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On periods of certain Eichler integrals for Kleinian groups

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Introduction

In 1949 G. Bol [5] discovered nice formulae concerning the differential operators and some operators induced by Möbius transformations (see §1). M. Eichler directed his attention to Bol's formulae and demonstrated their significance in his paper [9]. He took indefinite integrals of automorphic forms for Fuchsian groups to obtain a new class of functions which are now called Eichler integrals. Later L. Ahlfors [2] considered Eichler integrals for Kleinian groups.

In [15] I. Kra introduced two distinguished classes of Eichler integrals for a Kleinian group whose orbit space is a finite union of Riemann surfaces of finite type, and showed that Eichler integrals in these classes are uniquely determined by their periods (cf. [15; Theorem 2]; part of this theorem had earlier been shown by Ahlfors [2].). The proof is based on the theory of compact Riemann surfaces. The motivation of this paper is to generalize this fact to a wider class of Kleinian groups.

A Kleinian group G acts in a certain way on the vector space of polynomials in one complex variable of degree at most $2q-2$, where q is an integer greater than one. We can thus form the first cohomology space of G with coefficients in this vector space. The period of an Eichler integral of order $1-q$ for G naturally defines a cocycle, and we obtain a map, called the period map, that maps Eichler integrals to the cocycles determined by their periods. Let D be a G-invariant union of components of G. If the orbit space D/G is a union of compact Riemann surfaces, then the period map on the space of Eichler integrals holomorphic on D is injective. However, if D/G has a non-compact component, then the conclusion does not necessarily hold (even if D/G is a finite union of Riemann surfaces of finite type). Thus, to make the period map on a class of Eichler integrals for a general Kleinian group injective, we need some conditions on Eichler integrals in the class other than holomorphicity. In [15] Kra restricted the behavior of integrals at cusps. In this paper we introduce new classes of Eichler integrals in terms of Hardy classes. If the group is a finitely generated Fuchsian group of the first kind, then one of Kra's classes turns out to coincide with one of ours (Theorem 3).

Our main result is Theorem 4 which says that if every component of D/G is a parabolic Riemann surface, then the period maps on our classes are injective. Note that G may be infinitely generated. For the proof we use the E. Hopf-Tsuji theorem (see Theorem A) to investigate the boundary behavior of Eichler integrals in our classes.

In §1 we introduce our classes and give some properties of them. In §2 we prove the main theorem. In §3 we are concerned with a relation between fine limits and periods of Eichler integrals in our classes. In the final section we consider Fuchsian groups.

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§1. Spaces of Eichler integrals

Let Möb denote the group of all Möbius transformations acting on the Riemann sphere $\hat{C}=C\cup\{\infty\}$. Let $A\in$ Möb, and let k be an integer. When f is a function defined on a subset E of $\hat{C}$, we define a function $A_k^* f$ on $A^{-1}(E)$ by $A_k^* f(z)=(f\circ A)(z)\cdot A'(z)^k$ $(z\in A^{-1}(E))$. If B is also in Möb, then we have $(A\circ B)_k^* f=B_k^*(A_k^* f)$.

Let G be a Kleinian group. We denote by $\Omega(G)$ the set of discontinuity of G, and by $\Lambda(G)$ the limit set of G.

Fix an integer $q>1$. Let $\prod_{2q-2}$ be the vector space of all polynomials of one complex variable with degree at most $2q-2$. The group G acts on $\prod_{2q-2}$ from the right (not from the left) by $\prod_{2q-2}\times G\ni(v, A)\mapsto A_{1-q}^* v\in\prod_{2q-2}$. A *cocycle* is a map $\alpha\colon G\to\prod_{2q-2}$ that satisfies $\alpha(A\circ B)=B_{1-q}^*(\alpha(A))+\alpha(B)$ for each $A\in G$ and $B\in G$. The *coboundary* of $v\in\prod_{2q-2}$ is the cocycle $A\mapsto A_{1-q}^* v-v$. We denote by $Z^1(G, \prod_{2q-2})$ the vector space of all cocycles, and by $B^1(G, \prod_{2q-2})$ that of all coboundaries. The quotient vector space $H^1(G, \prod_{2q-2})=Z^1(G, \prod_{2q-2})/B^1(G, \prod_{2q-2})$ is called the *(first) cohomology group* of G.

Let D be a G-invariant open subset of $\Omega(G)$. Let F be a holomorphic function on $D\cap C$. If $\infty\in D$, we require that $F(z)=O(|z|^{2q-2})$ $(z\to\infty)$, so that F is meromorphic on D. We say that F is a *(holomorphic) Eichler integral* of order $1-q$ for G on D if for every $A\in G$ the function $A_{1-q}^* F-F$ is equal to an element, denoted by $(\mathrm{pd}\,F)(A)$, of $\prod_{2q-2}$ on D. The map $\mathrm{pd}\,F\colon G\to\prod_{2q-2}$ defines a cocyle, and is called the *period* of F. Two Eichler integrals F_1 and F_2 are *equivalent* if F_1-F_2 is the restriction of some polynomial in $\prod_{2q-2}$ to D. In this case $\mathrm{pd}\,(F_1-F_2)$ is a coboundary.

