HIROSHIMA MATH. J.
15 (1985), 429–436

Nakai's conjecture for invariant subrings

Yasunori ISHIBASHI

(Received November 12, 1984)

Let k be a field of characteristic 0 and R a locality over k. Let $\mathrm{Der}_k(R)$ denote the R-algebra of high order derivations on R/k (cf. [7]) and $\mathrm{der}_k(R)$ the subalgebra of $\mathrm{Der}_k(R)$ which is generated by ordinary derivations on R/k. It is known that if R is regular, then $\mathrm{Der}_k(R)=\mathrm{der}_k(R)$ ([1], (16.11.2)). Nakai proposed a conjecture which asserts that regularity of R is characterized by the condition $\mathrm{Der}_k(R)=\mathrm{der}_k(R)$. The conjecture has been shown to be true only in the case where either R is one-dimensional ([6], Theorem) or R is the local ring of a two-dimensional homogeneous complete intersection at the origin ([3], Theorem 4).

The main purpose of this paper is to verify the conjecture for certain invariant subrings under finite group actions. Let a finite group G act on a k-algebra R. In §1 we study the relation between G-invariant high order derivations on R and high order derivations on the invariant subring R^G (Proposition 1.2, Corollary 1.3). In §2 we deal with invariant subrings of polynomial rings and show that the conjecture is correct in the case of the local ring of an invariant subring at the origin (Theorem 2.3). In §3 the conjecture is also verified in the case of invariant subrings of regular local rings (Theorem 3.3).

Notation and terminology: Throughout this paper k denotes a field and any ring is assumed to be commutative and Noetherian. Let R be a k-algebra and M an R-module. A q-th order k-derivation of R into M is the same notion as that of a differential operator of R into M of order $\leqq q$ which vanishes on 1 (cf. [1], [7]). In the present paper we use the terminology in [7]. The R-module of q-th order k-derivations of R into M is denoted by $\mathrm{Der}_k^q(R, M)$. A q-th order k-derivation of R into itself is simply called a q-th order derivation on R/k and the module of such high order derivations is denoted by $\mathrm{Der}_k^q(R)$ instead of $\mathrm{Der}_k^q(R, R)$. We denote by $\mathrm{Der}_k(R)$ the R-algebra of high order derivations on R/k and by $\mathrm{der}_k(R)$ the subalgebra of $\mathrm{Der}_k(R)$ which is generated by the first order derivations on R/k. It follows from [7] that $\mathrm{Der}_k(R)=R\oplus\bigcup_{q=1}^{\infty}\mathrm{Der}_k^q(R)$. Let $\mathfrak{J}\subset R\otimes_k R$ be the diagonal ideal, that is, the ideal generated by $a\otimes 1-1\otimes a$, $a\in R$. The module of q-th order differentials on R/k is, by definition, $\Omega_{R/k}^q=\mathfrak{J}/\mathfrak{J}^{q+1}$ and then $\mathrm{Hom}_R(\Omega_{R/k}^q, M)\cong\mathrm{Der}_k^q(R, M)$. For $X=\mathrm{Spec}(R)$ let $\Omega_{X/k}^q$ denote the sheaf of q-th order differentials on X/k.

§1. Preliminaries

In this section k denotes a field of arbitrary characteristic. Let R be a k-algebra, G a group acting on R and R^G the invariant subring of R under the action of G. Suppose G acts trivially on k. Then G acts on $\mathrm{Der}_k^q(R)$ by the rule: $D \to D^\sigma = \sigma^{-1} D \sigma$ for $D \in \mathrm{Der}_k^q(R)$ and $\sigma \in G$. We denote by $\mathrm{Der}_k^q(R)^G$ the R^G-module of invariant q-th order derivations on R/k under the action of G. Then we have the following.

Lemma 1.1. *If k' is an extension field of k, the action of G can be extended to $R' = R \otimes_k k'$ and $\mathrm{Der}_{k'}^q(R')$ naturally. Then $R'^G = R^G \otimes_k k'$, $\mathrm{Der}_{k'}^q(R'^G) = \mathrm{Der}_k^q(R^G) \otimes_k k'$ and $\mathrm{Der}_{k'}^q(R')^G = \mathrm{Der}_k^q(R)^G \otimes_k k'$.*

The proof of this lemma is immediate and will be omitted.

