\dim_{C} Hol_{z}(\mathfrak{q}_{C}; I) = n - l$. Indeed, we can easily see that

$$Hol_{z}(\mathfrak{q}_{C}; I) = \{v\in Hol_{z}(\mathfrak{q}_{C}); (dp_{j})(v) = 0 \text{ for any } j\ (1\leq j\leq l)\}$$

from the definition of $Hol_{z}(\mathfrak{q}_{C}; I)$, where $P_{H}(\mathfrak{q}_{C}) = C[p_{1},\cdots,p_{l}]$. By the similar proof of Theorem 13 in [7], we have that if $z\in\mathscr{R}_{\mathfrak{q}_{C}}$ then $(dp_{1})_{z},\cdots,(dp_{l})_{z}$ are linearly independent. Thus we have the assertion. This implies that the map is surjective. So the proposition is proved.

For each $z\in\mathfrak{q}_{C}$, we define the linear map (over C) φ_{z} of $\mathfrak{gl}(\mathfrak{q}_{C})$ into $Hol_{z}(\mathfrak{q}_{C})$ such that $\varphi_{z}(X) = (\varphi(X))_{z}$ for $X\in\mathfrak{gl}(\mathfrak{q})_{C}$. Then we have the following.

PROPOSITION 4.2. (1) *$\varphi_{z}(\tilde{\mathfrak{h}}_{C}) \subset Hol_{z}(\mathfrak{q}_{C}; I)$ for any $z\in\mathfrak{q}_{C}$,*
(2) *If $z\in\mathscr{R}_{\mathfrak{q}_{C}}$, $\varphi_{z}(ad\ \mathfrak{h}_{C}) = Hol_{z}(\mathfrak{q}_{C}; I)$.*

PROOF. For any $X\in\tilde{\mathfrak{h}}_{C}$, $z\in\mathfrak{q}_{C}$, $p\in P_{H}(\mathfrak{q}_{C})$, we have

$$(dp)_{z}(\varphi(X)_{z}) = \varphi(X)(p)(z) = 0.$$

This implies (1). From the definition of φ, for any $z\in\mathfrak{q}_{C}$ and $w\in\mathfrak{h}_{C}$, we have $\varphi(adw)_{z} = (\partial[z, w])_{z}$. By Proposition 4.1, if $z\in\mathscr{R}_{\mathfrak{q}}$ then for any $v\in Hol_{z}(\mathfrak{q}_{C}; I)$ there exists $w\in\mathfrak{h}_{C}$ such that $i_{z}([z, w]) = v$. Hence $\varphi_{z}(adw) = \varphi(adw)_{z} = (\partial[z, w])_{z} = i_{z}([z, w]) = v$. This implies (2).

Let $P(\mathfrak{q}_{C})\varphi(ad\ \mathfrak{h}_{C})$ be the Lie subalgebra (of $\mathscr{D}_{P}(\mathfrak{q}_{C})$) of all elements D such that $D = \sum p_{i}\varphi(X_{i})$ for some $p_{i}\in P(\mathfrak{q}_{C})$ and $X_{i}\in ad\ \mathfrak{h}_{C}$. Indeed, we have $[p\varphi(X), q\varphi(Y)]\in P(\mathfrak{q}_{C})\varphi(ad\ \mathfrak{h}_{C})$ (for $p, q\in P(\mathfrak{q}_{C})$, $X, Y\in ad\ \mathfrak{h}_{C}$), because

$$[p\varphi(X), q\varphi(Y)] = pq\varphi([X, Y]) + p\varphi(X)(q)\varphi(Y) - q\varphi(Y)(p)\varphi(X).$$

Then we have the following.

LEMMA 4.3. *For any $X\in\tilde{\mathfrak{h}}_{C}$ and $z\in\mathscr{R}_{\mathfrak{q}_{C}}$, there exist a polynomial $p\in P(\mathfrak{q}_{C})$ and a domain $W\subset\mathfrak{q}_{C}$ such that $z\in W$, $p(w)\neq 0$ for any $w\in W$ and $p\varphi(X)\in P(\mathfrak{q}_{C})\varphi(ad\ \mathfrak{h}_{C})$.*

