HIROSHIMA MATH. J.
21 (1991), 343–350

Vanishing of Im J classes in the stunted quaternionic projective spaces

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(Received March 26, 1990)

§ 1. Introduction

Let HP^n be the quaternionic projective space, and

$$i: S^{4m} \longrightarrow HP^l_m = HP^l/HP^{m-1} = S^{4m} \cup e^{4m+4} \cup \cdots \cup e^{4l}$$

be the inclusion to the bottom sphere in the stunted space. Then the purpose of this paper is to investigate the induced homomorphism

$$(1.1) \qquad i_*: \pi^s_{4n-1}(S^{4(n-r)}) \longrightarrow \pi^s_{4n-1}(HP^l_{n-r}) \quad (n - r = m \leqq l)$$

of i between the stable homotopy groups on $(\mathrm{Im}\, J)_2$, where

(1.2) $(\mathrm{Im}\, J)_2$ is the 2-primary component of the image of the stable J-homomorphism $J: \pi_{4r-1}(SO) \to \pi_{4r-1}(\Omega^\infty S^\infty) = \pi^s_{4n-1}(S^{4(n-r)})$ $(r \geqq 1)$, and is the cyclic group of order $2^{3+\nu(r)}$ by Adams [1] and Quillen [7].

Here and throughout this paper $\nu(r) = \nu_2(r)$ denotes the exponent of 2 in the prime power decomposition of a positive integer r. Also, we put

$$(1.3) \qquad a(n, r) = \binom{n+r}{r}, \quad b(n, r) = \binom{n+r-1}{r-1}.$$

The main result is stated as follows:

THEOREM A. *The induced homomorphism i_* in* (1.1) *satisfies the following properties on* $(\mathrm{Im}\, J)_2$ *in* (1.2).

(i) *If $l < n$, then i_* is injective on* $(\mathrm{Im}\, J)_2$.

(ii) *Let $l = n$ and r be odd $\geqq 1$. Then $i_*((\mathrm{Im}\, J)_2)$ is* 0 *if $a(n, r)$ is odd,*
$Z/2$ *if $b(r, r)$ is odd and $r + n \equiv 0$* mod 4,
$Z/4$ *if $r > 1$, $b(n, r)$ is odd and $n \equiv r \equiv 1$* mod 8,
$Z/8$ *if $r = 1$ and $n \equiv 1$* mod 8.

(iii) *Let $l = n$ and r be even $\geqq 2$, and assume that $b(n, r)$ is odd. Then* $\mathrm{Ker}(i_*) \cap (\mathrm{Im}\, J)_2$ *is*

$$0 \text{ if } \nu(n) = \nu(r), \quad Z/2 \text{ if } \nu(n) > \nu(r).$$

The connecting homomorphism $\partial_*: \pi_i^s(HP_{m+1}^l) \to \pi_{i-1}^s(S^{4m})$ associated with the cofibering $S^{4m} \to HP_m^l \to HP_{m+1}^l$ often plays an important role in the study of the stunted projective spaces; and Theorem A yields a partial result on ∂_*.

Moreover, we have a similar result for the quaternionic quasi-projective space Q_n (cf. [5]) instead of HP^n. In this case, we have the inclusion

$$i: S^{4m+3} \longrightarrow Q_m^l = Q_{l+1}/Q_m = S^{4m+3} \cup e^{4m+7} \cup \cdots \cup e^{4l+3}$$

to the bottom sphere and the induced homomorphism

(1.4) $$i_*: \pi_{4n+2}^s(S^{4(n-r)+3}) \longrightarrow \pi_{4n+2}^s(Q_{n-r}^l) \quad (r \geqq 1,\ l \geqq n-r).$$

THEOREM B. i_* *in* (1.4) *satisfies* (i) *and* (iii) *in Theorem A and the following* (ii)$'$:

(ii)$'$ *Let* $l = n$ *and* r *be odd* $\geqq 1$. *Then* $i_*((\mathrm{Im}\, J)_2)$ *is* 0 *if* $a(n, r)$ *is odd,*
0 *if* $a(n, r)$ *is odd,*
$Z/2$ *if* $b(n, r)$ *is odd and* $n \equiv r \equiv 1 \bmod 4$,
$Z/4$ *if* $b(n, r)$ *is odd,* $n + r \equiv 4 \bmod 8$ *and* $r \equiv 1$ *or* $3 \bmod 8$.