A G-invariant q-th order derivation on R/k induces a q-th order derivation on R^G/k, and hence there is a natural homomorphism: $\mathrm{Der}_k^q(R)^G \to \mathrm{Der}_k^q(R^G)$ as an R^G-module.

Proposition 1.2. *Let R be a k-algebra and G a finite subgroup of $\mathrm{Aut}(R)$ such that its action on k is trivial. Assume that there exists a closed subset Z of $X = \mathrm{Spec}(R)$ such that $\operatorname{depth} R_\mathfrak{p} \geqq 2$ for every $\mathfrak{p} \in Z$ and a natural morphism $f: X \to Y = \mathrm{Spec}(R^G)$ is etale outside of Z. Then the homomorphism: $\mathrm{Der}_k^q(R)^G \to \mathrm{Der}_k^q(R^G)$ is an isomorphism.*

Proof. Consider a natural homomorphism of sheaves: $f^*\Omega_{Y/k}^q \to \Omega_{X/k}^q$ (see, [7], Chapter II, §3). Taking the dual of this homomorphism gives us a homomorphism: $(\Omega_{X/k}^q)^\vee \to (f^*\Omega_{Y/k}^q)^\vee$ between the dual sheaves. Since $\operatorname{depth} R_\mathfrak{p} \geqq 2$ for every $\mathfrak{p} \in Z$, we have $\operatorname{depth}_Z (\Omega_{X/k}^q)^\vee$, $\operatorname{depth}_Z (f^*\Omega_{Y/k}^q)^\vee \geqq 2$ ([12], Lemma 1) and hence there are isomorphisms

$$H^0(X-Z, (\Omega_{X/k}^q)^\vee) \cong H^0(X, (\Omega_{X/k}^q)^\vee),$$
$$H^0(X-Z, (f^*\Omega_{Y/k}^q)^\vee) \cong H^0(X, (f^*\Omega_{Y/k}^q)^\vee)$$

([2], Theorem 3.8). By the assumption, $f: X \to Y$ is etale outside of Z and consequently

$$(\Omega_{X/k}^q)^\vee | X-Z \longrightarrow (f^*\Omega_{Y/k}^q)^\vee | X-Z$$

is an isomorphism (cf. [8]). Thus we get an isomorphism

$$H^0(X, (\Omega_{X/k}^q)^\vee) \cong H^0(X, (f^*\Omega_{Y/k}^q)^\vee),$$

that is,

$$\mathrm{Der}_k^q(R) \cong \mathrm{Der}_k^q(R^G, R).$$

Then it is easy to show that $\mathrm{Der}_k^q(R)^G \cong \mathrm{Der}_k^q(R^G)$. Q. E. D.

COROLLARY 1.3. *Let R be a k-algebra and G a finite subgroup of* $\mathrm{Aut}(R)$ *such that its action on k is trivial. Assume that there exists a closed subset Z of $X = \mathrm{Spec}(R)$ such that* $\operatorname{depth} R_{\mathfrak{p}} \geqq 2$ *for every $\mathfrak{p} \in Z$ and the inertia group of $\mathfrak{p}$ is the unit group for every $\mathfrak{p} \in X - Z$. Then the homomorphism:* $\mathrm{Der}_k^q(R)^G \to \mathrm{Der}_k^q(R^G)$ *is an isomorphim.*

PROOF. Since the inertia group of $\mathfrak{p}$ is the unit group for every $\mathfrak{p} \in X - Z$, the morphism f: $X \to Y = \mathrm{Spec}(R^G)$ is etale outside of Z ([10], Chapitre X, Théorème 1). Thus the assertion follows from Proposition 1.2.