PROOF. Choose a basis (over C) $v_{1},\cdots,v_{n}$ of $\mathfrak{q}_{C}$ which is a basis (over $\boldsymbol{R}$) of $\mathfrak{q}$. So we identify $\mathfrak{q}_{C}$ with C^{n} by the mapping;

$$\mathfrak{q}_{C}\ni z = z_{1}v_{1} + \cdots + z_{n}v_{n}\longmapsto(z_{1},\cdots,z_{n})\in C^{n}.$$

Under this identification, for any $X \in \mathfrak{gl}(\mathfrak{q})_C$, we have

$$\varphi(X)_z = \sum_{1 \leq j \leq n} g_j(z; X)\left(\frac{\partial}{\partial z_j}\right)_z \qquad \text{for any } z \in \mathfrak{q}_C,$$

where $g_j(z; X) = -\sum_{1 \leq i \leq n} a_{ij}(X) z_i$ $(1 \leq j \leq n)$ (see Proposition 2.1). From Proposition 4.2, if $z_0 \in \mathscr{R}_{\mathfrak{q}_C}$ then there exist $H_1, \cdots, H_t \in \mathfrak{h}_C$ $(t = n - l)$ such that $\varphi(ad\ H_1)_{z_0}, \cdots, \varphi(ad\ H_t)_{z_0}$ is a C-basis of $Hol_{z_0}(\mathfrak{q}_C; I)$. That is,

$$\operatorname{rank}\begin{bmatrix} g_1(z; ad\ H_1) \cdots g_n(z; ad\ H_1) \\ \vdots \qquad\qquad \vdots \\ g_1(z; ad\ H_t) \cdots g_n(z; ad\ H_t) \end{bmatrix}_{z = z_0} = t.$$

Since $g_j(z; ad\ H_i)$ $(1 \leq j \leq n,\ 1 \leq i \leq t)$ is a continuous map on $\mathfrak{q}_C$, there exists a domain W of $\mathfrak{q}_C$ such that $z_0 \in W$ and for any $z \in W$, $\operatorname{rank}(g_j(z; ad\ H_i)) = t$. Thus for any $z \in W$, $\varphi(ad\ H_1)_z, \cdots, \varphi(ad\ H_t)_z$ is a C-basis of $Hol_z(\mathfrak{q}_C; I)$. Since, for any $X \in \tilde{\mathfrak{h}}_C$ and $z \in W$, $\varphi(X)_z \in Hol_z(\mathfrak{q}_C; I)$ from Proposition 4.2, there exists $h_i (1 \leq i \leq t) \in C^\infty(W)$ such that $\varphi(X)_z = \sum_{1 \leq i \leq t} h_i(z) \varphi(ad\ H_i)_z$ for any $z \in W$. So

$$\sum_{1 \leq j \leq n} g_j(z; X)\left(\frac{\partial}{\partial z_j}\right)_z = \sum_{\substack{1 \leq i \leq t \\ 1 \leq j \leq n}} g_j(z; ad\ H_i) h_i(z)\left(\frac{\partial}{\partial z_j}\right)_z$$

for any $z \in W$. Hence $g_j(z; X) = \sum_{1 \leq i \leq t} g_j(z; ad\ H_i) h_i(z)$ $(1 \leq j \leq n)$ for any $z \in W$. This implies that there exists $g \in P^t(\mathfrak{q}_C)$ such that $gh_i \in P(\mathfrak{q}_C)$ $(1 \leq i \leq t)$ and $g(z) \neq 0$ for any $z \in W$, since $\operatorname{rank}(g_j(z; ad\ H_i)) = t$ for any $z \in W$. Hence

$$g(z)\varphi(X)_z = \sum_{1 \leq i \leq t} g(z) h_i(z) \varphi(ad\ H_i)_z \qquad \text{for any } z \in W.$$

Since $gh_i \in P(\mathfrak{q}_C)$, we have $g\varphi(X) \in P(\mathfrak{q}_C)\varphi(ad\ \mathfrak{h}_C)$. Thus g is a desired polynomial. Therefore the lemma is proved, because the above argument is independent of a choice of a basis.

For each Lie subalgebra $\mathfrak{a}$ of $\mathfrak{gl}(\mathfrak{q})_C$ and an open subset U of $\mathfrak{q}_C$, we denote by $\mathcal{O}_{\mathfrak{a}}(U)$ a vector space of all holomorphic functions on U such that $\varphi(X)f = 0$ for any $X \in \mathfrak{a}$. Then it is obvious that $\mathcal{O}_{\tilde{\mathfrak{h}}}(U) \subset \mathcal{O}_{ad\mathfrak{h}_C}(U)$. But we have the following.