To introduce some classes of Eichler integrals we shall use Hardy classes. Let V be an open subset of $\hat{C}$, and let p be a positive real number. The *Hardy* *class* $H^p(V)$ is the vector space of all holomorphic functions f on V such that $|f|^p \leq s$ for some positive superharmonic function s on V. The space $H^\infty(V)$ consists of all holomorphic functions on V that are bounded on each component of V. If $0<p<p'\leq\infty$, then $H^{p'}(V)\subset H^p(V)$. If V is the unit disk and if $0<p<\infty$, then it is known that $f\in H^p(V)$ if and only if $\sup_{0\leq r<1}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta < +\infty$ (cf. Duren [8; Theorem 2.12]).

Now, let G and D be as before, and let $0<p\leq\infty$. We denote by $E^p_{1-q}(D, G)$ the set of all Eichler integrals F of order $1-q$ for G on D such that $S^*_{1-q}F_1\in H^p(S^{-1}(D))$ for some $S\in$ Möb and for some Eichler integral F_1 equivalent to F. Note that $\prod_{2q-2}\subset E^p_{1-q}(D, G)$. The purpose of this paper is to investigate the properties of $E^p_{1-q}(D, G)$. First of all, we show that the set $E^p_{1-q}(D, G)$ forms a vector space.

LEMMA 1. *Let* $S\in$ *Möb. If* $F\in E^p_{1-q}(D, G)$, *then* $S^*_{1-q}F$ *belongs to* $E^p_{1-q}\cdot(S^{-1}(D), S^{-1}\circ G\circ S)$.

PROOF. It is easy to check that $S^*_{1-q}F$ is an Eichler integral for the group $S^{-1}\circ G\circ S$. By the definition there are $T\in$ Möb and an Eichler integral F_1 equivalent to F such that $T^*_{1-q}F_1\in H^p(T^{-1}(D))$. The Eichler integral $S^*_{1-q}F_1$ is equivalent to $S^*_{1-q}F$. Further, $(S^{-1}\circ T)^*_{1-q}(S^*_{1-q}F_1)=T^*_{1-q}F_1$ is an element of $H^p((S^{-1}\circ T)^{-1}\cdot(S^{-1}(D)))=H^p(T^{-1}(D))$. This completes the proof.

LEMMA 2. *For each* $F\in E^p_{1-q}(D, G)$, *there are* $F_1\in H^p(D)$ *and* $v_j\in\prod_{2q-2}$, $j=1, 2$, *such that* $F=F_1v_1+v_2$ *on* D.

PROOF. We can find $S\in$ Möb and $v_2\in\prod_{2q-2}$ such that $S^*_{1-q}(F-v_2)\in H^p(S^{-1}(D))$. Set $F_1=(S^*_{1-q}(F-v_2))\circ T$ and $v_1=(T')^{1-q}$, where $T=S^{-1}$. Then $F_1\in H^p(D)$ and $F-v_2=F_1v_1$. Since $v_1=T^*_{1-q}1\in\prod_{2q-2}$, we have the lemma.

THEOREM 1. (*a*) *Put* $S(z)=1/(z-a)$, *where* $a\in D-\{\infty\}$. *Then*

$$E^p_{1-q}(D, G) = S^*_{1-q}[E^p_{1-q}(S(D), S\circ G\circ S^{-1})\cap H^p(S(D))] + \prod_{2q-2}.$$

(*b*) *If* $\infty\in D$, *then*

$$E^p_{1-q}(D, G) = (E^p_{1-q}(D, G)\cap H^p(D)) + \prod_{2q-2}.$$

PROOF. We have only to prove the statement (*b*) since (*a*) follows at once from (*b*) and Lemma 1.

To prove (*b*), first note that the definition implies $E^p_{1-q}(D, G)\supset(E^p_{1-q}(D, G)\cap H^p(D))+\prod_{2q-2}$.

To show the converse, let $F\in E^p_{1-q}(D, G)$. By Lemma 2, there are $F_1\in H^p(D)$ and $v_j\in\prod_{2q-2}$, $j=1, 2$, such that $F=F_1v_1+v_2$. Since F_1v_1 has a pole of order at most $2q-2$ at ∞, we can choose $v_3\in\prod_{2q-2}$ so that $F_2=F_1v_1-v_3$ is holomorphic on D.

We claim that $F_2 \in H^p(D)$. Set $r=p$ if $0<p<\infty$, and set $r=1$ if $p=\infty$. Since F_1 is in $H^p(D)$, there is a positive superharmonic function s, which is constant on every component of D if $p=\infty$, such that $|F_1|^r \leq s$ on D. Take a neighborhood V of ∞ whose closure is included in D, and let $L=\sup_{D-V}|v_1|$, $M=\sup_{D-V}|v_3|$, and $N=\sup_V |F_2|$. Then, on $D-V$, we have

$$|F_2|^r = |F_1 v_1 - v_3|^r \leq 2^r(|F_1|^r|v_1|^r + |v_3|^r) \leq 2^r(L^r s + M^r).$$

Hence, on D,

$$|F_2|^r \leq 2^r(L^r s + M^r) + N^r,$$

which means that F_2 is an element of $H^p(D)$.

Since $F=F_2+(v_2+v_3)$, we see that F_2 is an Eichler integral equivalent to F. We have proved the theorem.

COROLLARY. *The class $E^p_{1-q}(D, G)$ forms a vector space over* $\boldsymbol{C}$.