§2. The case of invariant subrings of polynomial rings

Let k be a field of characteristic 0. Let $R = k[x_1,\ldots, x_n]$ be a polynomial ring over k and G a finite subgroup of $\mathrm{GL}(n, k)$. Then G acts on 1-forms of R and thus G can be regarded as an automorphism group of R. An element $\sigma \in \mathrm{GL}(n, k)$ is called a pseudo-reflection if $\operatorname{rank}(\sigma - I) \leqq 1$ and σ has a finite order, where I is the identity matrix. We say that G contains no pseudo-reflections if G does not contain any pseudo-reflections other than the identity element.

LEMMA 2.1. *Let a finite group $G \subset \mathrm{GL}(n, k)$ act on $R = k[x_1,\ldots, x_n]$ naturally. If G contains no pseudo-reflections, then the homomorphism:* $\mathrm{Der}_k^q(R)^G \to \mathrm{Der}_k^q(R^G)$ *is an isomorphism.*

PROOF. The assertion is immediate in the case where either $n = 1$ or $G = \{I\}$. Assume that $n \geqq 2$ and $G \neq \{I\}$. Let σ $(\neq I) \in G$. Our hypothesis and desired conclusion are unaffected by tensoring, over k, with an algebraic closure of k (Lemma 1.1). Thus we may assume that k is algebraically closed. Then we can assume that σ is a diagonal matrix $\mathrm{diag}\{\zeta_1,\ldots, \zeta_n\}$, by a suitable linear change of variables. Since G contains no pseudo-reflections, we have $\zeta_i \neq 1$ for at least two i. Suppose $\zeta_i \neq 1$ for $i = 1, 2$. Put $X = \mathrm{Spec}(R)$ and $X_\sigma = \{\mathfrak{p} \in X \mid \sigma(\mathfrak{p}) = \mathfrak{p}\}$. If a maximal ideal $(x_1 - a_1,\ldots, x_n - a_n)$ belongs to X_σ, then $a_1 = a_2 = 0$. Thus $\bar{X}_\sigma$ is of codimension $\geqq 2$ in $\bar{X}$, where for a closed subset Y of X we denote by $\bar{Y}$ the set of closed points contained in Y. $\bar{Y}$ is dense in Y for every closed subset Y, because k is algebraically closed. Hence we obtain $\operatorname{codim}_X X_\sigma \geqq 2$ and thus $Z = \bigcup_{\sigma(\neq I) \in G} X_\sigma$ is a closed subset of X with $\operatorname{codim}_X Z \geqq 2$. Recalling the definition of Z the decomposition group of $\mathfrak{p}$ is the unit group and so is the inertia group of $\mathfrak{p}$ for every $\mathfrak{p} \in X - Z$. On the other hand, the fact that $\operatorname{codim}_X Z \geqq 2$ implies $\operatorname{depth} R_{\mathfrak{p}} \geqq 2$ for every $\mathfrak{p} \in Z$. Thus Corollary 1.3 completes the proof.

LEMMA 2.2. *Let G be a finite subgroup of* $\mathrm{GL}(n, k)$ *and G be of order g.*

If G acts on $R=k[x_1,\dots, x_n]$ naturally, then there exists $\Delta \in \mathrm{Der}_k^g(R)^G$ such that $\Delta(x_1^g)=1$.

Proof. Let $\sigma=(a_{ij}(\sigma))\in G$ and $G=\{\sigma_1,\dots,\sigma_g\}$. Since each σ_j is a non-singular matrix, we have $a_{i_j 1}(\sigma_j)\neq 0$ for some i_j $(1\leqq j\leqq g)$. Hence there exist $c_1,\dots, c_n\in k$ such that

$$(\textstyle\sum_{i_1=1}^n c_{i_1}a_{i_1 1}(\sigma_1))\cdots(\sum_{i_g=1}^n c_{i_g}a_{i_g 1}(\sigma_g))\neq 0.$$

For these c_i's let us consider a derivation $D=\sum_{i=1}^n c_i\partial/\partial x_i$ on R. Then $\Delta=\prod_{\sigma\in G}D^\sigma$ is a G-invariant g-th order derivation on R/k. A straightforward calculation yields

$$D^\sigma = \textstyle\sum_{j=1}^n(\sum_{i=1}^n c_i a_{ij}(\sigma))\partial/\partial x_j$$

and hence it is easy to check that the coefficient of $\partial^g/\partial x_1^g$ in Δ is

$$c = (\textstyle\sum_{i_1=1}^n c_{i_1}a_{i_1 1}(\sigma_1))\cdots(\sum_{i_g=1}^n c_{i_g}a_{i_g 1}(\sigma_g))\neq 0.$$

Then $c^{-1}\Delta$ is a desired high order derivation. Q.E.D.