These theorems are proved by applying the recent results given in [4].

To prove these theorems, we consider a finite CW-spectrum

(1.5) $$X = S^0 \cup e^{4a_1} \cup \cdots \cup e^{4a_t} \cup e^{4r} \quad (1 \leqq a_1 \leqq \cdots \leqq a_t < r)$$

in general, by noticing that $\Sigma^{-4(n-r)} HP_{n-r}^n$ and $\Sigma^{-4(n-r)-3} Q_{n-r}^n$ are such ones. We define $d(X)$ by

(1.6) $$d(X) = \nu \text{ (the order of } \mathrm{Coker}[h: \pi_{4r}(X) \longrightarrow H_{4r}(X; Z)]),$$

where h is the Hurewicz homomorphism. Then, by the theorem of Crabb and Knapp [2] on the maximal codegree, we have the inequality

(1.7) $d(X) \leqq m(r)$, where $m(r)$ is $2r$ if r is even and $2r + 1$ if r is odd.

Now, consider the inclusion $i: S^0 \to X$ and the induced homomorphism

(1.8) $$i_*: \pi_{4r-1}(S^0) \longrightarrow \pi_{4r-1}(X) \quad \text{for } X \text{ in (1.5).}$$

Then we have the following results which yield Theorems A and B of above as special cases, where

(1.9) j_r denotes the generator of $(\mathrm{Im}\, J)_2 = Z/2^{3+\nu(r)}$ in (1.2).

THEOREM C. i_* *in* (1.8) *satisfies the following properties*:

(i) *If* r *is odd and* $d(X) = m(r) - \varepsilon$ *for* $0 \leqq \varepsilon \leqq 2$, *then* $2^\varepsilon i_*(j_r) = 0$.

(ii) *If* r *is even and* $d(X) = m(r)$, *then* $2^{2+\nu(r)} i_*(j_r) = 0$.

(iii) *Put* $d = d(X) - d(X/S^0)$, *where* $d(X/S^0)$ *is defined similarly to* $d(X)$ *in* (1.6). *If* $d \leqq 2 + \nu(r)$, *then* $2^{2+\nu(r)-d} i_*(j_r) \neq 0$.

We prepare in § 2 some necessary properties for the proof of the theorems, and prove Theorem C in § 3 and Theorems A and B in § 4.

§ 2. Preliminaries

To treat the spectrum X in (1.5) and its $4(r-1)$-dimensional skeleton X' in the next section, we consider the following $4m$-dimensional finite CW-spectrum in this section:

(2.1) $$Y = S^0 \cup e^{4a_1} \cup \cdots \cup e^{4a_k} \quad \text{with } 1 \leqq a_1 \leqq \cdots \leqq a_k = m.$$

For the spectrum $V = Y$ or S^0, we will consider the mod 2 Adams spectral sequence which has the E_2-term

$$E_2^{s,t}(V) = \mathrm{Ext}_A^{s,t}(H^*(V; Z/2), Z/2)$$

and converges to $\pi_*(V)$, where A is the mod 2 Steenrod algebra. For $2 \leqq u \leqq \infty$, we denote by $E_u^{s,t}(V)$ the E_u-term of the spectral sequence. For the generator $j_r \in (\mathrm{Im}\, J)_2 = Z/2^{3+\nu(r)}$ in (1.9), $2^{3+\nu(r)-i} j_r$ represents a unique element

(2.2) $$\alpha_{2r/i} \in E_2^{q-i, q-i+4r-1}(S^0) \quad \text{for } 1 \leqq i \leqq 3$$

(cf. [6], [8]), where $q = 2r+2$ or $2r+1$ if r is odd or even respectively. Furthermore, for the homomorphism $i_*\colon E_2^{s,t}(S^0) \to E_2^{s,t}(V)$ induced from the inclusion $i\colon S^0 \to V$ to the bottom sphere, we put ${}_0\alpha_{2r/i} = i_*(\alpha_{2r/i})$. Then, by the similar way as in [3; Lemmas 3.6, 3.9], we have the following lemma for the spectrum Y in (2.1).