Let G and D be as before. The orbit space D/G is a union of Riemann surfaces. Assume that there is a union S of Riemann surfaces such that $D/G = S-\{a\}$, where $a \in S$. Assume further that there is a punctured neighborhood $\tilde{V}$ of a such that the natural projection $\pi\colon D \to D/G$ is unramified over $\tilde{V}$. Let V be a component of $\pi^{-1}(\tilde{V})$, and denote by G_V the stability subgroup of V, that is, the subgroup consisting of all elements of G that fix V. Then it is known that G_V is either trivial, or a cyclic group generated by an elliptic or parabolic transformation. We set $\nu(a)=\operatorname{ord} G_V$, and according as $\nu(a)=\infty$ or $\nu(a)<\infty$, we say that the puncture a is *parabolic* or *non-parabolic*. We will denote D/G plus all such punctures by $\overline{D/G}$. Note that $\overline{D/G}$ is also a union of Riemann surfaces.

Define an operator ∂ on meromorphic functions by $\partial f = df/dz$. Bol [5] discovered that $\partial^{2q-1} \circ A^*_{1-q} = A^*_q \circ \partial^{2q-1}$ for every Möbius transformation A. A proof of this formula can be also found in Kra [16; Lemma 4.1 in Chapter V].

Kra [15] used Bol's formula to introduce two classes of Eichler integrals. If F is an Eichler integral of order $1-q$ on D for G, then Bol's formula implies that $\partial^{2q-1}F$ is a holomorphic automorphic form of weight $-2q$ on D for G, and hence is projected to a meromorphic q-differential Φ on D/G. The integral F is said to be *bounded in Kra's sense* if Φ satisfies the following three conditions:

(i) Φ extends to a meromorphic q-differential $\tilde{\Phi}$ on $\overline{D/G}$,

(ii) if a puncture $a \in \overline{D/G} - D/G$ is non-parabolic, then $\operatorname{ord}_a \tilde{\Phi} \geq -[q(1-1/\nu(a))]$, where $[x]$ denotes the largest integer not greater than x,

(iii) if a puncture $a \in \overline{D/G} - D/G$ is parabolic, then $\operatorname{ord}_a \tilde{\Phi} \geq -q+1$.

We say that F is *quasibounded in Kra's sense* if Φ satisfies (i), (ii) and the following condition:

(iii)′ if a puncture $a \in \overline{D/G} - D/G$ is parabolic, then $\operatorname{ord}_a \tilde{\Phi} \geq -q$.

The spaces of Eichler integrals that are bounded or quasibounded in Kra's sense will be denoted by $E^{Kb}_{1-q}(D, G)$ or $E^{Kc}_{1-q}(D, G)$, respectively.

We ask whether Eichler integrals in $E^{p}_{1-q}(D, G)$ are (quasi-)bounded in Kra's sense.

LEMMA 3. *If V is a bounded open subset of D, then the restriction to V of every element of $E^{p}_{1-q}(D, G)$ is contained in $H^p(V)$.*

PROOF. Since V is bounded, it is true that $\prod_{2q-2}\subset H^{\infty}(V)$. Thus the conclusion follows from Lemma 2.

LEMMA 4. *Let U be the upper half plane, and set $A(z)=z+1$. Suppose that $F\in E^{p}_{1-q}(U, \langle A\rangle)$, where $\langle A\rangle$ denotes the cyclic group generated by A. Then F has a Fourier expansion of the form*

$$F(z) = \textstyle\sum_{n=1}^{\infty} a_n e^{2\pi i n z} + a_0 z^{2q-1} + v(z)$$

on U, where $v\in\prod_{2q-2}$. If $1\leq p\leq\infty$ in addition, then $a_0=0$ in the above expansion.

PROOF. Set $S(z)=i(1+z)/(1-z)$, $T=S^{-1}$, and $F_0=S^{*}_{1-q}F$. Then $T(U)$ is the unit disk, and $F_0\in E^{p}_{1-q}(T(U), \langle T\circ A\circ T^{-1}\rangle)$. By Lemma 3 we see that $F_0\in H^p(T(U))$.

Since $(\partial^{2q-1}F)(z+1)=(\partial^{2q-1}F)(z)$, we see that $\partial^{2q-1}F$ has a Fourier expansion

$$(\partial^{2q-1}F)(z) = \textstyle\sum_{n=-\infty}^{\infty} b_n e^{2\pi i n z}$$

on U. Hence F has a Fourier expansion of the form

$$F(z) = \textstyle\sum_{n\neq 0} a_n e^{2\pi i n z} + a_0 z^{2q-1} + v(z)$$

on U, where $v\in\prod_{2q-2}$. We must show that $a_n=0$ for every negative integer n.

For each r in the interval $(1/2, 1)$, define a curve C_r: $[-1/2, 1/2]\to U$ by

$$C_r(t) = t + i\{1+r^2+\sqrt{4r^2-(1-r^2)^2t^2}\}/(1-r^2).$$

We have chosen C_r so that the image of $T\circ C_r$ is a subarc of the circle $|z|=r$.