Let a finite group $G\subset \mathrm{GL}(n,k)$ act on $R=k[x_1,\dots,x_n]$ naturally. Put $\mathfrak{m}=(x_1,\dots,x_n)\cap R^G$. Nakai's conjecture for the local ring $(R^G)_\mathfrak{m}$ asserts that if $\mathrm{Der}_k((R^G)_\mathfrak{m})=\mathrm{der}_k((R^G)_\mathfrak{m})$, then $(R^G)_\mathfrak{m}$ is regular. Now it is easy to show that the condition $\mathrm{Der}_k((R^G)_\mathfrak{m})=\mathrm{der}_k((R^G)_\mathfrak{m})$ is equivalent to $\mathrm{Der}_k(R^G)=\mathrm{der}_k(R^G)$ (cf. [3]). Hence Nakai's conjecture for $(R^G)_\mathfrak{m}$ is equivalent to the assertion that if $\mathrm{Der}_k(R^G)=\mathrm{der}_k(R^G)$, then R^G is a polynomial ring.

Theorem 2.3. *Let k be a field of characteristic* 0. *If a finite group $G\subset \mathrm{GL}(n,k)$ acts on $R=k[x_1,\dots,x_n]$ naturally and if $\mathrm{Der}_k(R^G)=\mathrm{der}_k(R^G)$, then R^G is a polynomial ring.*

Proof. We may assume that k is algebraically closed (Lemma 1.1). Let H be the subgroup of G generated by all its pseudo-reflections. Then H is normal in G and R^H is a polynomial ring over k([13], Théorème 1). Since $R^G=(R^H)^{G/H}$, we may assume that G contains no pseudo-reflections. Suppose $G\neq\{I\}$ and put $g=$the order of G. Let $\sigma(\neq I)\in G$. We can assume that σ is a diagonal matrix $\mathrm{diag}\{\zeta_1,\dots,\zeta_n\}$ with $\zeta_1\neq 1$, by a suitable linear change of variables. Let $D=\sum_{i=1}^n c_i\partial/\partial x_i$ $(c_i\in R)$ be a derivation on R/k. Since $D^\sigma=\sum_{i=1}^n\sigma^{-1}(c_i)\zeta_i\partial/\partial x_i$ and $\zeta_1\neq 1$, $D^\sigma=D$ implies that c_1 is a polynomial without the constant term. As G contains no pseudo-reflections, it follows from Lemma 2.1 that every ordinary derivation on R^G/k is the restriction of a derivation $\sum_{i=1}^n c_i\partial/\partial x_i$ on R, where c_1 has no constant term. Thus every g-th order derivation on R^G contained in $\mathrm{der}_k(R^G)$ is induced by a G-invariant g-th order derivation on R of the form $c\partial^g/\partial x_1^g+\cdots$, where c has no constant term. On the other hand, it follows from Lemma 2.2

that there exists $\Delta \in \mathrm{Der}_k^q(R)^G$ such that $\Delta(x_1^q)=1$. Since $\mathrm{Der}_k^q(R)^G \cong \mathrm{Der}_k^q(R^G)$, we see that the restriction $\Delta | R^G \notin \mathrm{der}_k(R^G)$. This contradicts the assumption. Hence $G=\{I\}$ and this completes the proof.

§3. The case of invariant subrings of regular local rings

Throughout this section let R be a local ring containing a field of characteristic 0 and $\mathfrak{m}$ the maximal ideal of R, where R is not necessarily a locality. Let G be a finite subgroup of $\mathrm{Aut}(R)$ satisfying the following condition:

(C) The automorphisms of $R/\mathfrak{m}$ induced by the elements of G are the identity.