Lemma 2.3. *Assume that $r > m$.*

(i) *When r is odd, $E_2^{s,s+4r-1}(Y) = 0$ if $s \geqq 2r+2$, and $= Z/2\{{}_0\alpha_{2r/i}\}$ if $s = 2r+2-i$ and $1 \leqq i \leqq 3$.*

(ii) *When r is even, $E_2^{s,s+4r-1}(Y) = 0$ if $s \geqq 2r+1$, and $= Z/2\{{}_0\alpha_{2r/1}\}$ if $s = 2r$.*

In fact, the elements ${}_0\alpha_{2r/i}$ in Lemma 2.3 are non zero permanent cycles, that is ${}_0\alpha_{2r/i} \neq 0$ in $E_\infty(Y)$, by the following lemma.

Lemma 2.4. *If $r > m$, then $i_*\colon \pi_{4r-1}(S^0) \to \pi_{4r-1}(Y)$ is injective on* $(\mathrm{Im}\, J)_2$.

Proof. Let $W = Y^*$ be the S-dual of Y with $\dim W = 4m$, and $p\colon W \to S^{4m}$ the collapsing map to the top cell. Then by the same reason as in the proof of [3; Prop. 2.1], it suffices to show the following:

(*) $$\nu(|p^*(\iota)|) = 3 + \nu(r) \quad \text{for } p^*(\iota) \in KO^{4(m-r)}(W)_{(2)}/\mathrm{Im}(\psi^3 - 1).$$

Here, $\iota \in KO^{4(m-r)}(S^{4m}) = Z$ is the generator, and $\psi^3\colon KO^i(W)_{(2)} \to KO^i(W)_{(2)}$ is the stable Adams operation on the KO-cohomology groups localized at 2. But, by the cell structure of W, we have an isomorphism

$$KO^{4(m-r)}(W) \cong \sum_{i=0}^{m} H^{4i}(W; Z) \otimes KO^{4(m-r-i)}(S^0),$$

and we can take $p^*(\imath) \in KO^{4(m-r)}(W)$ as one element in a basis of the free part. Then we see that, for an integer c, $cp^*(\imath) \in \mathrm{Im}(\psi^3 - 1)$ if and only if c is divisible by $9^r - 1$. This implies (*), and we have the desired result. Q.E.D.

Assume that a map $f: S^{k-1} \to Y$ with $k > 4m$ is given for the spectrum Y in (2.1). We denote the cofiber of f by $C(f)$ and the inclusion $S^0 \to C(f)$ by i, and we put $e = d(C(f)) - d(C(f)/S^0)$, where each $d(\ \ )$ is defined similarly as in (1.6). Then we have the following lemma, in which all homotopy groups are assumed to be cocalized at 2.

LEMMA 2.5. *If $i_*(2^e\gamma) \neq 0$ in $\pi_{k-1}(Y)$ for some $\gamma \in \pi_{k-1}(S^0)$, then $i_*(\gamma) \neq 0$ in $\pi_{k-1}(C(f))$.*

PROOF. Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_k(C(f)) \arrow[r, "q_*"] & \pi_k(C(f)/S^0) \arrow[r, "\partial"] & \pi_{k-1}(S^0) \arrow[r, "i_*"] & \pi_{k-1}(C(f)) \\
 & \pi_k(Y/S^0) \arrow[r, "\partial"] \arrow[u, "j_*"] & \pi_{k-1}(S^0) \arrow[r, "i_*"] \arrow[u, equal] & \pi_{k-1}(Y). \arrow[u, "j_*"]
\end{tikzcd}$$