Fix a negative integer n. If we set $V(z)=a_0z^{2q-1}+v(z)$ and $m(r)=\text{Im } C_r(1/2)$, then, by Cauchy's theorem,

$$\begin{aligned} a_n &= \int_{-1/2}^{1/2} e^{-2\pi i n(x+im(r))}\{F(x+im(r)) - V(x+im(r))\}\,dx \\ &= \int_{C_r} e^{-2\pi i n z}F(z)\,dz - \int_{-1/2}^{1/2} e^{-2\pi i n(x+im(r))}V(x+im(r))\,dx \\ &= I_1(r) - I_2(r). \end{aligned}$$

Since

$$|I_2(r)| \leq e^{2\pi nm(r)} \int_{-1/2}^{1/2} |V(x+im(r))| dx,$$

we see that $\lim_{r\to 1-0} I_2(r)=0$. To estimate $I_1(r)$, observe that if z is on the curve C_r, then

$$|T'(z)|^{-1} = |z+i|^2/2 \leq |C_r(0)+i|^2/2 = 2(1-r)^{-2}$$

Since $F(z)=(F_0\circ T)(z)\cdot T'(z)^{1-q}$, we have

$$\begin{aligned}|I_1(r)| &\leq e^{2\pi nm(r)} \int_{C_r} |F_0(T(z))|\,|T'(z)|^{1-q}|dz| \\ &\leq 2^q(1-r)^{-2q}e^{2\pi nm(r)} \int_{C_r} |F_0(T(z))|\,|T'(z)|\,|dz| \\ &\leq 2^q(1-r)^{-2q}e^{2\pi nm(r)}r \int_0^{2\pi} |F_0(re^{i\theta})| d\theta.\end{aligned}$$

Denote the last integral by $I_3(r)$. If $1\leq p\leq\infty$, then $I_3(r)=O(1)$ $(r\to 1-0)$ since $F_0\in H^1(T(U))$. If $0<p<1$, it follows from a theorem of Hardy-Littlewood (cf. Duren [8; Theorem 5.9]) that $I_3(r)=o((1-r)^{1-1/p})$ $(r\to 1-0)$. In any case we see that $\lim_{r\to 1-0} I_1(r)=0$. Thus, we have proved that $a_n=0$ for every negative integer n.

Finally, it follows from the same theorem of Hardy-Littlewood quoted above that $\sup_{0\leq\theta\leq 2\pi}|F_0(re^{i\theta})|=o((1-r)^{-1/p})$ $(r\to 1-0)$ if $0<p<\infty$. Since, for $r\in(0, 1)$,

$$\begin{aligned}a_0 = {} & 2^{q-1}i^{-q}(1+r)^{1-2q}(1-r)F_0(r) \\ & - i^{1-2q}(1+r)^{1-2q}(1-r)^{2q-1}\textstyle\sum_{n=1}^{\infty} a_n\, e^{-2\pi n(1+r)/(1-r)} \\ & - 2^{q-1}i^{-q}(1+r)^{1-2q}(1-r)(S_{1-q}^{*}v)\,(r),\end{aligned}$$

it follows that $a_0=0$ if $1\leq p\leq\infty$. The proof is complete.

THEOREM 2. *Let $F\in E_{1-q}^{p}(D, G)$. Then F is quasibounded in Kra's sense. If $1\leq p\leq\infty$ in addition, then F is bounded in Kra's sense.*

PROOF. Let Φ be the meromorphic q-differential on D/G induced by the automorphic form $\partial^{2q-1}F$. We have to verify that Φ satisfies the three conditions (i), (ii) and (iii)′ (or (iii) when $1\leq p\leq\infty$) in the definition. Let $\pi\colon D\to D/G$ be the natural projection, and let $a\in\overline{D/G}-D/G$. There is a punctured neighborhood $\tilde{V}$ of a in $\overline{D/G}$ such that π is unramified over $\tilde{V}$. Take a component V of $\pi^{-1}(\tilde{V})$.

Suppose first that a is non-parabolic. Then V is also a punctured neighborhood of some point $c \in \Omega(G)$. By taking conjugation if necessary, we may assume that V is bounded (and so $c \neq \infty$). Then c is a removable singularity of F by Lemma 3. Thus Φ is extended meromorphically to $(D/G) \cup \{a\}$ with $\mathrm{ord}_a \Phi \geq -[q(1-1/\nu(a))]$.

If a is a parabolic puncture, then we can choose $\tilde{V}$ so that V is a disk or a half plane in D. Let G_V be the stability subgroup of V. Then there is some $S \in$ Möb such that $S^{-1}(V)$ is the upper half plane and that $S^{-1} \circ G_V \circ S$ is the cyclic group generated by the translation $z \mapsto z+1$. Since $S^*_{1-q} F \in E^p_{1-q}(S^{-1}(V), S^{-1} \circ G_V \circ S)$, we can apply Lemma 4 and see that Φ is extended meromorphically to $(D/G) \cup \{a\}$ with $\mathrm{ord}_a \Phi \geq -q$ (or $\mathrm{ord}_a \Phi \geq -q+1$ when $1 \leq p \leq \infty$). This completes the proof.