COROLLARY 4.4. *For any domain U of $\mathfrak{q}_C$, $\mathcal{O}_{\tilde{\mathfrak{h}}}(U) = \mathcal{O}_{ad\mathfrak{h}_C}(U)$.*

PROOF. Let U is a domain of $\mathfrak{q}_C$. Since it is well known that $\mathscr{R}_{\mathfrak{q}_C}$ is an open dense subset of $\mathfrak{q}_C$, $\mathscr{R}_{\mathfrak{q}_C} \cap U \neq \phi$. From Lemma 4.3, for any $X \in \tilde{\mathfrak{h}}_C$ and $z_0 \in U$ there exist a polynomial $p \in P(\mathfrak{q}_C)$ and a domain W of $\mathfrak{q}_C$ such that $z_0 \in W$,

$p(z) \neq 0$ for any $z \in W$ and $p\varphi(X) \in P(\mathfrak{q}_C)\varphi(ad\, \mathfrak{h}_C)$. Hence for any $f \in \mathcal{O}_{ad\mathfrak{h}_C}(U)$, we have $p(z)(\varphi(X)f)(z) = 0$ for any $X \in \tilde{\mathfrak{h}}_C$ and $z \in U \cap W$. But since $p(w) \neq 0$ for any $w \in W$, $(\varphi(X)f)(z) = 0$ for any $X \in \tilde{\mathfrak{h}}_C$ and $z \in U \cap W$. Since f is holomorphic on U, $\varphi(X)f$ is so. Hence, from the identity theorem for an analytic functions, $\varphi(X)f = 0$ on U. This implies that $f \in \mathcal{O}_{\tilde{\mathfrak{h}}_C}(U)$. Thus the corollary is proved.

§ 5. $\tilde{\mathbf{H}}$-invariantness

In this section, we prove the following theorem.

THEOREM 5.1. *If* $\lambda \in \tilde{\mathscr{R}}_{\mathfrak{q}_C}$, *then*

$$\mathscr{B}_\lambda^H(\mathfrak{q}) = \mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q}).$$

PROOF. From the definition of $\mathscr{B}_\lambda^H(\mathfrak{q})$ and $\mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q})$, it is obvious that $\mathscr{B}_\lambda^H(\mathfrak{q}) \supset \mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q})$. Thus we must show that $\mathscr{B}_\lambda^H(\mathfrak{q}) \subset \mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q})$. For any element $X \in \tilde{\mathfrak{h}}$, we denote by $P_X(\mathfrak{q}_C)$ the ideal of all polynomials $p \in P(\mathfrak{q}_C)$ such that $p\varphi(X) \in P(\mathfrak{q}_C)\varphi(ad\, \mathfrak{h}_C)$. Let V_X be the algebraic subvariety of $\mathfrak{q}_C$ defining by $P_X(\mathfrak{q}_C)$. That is; V_X is the set of all elements $z \in \mathfrak{q}_C$ such that $p(z) = 0$ for any $p \in P_X(\mathfrak{q}_C)$. If there exists an element $z \in V_X \cap \mathscr{R}_{\mathfrak{q}_C}$, then $p(z) = 0$ and $z \in \mathscr{R}_{\mathfrak{q}_C}$ for any $p \in P_X(\mathfrak{q}_C)$. This contradicts Lemma 4.3. Thus $V_X \cap \mathscr{R}_\mathfrak{q} = \phi$. From Proposition 1.2, we have

$$V_X \subset \mathfrak{q}_C \backslash \mathscr{R}_{\mathfrak{q}_C} \subset \mathfrak{q}_C \backslash \tilde{\mathscr{R}}_{\mathfrak{q}_C} = \{z \in \mathfrak{q}_C;\, \Delta(z) = 0\}.$$

By Hilbert's Nullstellensatz,

$$\sqrt{(\Delta)} \subset \sqrt{P_X(\mathfrak{q}_C)},$$

where (Δ) is the ideal of $P(\mathfrak{q}_C)$ generated by Δ and $\sqrt{\mathfrak{a}}$ is the radical of an ideal $\mathfrak{a}$ of $P(\mathfrak{q}_C)$ that is; $p \in P(\mathfrak{q}_C)$ then $p \in \sqrt{\mathfrak{a}}$ if and only if $p^k \in \mathfrak{a}$ for some positive integer k. Therefore for any $X \in \tilde{\mathfrak{h}}$ there exists a positive integer k such that $\Delta^k \in P_X(\mathfrak{q}_C)$. That is; $\Delta^k \varphi(X) \in P(\mathfrak{q}_C)\varphi(ad\, \mathfrak{h}_C)$.