Here the homotopy groups are all assumed to be localized at 2, the horizontal two sequences are the exact sequences associated with the respective cofiberings and each j_* is the homomorphism induced from the canonical inclusion. By the definition of e, we can take the generators x and y of the respective free parts of $\pi_k(C(f))$ and $\pi_k(C(f)/S^0)$ to satisfy

$$q_*(x) = 2^e y + v \quad \text{for some torsion element } v.$$

Assume that $i_*(\gamma) = 0$ in $\pi_{k-1}(C(f))$. Then $\gamma = \partial(ty + w)$ for some integer t and some torsion element w, and thus $2^e\gamma = \partial(2^e w - tv)$. Since any torsion element of $\pi_k(C(f)/S^0)$ is in $\mathrm{Im}(j_*)$, $i_*(2^e\gamma) = 0$ in $\pi_{k-1}(Y)$, and thus we have the desired result. Q.E.D.

§ 3. Proof of Theorem C

Let X be a CW-spectrum in (1.5). For a non zero element $y \in \pi_{t-s}(X)$, we put $F(y) = s$ if y represents a non zero element of $E_\infty^{s,t}(X)$. That is, $F(y)$ denotes the mod 2 Adams filtration of y. Consider the S-dual of X, and apply

Theorem 1 and Proposition 3.2 in [4] to it. Then we have the following proposition, where $h: \pi_{4r}(X) \to H_{4r}(X; Z)$ is the Hurewicz homomorphism and the image $h(x)$ of an element $x \in \pi_{4r}(X)$ is regarded as an integer through the isomorphism $H_{4r}(X; Z) \cong Z$.

PROPOSITION 3.1. *Let $\varepsilon(r) = 2$ or 4 for even or odd $r \geqq 1$ respectively. If $d(X) \geqq 2r - \varepsilon(r)$, then there is an element $x \in \pi_{4r}(X)$ satisfying $\nu(h(x)) = F(x) = d(X)$.*

Let X' be the $4(r-1)$-skeleton of X. Then we have the exct sequence

$$(3.2) \qquad E_2^{s-1,t}(X) \xrightarrow{p_*} E_2^{s-1,t}(S^{4r}) \xrightarrow{\partial} E_2^{s,t}(X') \xrightarrow{i_*} E_2^{s,t}(X) \longrightarrow 0$$

for $t - s = 4r - 1$, where each $E_2^{s,t}(\quad)$ is the E_2-term of the spectral sequence as in §2. Let $h_0 \in E_2^{1,1}(S^0) = Z/2$ be the generator. Then $E_2^{s-1,t}(S^{4r}) = E_2^{s-1,s-1}(S^0) = Z/2\{h_0^{s-1}\}$, and we have the following proposition, where $m(r)$ is te integer in (1.7).

PROPOSITION 3.3. (i) *If $d(X) = m(r)$, then*

$$\partial(h_0^i) \neq 0 \textit{ for } 1 \leqq i \leqq m(r) - 1.$$

(ii) *If r is odd and $d(X) = m(r) - \varepsilon$ for $1 \leqq \varepsilon \leqq 2$, then*

$${}_0\alpha_{2r/t} = 0 \in E_3^{2r-t+2,6r-t+1}(X) \quad \textit{for } \varepsilon = 1 \textit{ and } 1 \leqq t \leqq 2, \textit{ and}$$

$${}_0\alpha_{2r/1} = 0 \in E_4^{2r+1,6r}(X) \quad \textit{for } \varepsilon = 2.$$