Let us introduce another class of Eichler integrals. Let G and D be as before, and suppose that ∂D contains more than two points. Let $\lambda_D(z)|dz|$ denote the Poincaré metric on D. We denote by $E^A_{1-q}(D, G)$ the space of all Eichler integrals F of order $1-q$ on D for G such that $\sup_{z \in D} \lambda_D(z)^{-q} |(\partial^{2q-1} F)(z)| < +\infty$. If D is a G-invariant union of components of $\Omega(G)$ and D/G is a finite union of Riemann surfaces of finite type, then it is known that $E^A_{1-q}(D, G) = E^{Kb}_{1-q}(D, G)$ (cf. Kra [16; Remark in p. 199]).

THEOREM 3. *If G is a Fuchsian group acting on a disk or a half plane D, then $E^A_{1-q}(D, G) \subset E^p_{1-q}(D, G)$. If, in addition, the Riemann surface D/G is of finite type and if $1 \leq p \leq \infty$, then $E^p_{1-q}(D, G) = E^{Kb}_{1-q}(D, G) = E^A_{1-q}(D, G)$.*

PROOF. We may assume that D is the unit disk. Every $F \in E^A_{1-q}(D, G)$ extends to a continuous function on $\bar{D}$. A proof of this fact can be found in [16; pp. 215–217]. However, we may prove as follows.

It follows from the definition that $\sup_{z \in D} (1-|z|)^q |(\partial^{2q-1} F)(z)| < +\infty$. Hence, we see that $\sup_{z \in D} (1-|z|) |(\partial^q F)(z)| < +\infty$ by a theorem of Hardy-Littlewood (cf. Duren [8; Theorem 5.5]), and so $\partial^{q-2} F$ is extended continuously to $\bar{D}$ by a theorem of Zygmund (cf. [8; Theorem 5.3]), which implies that F also extends to a continuous function on $\bar{D}$ (cf. [8; Theorem 3.11 or Theorem 5.1]).

The last statement is an immediate consequence of the first and Theorem 2.

EXAMPLE 1. Choose $2q-1$ distinct points $a_1, \ldots, a_{2q-1}$ in $\boldsymbol{C}$. Let μ be a locally bounded measurable function on $\boldsymbol{C}$ such that $\mu(z) = O(|z|^{2q-4})$ $(z \to \infty)$, $\mu|_D = 0$, and $\mu(A(z)) A'(z)^{1-q} \overline{A'(z)} = \mu(z)$ (a.e.) for all $A \in G$. Then, the function F on $\boldsymbol{C}$ defined by

$$F(z) = \frac{1}{2\pi i} \iint_{\boldsymbol{C}} \frac{\mu(\zeta)}{\zeta - z} \left(\prod_{j=1}^{2q-1} \frac{z-a_j}{\zeta - a_j} \right) d\zeta \wedge d\bar{\zeta}$$

is in $E^\infty_{1-q}(D, G)$. (Define $F(\infty) = \lim_{z \to \infty} F(z)$ if $\infty \in D$.)

This example shows that if G is finitely generated and if D is a G-invariant union of components of $\Omega(G)$ with $D \neq \Omega(G)$, then $E^p_{1-q}(D, G) \supsetneqq \prod_{2q-2}$. However, if G is a finitely generated Schottky group or a finitely generated Fuchsian group or a finitely generated quasi-Fuchsian group of the first kind and if $D = \Omega(G)$, then $E^p_{1-q}(D, G) = \prod_{2q-2}$ for $1 \leq p \leq \infty$ since $E^{Kb}_{1-q}(D, G) = \prod_{2q-2}$. For details, see Kra [16; Chapter V].

EXAMPLE 2. We assume that ∞ is in $\Omega(G)$ and is not fixed by any element in $G - \{\mathrm{id}\}$. Set $S = \cup_{A \in G} \{A(\infty)\}$. If $\zeta \in \Omega(G) - S$, then the Poincaré series

$$F(z, \zeta) = \sum_{A \in G} \frac{A'(\zeta)^q}{z - A(\zeta)}$$

converges and represents a meromorphic function of z on $\Omega(G)$. If $\zeta \notin D$, then it is known that $F(\cdot, \zeta)$ is an Eichler integral of order $1-q$ on D (cf. Ahlfors [2; Section 6.2] or Kra [16; Lemma 7.2 in Chapter V]). We set

$$F_\nu(z, \zeta) = \frac{\partial^{\nu-1} F}{\partial \zeta^{\nu-1}}(z, \zeta)$$

for every positive integer ν. It is clear that $F_\nu(\cdot, \zeta)$ is also an Eichler integral. If $\zeta \notin \bar{D}$ and $q \geq 3$, then $F_\nu(\cdot, \zeta) \in E^\infty_{1-q}(D, G)$.