Then G acts linearly on $\mathfrak{m}/\mathfrak{m}^2$ and hence there is a natural homomorphism $\lambda: G \to \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$. An element σ of G is called a pseudo-reflection if rank $(\lambda(\sigma)-I) \leqq 1$, where I is the identity automorphism of $\mathfrak{m}/\mathfrak{m}^2$. We say that G contains no pseudo-reflections if G does not contain any pseudo-reflections other than the identity element.

LEMMA 3.1. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field k of characteristic* 0. *Let G be a finite subgroup of* $\mathrm{Aut}(R)$ *satisfying the condition* (C). *If G contains no pseudo-reflections, then the homomorphism*: $\mathrm{Der}_k^q(R)^G \to \mathrm{Der}_k^q(R^G)$ *is an isomorphism.*

PROOF. In the case of $\dim R=1$, the assumption implies that G is the unit group, and the assertion is trivial. Assume that $\dim R \geqq 2$. Let $X=\mathrm{Spec}(R)$ and Z be the set of $\mathfrak{p} \in X$ at which R/R^G is not etale. Since G contains no pseudo-reflections, for every $\mathfrak{p} \in X$ of height 1 the inertia group of $\mathfrak{p}$ is the unit group (cf. [13]) and consequently R/R^G is etale at $\mathfrak{p}$ ([10], Chapitre X, Théorème 1). Hence every prime ideal in Z is of height $\geqq 2$. Thus Proposition 1.2 completes the proof.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing a field. Let $\hat{R}$ be the completion of R and K a coefficient field of $\hat{R}$ containing a quasi-coefficient field k_0 of R (cf. [5]). Let $t_1,\dots, t_n$ be a regular system of parameters of R. Then $\hat{R}=K[[t_1,\dots, t_n]]$, a formal power series ring over K, and $\mathrm{Der}_k^1(\hat{R})$ is a free $\hat{R}$-module with the partial derivations $\partial/\partial t_1,\dots, \partial/\partial t_n$ as a basis. Then the following conditions are equivalent:

(1) $\partial/\partial t_i$ $(1 \leqq i \leqq n)$ map R into itself, i.e. $\partial/\partial t_i \in \mathrm{Der}_{k_0}^1(R)$;

(2) $\mathrm{Der}_{k_0}^1(R)$ is a free R-module of rank n;

([5], Theorem 99). If a regular local ring $(R, \mathfrak{m})$ satisfies these equivalent conditions, then we say that (WJ) holds at $\mathfrak{m}$ (cf. [5]).

LEMMA 3.2. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field k of characteristic* 0 *and assume that* (WJ) *holds at* $\mathfrak{m}$. *Let G be a finite subgroup of* $\mathrm{Aut}(R)$ *satisfying the condition* (C). *Then, for any $t \in \mathfrak{m}-\mathfrak{m}^2$ there exists*

$\Delta \in \mathrm{Der}_k^g(R)^G$ such that $\Delta(t^g)$ is a unit in R, where g is the order of G.

PROOF. Since $t \in \mathfrak{m}-\mathfrak{m}^2$, there exist $t_2, \dots, t_n$ such that $\{t_1=t, t_2, \dots, t_n\}$ is a regular system of parameters of R. By the assumption, each $\sigma(\in G)$ induces an $R/\mathfrak{m}$-linear action on $\mathfrak{m}/\mathfrak{m}^2$ and so we have

$$\sigma(t_i) \equiv \textstyle\sum_{j=1}^n a_{ji}(\sigma)t_j \pmod{\mathfrak{m}^2} \qquad (1 \leqq i \leqq n),$$

where $(a_{ij}(\sigma)) \in \mathrm{GL}(n, R)$. Let $G=\{\sigma_1, \dots, \sigma_g\}$. Since $a_{i_j 1}(\sigma_j) \notin \mathfrak{m}$ for some i_j $(1 \leqq j \leqq g)$, there exist $c_1, \dots, c_n \in k$ satisfying

$$(\textstyle\sum_{i_1=1}^n c_{i_1}a_{i_1 1}(\sigma_1)) \cdots (\sum_{i_g=1}^n c_{i_g}a_{i_g 1}(\sigma_g)) \notin \mathfrak{m}.$$