PROOF. (i) We put $M = m(r) - 1$. Suppose that $\partial(h_0^i) = 0$ for some $1 \leqq i \leqq M$. Then there is an element $z \in E_2^{i,i+4r}(X)$ satisfying $p_*(z) = h_0^i$ by the exactness of (3.2), and we have $p_*(h_0^{M-i}z) = h_0^M$. We use Lemma 2.3 to the case of $Y = X'$. Then $E_2^{s,t}(X') = 0$ for $s \geqq M + 2$ and $t - s = 4r - 1$, and we have $E_2^{s,t}(X) = 0$ for the same s and t by (3.2). Therefore we have $h_0^{M-i}z \neq 0 \in E_\infty(X)$, and $\nu(h(\gamma)) = M$ for an element $\gamma \in \pi_{4r}(X)$ which represents $h_0^{M-i}z$, where h is the Hurewicz homomorphism and we regard $h(\gamma)$ as an integer as in Proposition 3.1. But this contradicts the assumption $d(X) = M + 1$, and thus we have $\partial(h_0^i) \neq 0$ for any $1 \leqq i \leqq M$.

(ii) Since $d(X) = 2r - \varepsilon + 1$ for some $1 \leqq \varepsilon \leqq 2$ by the assumption, we have $h_0^{2r-\varepsilon+1} \in \mathrm{Im}(p_*)$ by Proposition 3.1. Then, using (3.2), we have ${}_0\alpha_{2r/i} \neq 0$ in $E_2^{2r-i+2,6r-i+1}(X)$ for $1 \leqq i \leqq \varepsilon$. Suppose that $\partial(h_0^{2r-2}) \neq 0$. Then we have $\partial(h_0^{2r-2}) = {}_0\alpha_{2r/3}$ by Lemma 2.3. But it contradicts the above, since then ${}_0\alpha_{2r/1} = 0$ in $E_2^{2r+1,6r}(X)$. Hence we have $\partial(h_0^{2r-2}) = 0$, and so there is an element $y \in E_2^{2r-2,6r-2}(X)$ satisfying $p_*(y) = h_0^{2r-2}$ by (3.2).

Now consider the case of $\varepsilon = 1$. We get the first required equality if we show $d_2(y) \neq 0$, because $d_2(y) \in E_2^{2r,6r-1}(X) \cong Z/2\{{}_0\alpha_{2r/2}\}$ and then $d_2(h_0 y)$

$= {}_0\alpha_{2r/1}$. Suppose that $d_2(y) = 0$. Then we have $d_2(h_0 y) = 0$ also, and $d_u(h_0 y) = 0$ for any $u \geqq 3$ since it is an element of 0-group by Lemma 2.3 and (3.2). Since $p_*(h_0 y) = h_0^{2r-1}$, $h_0 y$ cannot be the image of d_u for ay $u \geqq 2$. Thus $h_0 y$ is represented by an element $\beta \in \pi_{4r}(X)$ with $v(h(\beta)) = 2r - 1$, and this contradicts the assumption $d(X) = 2r$. Hence we have $d_2(y) \neq 0$, and the desired result. For the case of $\varepsilon = 2$, we have $d_2(y) = 0$ and $d_3(y) = {}_0\alpha_{2r/1}$ by the similar argument to $\varepsilon = 1$, and the second required equality. Q.E.D.

PROOF OF THEOREM C. (i) and (ii). Let $1 \leqq i \leqq 3$. Then the element $2^{3+v(r)-i} i_*(j_r) \in \pi_{4n-1}(X)$ represents the element ${}_0\alpha_{2r/i} \in E_2^{q-i, q-i+4r-1}(X)$, and $E_2^{s,s+4r-1}(X) = 0$ for any $s \geqq q$, by Lemma 2.3 and (3.2), where $q = 2r + 2$ or $2r + 1$ for odd or even r respectively. Thus to prove $2^{3+v(r)-i} i_*(j_r) = 0$ we may show that ${}_0\alpha_{2r/t} = 0$ for $1 \leqq t \leqq i$.