To show this, first note that if the Möbius transformation $A: z \mapsto \dfrac{az+b}{cz+d}$ $(ad - bc = 1)$ belongs to $G - \{\mathrm{id}\}$, then $\dfrac{\partial^{\nu-1}}{\partial \zeta^{\nu-1}}\left(\dfrac{A'(\zeta)^q}{z - A(\zeta)}\right)$ is a (finite) linear combination of functions in

$$\left\{ \frac{c^l A'(\zeta)^{l/2+m+q-1}}{(z - A(\zeta))^m} \,\middle|\, l \in \boldsymbol{N} \cup \{0\},\ m \in \boldsymbol{N} \right\}.$$

The coefficients in this linear combination do not depend on A. Let δ (>0) be the distance from ζ to S. Then, since $-d/c = A^{-1}(\infty) \in S$,

$$\delta |c| \leq \left| \zeta + \frac{d}{c} \right| |c| = |c\zeta + d| = |A'(\zeta)|^{-1/2}$$

Next, let V_0 be a disk containing ζ such that $\bar{V}_0 \cap \bar{D} = \emptyset$. We can choose V_0 so that $B(V_0) = V_0$ if $B \in G_\zeta$ and $B(V_0) \cap V_0 = \emptyset$ if $B \in G - G_\zeta$, where G_ζ is the stability subgroup of $\{\zeta\}$. Set $V = \cup_{B \in G} B(V_0)$, and denote the distance from $w \in V$ to ∂V by $\tau(w)$. Then, we have $\lambda_V(w)\tau(w) \geq K > 0$, where $\lambda_V(w)|dw|$ is the Poincaré metric on V and K is an absolute constant (cf. [16; Proposition 1.1 (d) in Chapter II]). Hence,

$$|z - A(\zeta)| \geq \tau(A(\zeta)) \geq K\lambda_V(A(\zeta))^{-1} = K\lambda_V(\zeta)^{-1}|A'(\zeta)|$$

for every $z \in D$. Thus, we have

$$\left|\frac{c^l A'(\zeta)^{l/2+m+q-1}}{(z-A(\zeta))^m}\right| \leq \frac{\lambda_V(\zeta)^m}{K^m \delta^l} |A'(\zeta)|^{q-1}.$$

Since $\infty \in \Omega(G)$ and $q-1 \geq 2$, the series $\sum_{A\in G}|A'(\zeta)|^{q-1}$ converges (cf. [16; Lemma 9.2 in Chapter III]). Therefore, we see that $F_\nu(\cdot, \zeta)$ is bounded in D and hence $F_\nu(\cdot, \zeta) \in E_{1-q}^{\infty}(D, G)$.

§2. Injectivity of period maps

Let G be a Kleinian group, and let D be a G-invariant open subset of $\Omega(G)$. The period map pd: $E_{1-q}^{p}(D, G) \to Z^1(G, \prod_{2q-2})$ is C-linear. When is this map injective?

The map is not always injective even if D is a union of components of $\Omega(G)$. For instance, let D_0 be a component of $\Omega(G)$, and assume that $\infty \in D_0$ and that the stability subgroup of D_0 is trivial. (It is known that there are Kleinian groups satisfying this hypothesis. We can even assume that D_0 is simply connected. See Accola [1]. Of course, we are assuming that G itself is not trivial). Set $D = \bigcup_{A\in G} A(D_0)$. Take F_0 in $H^p(D_0)$, and define a function F on D by $F = (A^{-1})_{1-q}^{*} F_0$ on $A(D_0)$ for each $A \in G$. Then, it is easy to see that $F \in E_{1-q}^{p}(D, G)$ and that pd $F = 0$. Therefore the period map is not injective in this case.

The purpose of this section is to prove the following theorem, which gives a sufficient condition that the period map is injective.

THEOREM 4. *Let G be a non-elementary Kleinian group, and let D be a G-invariant open subset of $\Omega(G)$. If D/G is a union of parabolic Riemann surfaces, then the period map* pd: $E_{1-q}^{p}(D, G) \to Z^1(G, \prod_{2q-2})$ *is injective.*

To prove this theorem we need the theorem of E. Hopf-Tsuji. Let Γ be a Fuchsian group which keeps the unit disk U fixed. The limit set $\Lambda(\Gamma)$ lies in the unit circle ∂U. A limit point $e^{i\theta} \in \Lambda(\Gamma)$ is said to be a *conical limit point* if there is a sequence $\{S_n\}$ in Γ such that $S_n(0) \to e^{i\theta}$ $(n \to \infty)$ in some Stolz domain with vertex $e^{i\theta}$. In this case the same sequence $\{S_n\}$ has the property that for any $\zeta \in U$ the sequence $\{S_n(\zeta)\}$ converges to $e^{i\theta}$ in some Stolz domain with vertex $e^{i\theta}$. The set of all conical limit points is denoted by $\Lambda_c(\Gamma)$.

THEOREM A (E. Hopf-Tsuji). *Let Γ be a Fuchsian group acting on the unit disk U. Then, the Lebesgue linear measure of $\Lambda_c(\Gamma)$ is 0 or 2π. The latter occurs if and only if the orbit space U/Γ is a parabolic Riemann surface.*

For a proof, see Hopf [14] and Sullivan [18; Section III]. See also Ahlfors [3; Chapter VII].

In general, let H be a group of conformal automorphisms of a Riemann surface R. Assume that H acts properly discontinuously on R, and that the universal covering surface of R is conformally equivalent to the unit disk U. Let $\pi: U \to R$ be a holomorphic universal covering map. Denote by Möb(U) the group of all Möbius transformations that fix U. Let Γ be the set of all S in Möb(U) such that $\pi \circ S = \rho(S) \circ \pi$ for some $\rho(S) \in H$. Then, it is known that Γ is a Fuchsian group and the covering map π induces a (bijective) conformal mapping $U/\Gamma \to R/H$. Further, ρ is a group homomorphism of Γ onto H. The kernel K of ρ is the covering group of π, and thus U/K is conformally equivalent to R. We call Γ the *Fuchsian model* of H via π. For details, see Kra [16; pp. 48–50].