For these c_i's let us consider a derivation $D=\sum_{i=1}^n c_i\partial/\partial t_i$ on R and put

$$\Delta = \textstyle\sum_{i_1,\dots,i_g} D^{\sigma_{i_1}} \cdots D^{\sigma_{i_g}},$$

where $i_1, \dots, i_g$ run over all permutations of $1, \dots, g$. Then Δ is a G-invariant g-th order derivation on R. By the condition (C),

$$D^\sigma = \textstyle\sum_{j=1}^n (\sum_{i=1}^n c_i a_{ij}(\sigma) + u_j)\partial/\partial t_j$$

for some $u_j \in \mathfrak{m}$. Now a straightforward calculation shows that the coefficient of $\partial^g/\partial t_1^g$ in Δ is of the form

$$g!(\textstyle\sum_{i_1=1}^n c_{i_1}a_{i_1 1}(\sigma_1)) \cdots (\sum_{i_g=1}^n c_{i_g}a_{i_g 1}(\sigma_g)) + v$$

for some $v \in \mathfrak{m}$. This element is a unit in R by the choice of c_i. Thus Δ is a desired high order derivation on R. Q.E.D.

THEOREM 3.3. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field k of characteristic 0 and assume that (WJ) holds at $\mathfrak{m}$. Let G be a finite subgroup of $\mathrm{Aut}(R)$ satisfying the condition (C). If $\mathrm{Der}_k(R^G)=\mathrm{der}_k(R^G)$, then R^G is regular.*

PROOF. Let H be the subgroup of G generated by all its pseudo-reflections. Then H is normal in G and R^H is regular ([13], Théorème 1′). Since $R^G=(R^H)^{G/H}$, we may assume that G contains no pseudo-reflections. Let G be of order g and suppose $\lambda(\sigma) \neq I$ for some $\sigma \in G$. Let $t_1, \dots, t_n$ be a regular system of parameters of R. Since $\lambda(\sigma) \neq I$, $\sigma(t_i) \not\equiv t_i \pmod{\mathfrak{m}^2}$ for some i. Hence it holds that either $\sigma(t_i) \equiv at_i \pmod{\mathfrak{m}^2}$ with $a-1 \notin \mathfrak{m}$, or $\sigma(t_i) \equiv t_i + u \pmod{\mathfrak{m}^2}$, where the residue classes $\bar{t}_i$, $\bar{u} \pmod{\mathfrak{m}^2}$ are linearly independent over $R/\mathfrak{m}$. Thus we can choose a regular system $\{t_1, \dots, t_n\}$ of parameters of R so as to satisfy the following: If we put

$$\sigma(t_i) \equiv \textstyle\sum_{j=1}^n a_{ji}(\sigma)t_j \pmod{\mathfrak{m}^2} \qquad (1 \leqq i \leqq n),$$

then either one of the following holds:

(1) $a_{11}(\sigma)-1 \notin \mathfrak{m}$ and $a_{j1}(\sigma) \in \mathfrak{m}$ for $j \geqq 2$;

(2) $a_{12}(\sigma)-1$, $a_{22}(\sigma)-1 \in \mathfrak{m}$ and $a_{j2}(\sigma) \in \mathfrak{m}$ for $j \geqq 3$.