We consider the following system of differential equations on $\mathfrak{q}$, for fixed λ and k.

$$(\#)\begin{cases} (\partial e)u = v(e)(\lambda)u & \text{for any } e \in S_H(\mathfrak{q}_C), \\ \Delta^k u = 0. \end{cases}$$

We put $m = k(N - L)$ (see § 1, for N and L). From Proposition 2.2, for any $e \in S^d(\mathfrak{q}_C)$ there exists unique element $D(e, \Delta^k) \in S^{m(d-1)}(\mathfrak{q}_C)$ such that $\mu_e^m(\Delta^k) = \partial D(e, \Delta^k)$, since $\deg \Delta^k = m$. Let $e \in S_H(\mathfrak{q}_C)$ such that $\deg e = d$. Then $\mu_e^m(\Delta^k)$ is obviously an H-invariant differential operator on $\mathfrak{q}$. So $D(e, \Delta^k)$ is

H-invariant. When u is a solution of the above differential equations (#), it is easily seen that $\mu_e^m(\Delta^k)u = (\partial e - \nu(e)(\lambda))^m \Delta^k u = 0$. So $\partial(D(e, \Delta^k))u = \nu(D(e, \Delta^k))(\lambda)u = 0$. Hence, if there exists a homogeneous element $e \in S_H(\mathfrak{q}_C)$ for fixed $\lambda \in \mathfrak{q}_C$ and $k \in N$ such that $\nu(D(e, \Delta^k))(\lambda) \neq 0$, then $u = 0$. From Lemma 2.3, when $e = \omega$ (ω is the Casimir element), we have $\nu(D(\omega, \Delta^k))(\lambda) = \Delta^k(\lambda)$. Therefore if $\lambda \in \widetilde{\mathscr{R}}_{\mathfrak{q}_C}$, then any solution u of the differential equations (#) is zero.

Finally, for any $f \in \mathscr{B}_\lambda^H(\mathfrak{q})$ and $X \in \tilde{\mathfrak{h}}$, we put $g = \varphi(X)f$. Then there exists a positive integer k such that $\Delta^k \in P_X(\mathfrak{q}_C)$ and g is a solution of the system of the differential equations (#), because $\varphi(ad\, \mathfrak{h}_C)f = 0$ and $[\partial e, \varphi(X)] = 0$ for any $e \in S_H(\mathfrak{q}_C)$. Hence if $\lambda \in \widetilde{\mathscr{R}}_{\mathfrak{q}_C}$, then $g = 0$. Thus $f \in \mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q})$. This proves that $\mathscr{B}_\lambda^H(\mathfrak{q}) \subset \mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q})$ for any $\lambda \in \widetilde{\mathscr{R}}_{\mathfrak{q}_C}$. Therefore the theorem is proved.

We consider Theorem 5.1 in the case when $l = \text{rank } \mathfrak{q} = 1$. In the case, the polynomial Δ is a homogeneous polynomial of $\mathfrak{q}_C$ such that the homogeneous degree of Δ is $\dim \mathfrak{g} - \text{rank } \mathfrak{g}$ (see §1). Since $\text{rank } \mathfrak{g} = \dim \mathfrak{h} - \dim \mathfrak{q} + 2 \text{rank } \mathfrak{q}$, $\dim \mathfrak{g} - \text{rank } \mathfrak{q} = \dim \mathfrak{h} + \dim \mathfrak{q} - \text{rank } \mathfrak{g} = 2(\dim \mathfrak{q} - \text{rank } \mathfrak{q}) = 2(\dim \mathfrak{q} - 1)$. On the other hand, Δ is a polynomial of the Casimir polynomial ω, because Δ is an H-invariant polynomial (we may use the same notation ω for the Casimir element ω in $S^2(\mathfrak{q}_C)$). Hence there is a non zero constant c such that $\Delta = c\omega^{\dim \mathfrak{q} - 1}$. Let $\mathscr{N}$ be the variety of all elements $z \in \mathfrak{q}_C$ such that $\omega(z) = 0$. Then we have the following.