We are now ready to prove Theorem 4.

PROOF of THEOREM 4. Suppose first that D is connected. By Lemma 1, we may assume that the point ∞ is in D and is not a fixed point of any elliptic element in G. Let $F \in E^p_{1-q}(D, G)$ and assume that pd $F = 0$. Then, we have

$$F(A(z)) = F(z)A'(z)^{q-1} \tag{1}$$

for all $A \in G$.

Let $\pi: U \to D$ be a holomorphic universal covering map with $\pi(0) = \infty$, where U denotes the unit disk. Let K be the covering group of π. Since G is non-elementary, the (planar) Riemann surface D is hyperbolic (Myrberg [17]. See also Dodziuk [7; Theorem 3.3].). This means that K is of convergence type, that is, $\sum_{T \in K} (1 - |T(0)|) < +\infty$, and hence the Blaschke product $B(\zeta) = \zeta \prod_{T \in K - \{\mathrm{id}\}} |T(0)|(T(0) - \zeta)/T(0)(1 - \overline{T(0)}\zeta)$ converges uniformly on every compact subset of U. We set $f(\zeta) = F(\pi(\zeta))B(\zeta)^{2q-2}$. Note that f is holomorphic on U.

By Theorem 1, there is $v \in \prod_{2q-2}$ such that $F_0 = F - v$ belongs to $H^p(D)$ and has a zero at ∞. If $g(z)$ is Green's function on D with pole at ∞, then the function $h(z) = |z|e^{-g(z)}$ is apparently bounded on D. Since $|B(\zeta)| = e^{-g \circ \pi(\zeta)}$, we have $|\pi(\zeta)B(\zeta)| = h \circ \pi(\zeta)$, which is also bounded. Thus, we have $(v \circ \pi) \cdot B^{2q-2} \in H^\infty(U)$. Since $(F_0 \circ \pi) \cdot B^{2q-2} \in H^p(U)$, we see that $f \in H^p(U)$. Hence, at almost every point $e^{i\theta} \in \partial U$, f has a non-tangential limit, which will be denoted by $\hat{f}(e^{i\theta})$.

We want to show that $\hat{f} = 0$ a.e. on ∂U. Let Γ be the Fuchsian model of G via π. Let E denote the set of all points $e^{i\theta} \in \Lambda_c(\Gamma) - \Lambda_c(K)$ at which f has a non-tangential limit. Since U/Γ is a parabolic Riemann surface, and since U/K is hyperbolic, the Lebesgue linear measure of E is equal to 2π by Theorem A. Let $e^{i\theta} \in E$. Then, there is a sequence $\{S_n\}$ in Γ such that for every $\zeta \in U$ the sequence $\{S_n(\zeta)\}$ converges to $e^{i\theta}$ in some Stolz domain with vertex $e^{i\theta}$. Since $e^{i\theta} \notin \Lambda_c(K)$, there must be infinitely many cosets in $\{K \circ S_n\}_n$, and so we may assume that $A_n = \rho(S_n)$ $(n = 1, 2, \ldots)$ are all distinct, where $\rho: \Gamma \to G$ is the canonical homomorphism. Because $\infty \in D$, the series $\sum_{A \in G} |A'(z)|^q$ converges uniformly on each compact subset of $D' = D - \bigcup_{A \in G} \{A(\infty)\}$ (cf. Kra [16; Lemma 9.2 in

Chapter III]). In particular $\lim_{n\to\infty} A_n'(z)=0$ if $z\in D'$. On the other hand, using (1), we see that

$$f(S_n(\zeta)) = F(\pi(S_n(\zeta)))B(S_n(\zeta))^{2q-2} = F(A_n(\pi(\zeta)))B(S_n(\zeta))^{2q-2}$$
$$= F(\pi(\zeta))A_n'(\pi(\zeta))^{q-1}B(S_n(\zeta))^{2q-2}.$$

Fixing $\zeta\in\pi^{-1}(D')$ and letting $n\to\infty$ in the above equality, we have $\hat{f}(e^{i\theta})=0$ since $\sup_{\zeta\in U}|B(\zeta)|=1$. Thus, we have shown that $\hat{f}=0$ a.e. on ∂U. Since $f\in H^p(U)$, this implies that $f=0$ and hence $F=0$. We have proved the theorem in the case where D is connected.

Finally, assume that D is not connected. If $F\in E_{1-q}^p(D, G)$, then $F\in E_{1-q}^p(D_0, G_0)$ for each component D_0 of D, where G_0 denotes the stability subgroup of D_0. Note that D_0/G_0 is a component of D/G. Thus, what we have proved shows that $F=0$ on D_0 if $\operatorname{pd} F=0$. Since D_0 is arbitrary, the injectivity of the period map follows.

REMARK. Let D be a G-invariant union of components of $\Omega(G)$. It is known that the period map pd: $E_{1-q}^{Kc}(D, G)\to Z^1(G, \prod_{2q-2})$ is injective if every component of D/G is a Riemann surface of finite type. Its proof depends on the theory of compact Riemann surfaces (cf. Ahlfors [2] or Kra [15; Theorem 2]). Hence, in this case, Theorem 4 is a trivial consequence of Theorem 2. If one could show that $E_{1-q}^{Kc}(D, G)=E_{1-q}^p(D, G)$ for some $p>0$, then Theorem 4 would be a generalization of the finite type case. However, at present the author does not know whether $E_{1-q}^{Kc}(D, G)=E_{1-q}^p(D, G)$ holds or not.