Let $\{z_\nu\}$ be a transcendence basis of $R/\mathfrak{m}$ over k and choose a representative x_ν in R for each z_ν. Then R contains the quotient field k' of $k[\{x_\nu\}]$. Let K be the coefficient field of the completion $\hat{R}$ containing k'. Then $\hat{R}=K[[t_1,\dots, t_n]]$. Let $\partial/\partial x_\nu$ denote a derivation on $\hat{R}$ defined by $\partial x_\mu/\partial x_\nu=\delta_{\mu\nu}$ and $\partial t_i/\partial x_\nu=0$, where $\delta_{\mu\nu}$ is the Kronecker delta. Then every derivation D on R/k is written

$$D = \textstyle\sum_{i=1}^n c_i\partial/\partial t_i + \sum_\nu d_\nu\partial/\partial x_\nu,$$

where c_i, $d_\nu \in R$ and $(\sum_\nu d_\nu\partial/\partial x_\nu)(R)\subset R$. By the condition (C),

$$D^\sigma = \textstyle\sum_{j=1}^n (\sum_{i=1}^n c_i a_{ij}(\sigma)+u_j)\partial/\partial t_j + \sum_\nu (d_\nu+v_\nu)\partial/\partial x_\nu$$

with u_j, $v_\nu \in \mathfrak{m}$. Hence $D^\sigma=D$ implies $c_1 \in \mathfrak{m}$ by the choice of a regular system $\{t_1,\dots, t_n\}$ of parameters of R. As G contains no pseudo-reflections, it follows from Lemma 3.1 that a derivation on R^G/k is induced by a G-invariant derivation on R. Hence every g-th order derivation on R^G contained in $\mathrm{der}_k(R^G)$ is the restriction of a G-invariant g-th order derivation on R of the form $c\partial^g/\partial t_1^g+\cdots$ with $c \in \mathfrak{m}$. On the other hand, Lemma 3.2 implies that there exists $\Delta \in \mathrm{Der}_k^g(R)^G$ such that $\Delta(t_1^g)$ is a unit in R. Since $\mathrm{Der}_k^g(R)^G \cong \mathrm{Der}_k^g(R^G)$, $\Delta|R^G \notin \mathrm{der}_k(R^G)$. This contradiction shows $\lambda(G)=\{I\}$. Thus, the proof is completed.

COROLLARY 3.4. *Let $(R, \mathfrak{m})$ be a regular locality over a field k of characteristic* 0. *Let G be a finite subgroup of* $\mathrm{Aut}(R)$ *satisfying the condition* (C). *If* $\mathrm{Der}_k^1(R^G)$ *is a free R^G-module, then R^G is regular.*

PROOF. The proof is immediate from Theorem 3.3, [4], Theorem 1 and [11], Proposition 2.

REMARK. Platte [9] proved a result similar to Corollary 3.4 in the case of analytic algebras.

References

[1] A. Grothendieck, Éléments de géométrie algébrique IV (quatrième partie), Publ. Math. I.H.E.S. **32** (1967).

[2] A. Grothendieck, Local Cohomology, Lecture Notes in Math. No. 41, Springer–Verlag, Berlin/Heidelberg/New York, 1967.

[3] Y. Ishibashi, Remarks on a conjecture of Nakai, J. Algebra, in press.

[4] J. Lipman, Free derivation modules on algebraic varieties, Amer. J. Math. **87** (1965), 874–898.

[5] H. Matsumura, Commutative Algebra, 2nd Ed. Benjamin, 1980.

[6] K. R. Mount and O. E. Villamayor, On a conjecture of Y. Nakai, Osaka J. Math. **10** (1973), 325–327.
[7] Y. Nakai, High order derivations I, Osaka J. Math. **7** (1970), 1–27.
[8] Y. Nakai and Y. Ishibashi, Extensions of high order derivations, in preparation.
[9] E. Platte, Differentielle Eigenschaften der Invarianten regulärer Algebren, J. Algebra **62** (1980), 1–12.
[10] M. Raynaud, Anneaux Locaux Henséliens, Lecture Notes in Math. No. 169, Springer-Verlag, Berlin/Heidelberg/New York, 1970.
[11] C. J. Rego, Remarks on differential operators on algebraic varieties, Osaka J. Math. **14** (1977), 481–486.
[12] M. Schlessinger, Rigidity of quotient singularities, Invent. Math. **14** (1971), 17–26.
[13] J. P. Serre, Groupes finis d'automorphismes d'anneaux locaux régulieres, Colloq. d'Alg. E.N.S. (1967).

Department of Mathematics,
Faculty of School Education,
Hiroshima University