COROLLARY 5.2. *When* $\text{rank } \mathfrak{q} = 1$, *if* $\lambda \notin \mathscr{N}$, *then*

$$\mathscr{B}_\lambda^H(\mathfrak{q}) = \mathscr{B}_\lambda^{\tilde{H}}(\mathfrak{q}).$$

REMARK. In this case, the system of differential equations

$$(\partial e)f = \nu(e)(\lambda)f \qquad \text{for any } e \in S_H(\mathfrak{q}_C)$$

are written simplify so that $(\partial\omega)f = \mu f$, where we set $\mu = \nu(\omega)(\lambda)$. Under the new parametrization ($\mu \in C$), Corollary 5.2 can be rewritten such that;

$$\text{If } \mu \neq 0, \text{ then } \mathscr{B}_\mu^H(\mathfrak{q}) = \mathscr{B}_\mu^{\tilde{H}}(\mathfrak{q}).$$

On the other hand, we consider about $\tilde{H}$. In this case, any H-invariant polynomial is a polynomial of the Casimir polynomial ω. We can choose a basis $X_1, \cdots, X_p, \cdots, Y_1, \cdots, Y_q$ of $\mathfrak{q}$ such that $X_i \in \mathfrak{k} \cap \mathfrak{q}$, $Y_i \in \mathfrak{p} \cap \mathfrak{q}$, $B(X_i, X_j) = -\delta_{i,j}$ and $B(Y_i, Y_j) = \delta_{i,j}$. Then the Casimir polynomial is written as such;

$$\omega(X) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2,$$

where $X = \sum_{1 \le i \le p} x_i X_i + \sum_{1 \le i \le q} y_i Y_i$. Then from the definition of $\tilde{H}$, we have $\tilde{H}$

$\simeq SO_0(p, q)$. On the other hand, in [1], Cerezo proved the following assertion;

(1) $p = q = 1$ case, $\dim \mathscr{B}_{\mu}^{\tilde{H}}(\mathfrak{q}) = 4$,

(2) $p = 1$ or $q = 1$ case, (except for case (1)) $\dim \mathscr{B}_{\mu}^{\tilde{H}}(\mathfrak{q}) = 3$,

(3) $p > 2$ and $q > 2$ case, $\dim \mathscr{B}_{\mu}^{\tilde{H}}(\mathfrak{q}) = 2$,

for any complex number μ.

Therefore we have the following.

THEOREM 5.3. *When* rank $\mathfrak{q} = 1$, *if* $\mu \neq 0$, *then*

(1) $p = q = 1$ *case*, $\dim \mathscr{B}_{\mu}^{H}(\mathfrak{q}) = 4$,

(2) $p = 1$ *or* $q = 1$ *case*, (*except for case* (1)) $\dim \mathscr{B}_{\mu}^{H}(\mathfrak{q}) = 3$,

(3) $p > 2$ *and* $q > 2$ *case*, $\dim \mathscr{B}_{\mu}^{H}(\mathfrak{q}) = 2$,

where $p = \dim(\mathfrak{q} \cap \mathfrak{k})$ *and* $q = \dim(\mathfrak{q} \cap \mathfrak{p})$.

REMARK. In [2], Van Dijk listed up the dimension of invariant eigen distributions. Since $\mathscr{D}'_{\lambda,H}(\mathfrak{q}) \subset \mathscr{B}_{\lambda}^{H}(\mathfrak{q})$ (see [2] for the definition of $\mathscr{D}'_{\lambda,H}(\mathfrak{q})$), it is clear that $\dim \mathscr{D}'_{\lambda,H}(\mathfrak{q}) \leq \dim \mathscr{B}_{\lambda}^{H}(\mathfrak{q})$. But from Theorem 5.3 and [2], if $\lambda \neq 0$, then we have $\mathscr{D}'_{\lambda,H}(\mathfrak{q}) = \mathscr{B}_{\lambda}^{H}(\mathfrak{q})$.

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