When $d(X) = m(r) - \varepsilon$ for odd r and $1 \leqq \varepsilon \leqq 2$, the assertion follows from Proposition 3.3 (ii). Now assume that $d(X) = m(r)$. By Proposition 3.3 (i) and Lemma 2.3, we have

$$\partial(h_0^{2r-i+1}) = {}_0\alpha_{2r/i} \quad \text{for odd } r \text{ and } 1 \leqq i \leqq 3, \text{ and}$$

$$\partial(h_0^{2r-1}) = {}_0\alpha_{2r/1} \quad \text{for even } r,$$

where ∂ is the homomorphism in (3.2). Therefore, ${}_0\alpha_{2r/i} = 0 \in E_2^{2r-i+2, 6r-i+1}(X)$ for odd r and $1 \leqq i \leqq 3$, and ${}_0\alpha_{2r/1} = 0 \in E_2^{2r, 6r-1}(X)$ for even r. Thus we have (i) and (ii).

(iii) We apply Lemmas 2.4 and 2.5 to the case of $(C(f), Y, e) = (X, X', d)$, where $d = d(X) - d(X/S^0)$. Lemma 2.4 implies that $2^{2+v(r)} i_*(j_r) \neq 0$ in $\pi_{4r-1}(X')$, and thus we have the desired result by Lemma 2.5.

§4. Proof of Theorems A and B

We will apply Theorem C to the spectra $\Sigma^{-4(n-r)} HP_{n-r}^n$ and $\Sigma^{-4(n-r)-3} Q_{n-r}^n$. Recall the integers $a(n, r)$ and $b(n, r)$ defined for given integers n and $r \geqq 1$ in (1.3). Also, we put $c(n, r) = \binom{n+r-2}{r-1}$. Then by [4; Th. 2, 3] we have the following:

LEMMA 4.1. (i) *$d(\Sigma^{-4(n-r)} HP_{n-r}^n) = m(r)$ if and only if $a(n, r)$ is odd.*

(ii) *Assume that $r \geqq 3$ is odd. Then, $d(\Sigma^{-4(n-r)} HP_{n-r}^n) = m(r) - 1$ or $m(r) - 2$ if the following* (1) *or* (2) *holds respectively:*

(1) $a(n, r) + 1 \equiv (a(n, r)/2) + b(n, r) \equiv 1 \mod 2$;

(2) $a(n, r) \equiv 2 \mod 4$, $b(n, r) \equiv 1 \mod 2$ *and* $(a(n, r)/2) + b(n, r) + 2c(n, r) \equiv 2 \mod 4$.

LEMMA 4.2. (i) *$d(\Sigma^{-4(n-r)-3}Q^n_{n-r}) = m(r)$ if and only if $a(n, r)$ is odd.*

(ii) *Assume that $r \geqq 1$ is odd. Then, $d(\Sigma^{-4(n-r)-3}Q^n_{n-r}) = m(r) - 1$ or $m(r) - 2$ if the following (1)′ or (2)′ holds respectively:*

(1)′ $a(n, r) \equiv 2 \mod 4$;

(2)′ $a(n, r) \equiv 4(1 + (n + 1)c(n, r)) \mod 8$.

Let $i_*: \pi^s_{4n-1}(S^{4(n-r)}) \to \pi^s_{4n-1}(HP^n_{n-r})$ be the homomorphism in (1.1). Then, by Theorem C and Lemma 4.1, we have the following theorem.

THEOREM 4.3. (i) *If $a(n, r)$ is odd, then $i_*((\operatorname{Im} J)_2) = 0$ for odd $r \geqq 1$, and $2^{2+\nu(r)}i_*(j_r) = 0$ for even $r \geqq 2$.*

(ii) *For odd $r \geqq 3$, $2i_*(j_r) = 0$ or $4i_*(j_r) = 0$ if (1) or (2) in Lemma 4.1 holds respectively.*

Similarly, for the homomorphism $i_*: \pi^s_{4n+2}(S^{4(n-r)+3}) \to \pi^s_{4n+2}(Q^n_{n-r})$ in (1.4), we have the following theorem by Theorem C and Lemma 4.2.

THEOREM 4.4. *i_* satisfies (i) in Theorem 4.3 and the following (ii)′:*

(ii)′ *For odd $r \geqq 1$, $2i_*(j_r) = 0$ or $4i_*(j_r) = 0$ if (1)′ or (2)′ in Lemma 4.2 holds respectively.*

PROOF OF THEOREM A. (i) follows from Lemma 2.4 by applying it to the case of $Y = \Sigma^{-4(n-r)}HP^l_{n-r}$ and $m = l - n + r$ for $l < n$.

(ii) Let $i_*: \pi^s_{4n-1}(S^{4(n-r)}) \to \pi^s_{4n-1}(HP^n_{n-r})$ be the homomorphism in (1.1) for $l = n$, and $(\operatorname{Im} J)_2 \subset \pi^s_{4n-1}(S^{4(n-r)})$ as in (1.2). We put $P = \Sigma^{-4(n-r)}HP^n_{n-r}$ and $d = d(P) - d(P/S^0)$. Then, by the same reason to Lemma 4.1 (i), we have

$$(4.5) \qquad d(P/S^0) = m(r-1) \quad \text{if and only if} \quad b(n, r) \text{ is odd.}$$

Assume that r is odd. If $a(n, r)$ is odd, then $i_*((\operatorname{Im} J)_2) = 0$ by Theorem 4.3 (i), and we have the first case of the required result. For the case of $r = 1$, since $\Sigma^{-4n+4}HP^n_{n-1}$ is homotopy equivalent to $S^0 \cup_{(n-1)\nu} e^4$, where $\nu \in \pi_3(S^0) = Z/24$ is the generator, $i_*((\operatorname{Im} J)_2)$ is a cyclic group of order g.c.m.$\{n - 1, 8\}$. Hence we have the desired result for $r = 1$. Now we assume further that $b(n, r)$ is odd and $r \geqq 3$. Then we see that the condition (1) in Lemma 4.1 (ii) is equivalent to that $r + n \equiv 0 \mod 4$. Thus, if $r + n \equiv 0 \mod 4$, then $2i_*(j_r) = 0$ by Theorem 4.3 (ii), and $i_*(j_r) \neq 0$ by Theorem C (iii) since $d = 2$ by Lemma 4.1 (ii) and (4.5). Hence we have the second case of the desired result. Similarly, the condition (2) in Lemma 4.1 (ii) is equivalent to that $n \equiv r \equiv 1 \mod 8$. Then, under this condition, $d = 1$ by Lemma 4.1 (ii) and (4.5), and we have $i_*((\operatorname{Im} J)_2) = Z/4$ by Theorem 4.3 (ii) and Theorem C (iii), which is the third case of the required result.

(ii) Assume that r is even. Then, $b(n, r)$ is odd if and only if $d(P/S^0) = 2r - 1$ by (4.5), and under this assumption we have the following:

$$\nu(n) > \nu(r) \quad \text{if and only if} \quad a(n, r) \text{ is odd},$$

$$\nu(n) = \nu(r) \quad \text{if and only if} \quad d(P) = 2r - 1,$$

Hence, if $b(n, r)$ is odd and $\nu(n) = \nu(r)$, then $2^{2+\nu(r)} i_*(j_r) \neq 0$ by Theorem C (iii) since $d = 0$ in this case, and thus Ker $(i_*) \cap (\mathrm{Im}\, J)_2 = 0$. If both $a(n, r)$ and $b(n, r)$ are odd, then $2^{2+\nu(r)} i_*(j_r) = 0$ by Theorem 4.3 (i), and $\mathrm{Ker}(i_*) \cap (\mathrm{Im}\, J)_2 = Z/2$ by Theorem C (iii) since $d = 1$ in this case. Thus we have completed the proof. Q.E.D.

The proof of Theorem B is similar to that of Theorem A, by using Lemma 4.2 and Theorem 4.4 instead of Lemma 4.1 and Theorem 4.3.

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