If G is a Fuchsian group of the first kind which fixes a disk or a half plane D, then Bers showed that the period map pd: $E_{1-q}^A(D, G)\to Z^1(G, \prod_{2q-2})$ is injective (cf. [4; Theorem 4]). He used the fact that every element in $E_{1-q}^A(D, G)$ extends to a continuous function on $\bar{D}\cap \boldsymbol{C}$, which is also used in the proof of our Theorem 3.

§3. Fine limits and periods of Eichler integrals

In this section we shall give a relation between fine limits and periods of Eichler integrals in Hardy classes. We first investigate in a slightly more general setting.

Let us recall the definition of the fine topology. For details, see Constantinescu–Cornea [6]. Let R^* denote the Martin compactification of a hyperbolic Riemann surface R, and let $\Delta=\Delta(R)=R^*-R$, the Martin boundary. To each $b\in\Delta$, there corresponds the Martin kernel k_b, which is a positive harmonic function on R. Let $\Delta_1=\Delta_1(R)$ be the set of all minimal points, that is, the set of all $b\in\Delta$ for which k_b is a minimal harmonic function. For each $b\in\Delta_1$, we denote by $\mathscr{G}_b$ the family of all open subsets D of R such that the balayaged function $(k_b)_{R-D}$ is a potential on R. Then $\mathscr{G}_b$ is a filter base. We set $\mathscr{U}(b)=\{D\cup\{b\}|D\in\mathscr{G}_b\}$. If $b\in R$, we let $\mathscr{U}(b)$ be the family of all open subsets of R that contain b. Then, the class $\{\mathscr{U}(b)|b\in R\cup\Delta_1\}$ satisfies the axiom of the fundamental neighborhood system. Thus, we can induce a new topology on $R\cup\Delta_1$, which is called the *fine topology*. Let ψ be a continuous map of R into a compact Hausdorff space X, and set $\psi^\wedge(b)=\bigcap_{D\in\mathscr{G}_b}\overline{\psi(D)}$. Denote by $\mathscr{F}(\psi)$ the set of all points $b\in\Delta_1$ such that $\psi^\wedge(b)$ consists of a single point. If $b\in\mathscr{F}(\psi)$, then we denote the element of $\psi^\wedge(b)$ by $\hat{\psi}(b)$. We may regard $\hat{\psi}$ as a map from $\mathscr{F}(\psi)$ into X.

Let $HP(R)$ denote the vector space generated by all positive harmonic functions on R. Let χ be the canonical measure of the constant function $1\in HP(R)$ (for the definition, see [6; p. 138]). Regarding every u in $HP(R)$ as a map into the extended real line $[-\infty, +\infty]$, we have $\chi(\Delta-\mathscr{F}(u))=0$. The function $\hat{u}$ is integrable with respect to χ. It is known that u can be represented uniquely in the form $u=u_q+u_s$, where u_q and u_s are quasibounded and singular, respectively, in Parreau's sense. Note that $u_q(a)=\int\hat{u}(b)k_b(a)d\chi(b)$ for all $a\in R$.

Let $\pi\colon U\to R$ be a holomorphic universal covering map, where U denotes the unit disk. As is well known, the Martin compactification U^* of U is homeomorphic to $\overline{U}$, the closure of U in $\boldsymbol{C}$, and every point on the boundary is minimal. If we take the origin 0 as a reference point, the canonical measure of $1\in HP(U)$ is the normalized Lebesgue linear measure $dm=\frac{1}{2\pi}d\theta$ on the unit circle ∂U. Since both U and R are hyperbolic, the covering map $\pi\colon U\to R$ is a Fatou map (cf. [6; Satz 10.2]). Hence, regarding π as a map into R^*, we see that $\hat{\pi}$ is defined almost everywhere on ∂U (cf. [6; Satz 14.4]).

THEOREM B (Hasumi). *Let U, R and π be as above, and let K be the group of covering transformations. Then the following two statements hold.*

(*a*) *There is a K-invariant Borel set $\mathscr{B}$ in ∂U with $m(\mathscr{B})=1$ such that $\mathscr{B}\subset\mathscr{F}(\pi)$ and $\hat{\pi}(\mathscr{B})\subset\Delta(R)$.*

(*b*) *If $v\in HP(R)$ is quasibounded in Parreau's sense, then $\widehat{v\circ\pi}=\hat{v}\circ\hat{\pi}$ a.e. on ∂U. Further, by the correspondence $\hat{v}\mapsto\hat{v}\circ\hat{\pi}$, the Banach space $L^p(d\chi)$ is isometrically isomorphic to the space $L^p(dm)_K$ of all K-invariant functions in $L^p(dm)$ $(1\leq p\leq\infty)$.*

For a proof, see Hasumi [10; §3 in Chapter III]. See also Hayashi [11; Proposition 4.4].

LEMMA 5. *Let R be a hyperbolic Riemann surface, and let H be a group of conformal automorphisms of R. Assume that H acts properly discontinuously on R, and that R/H is a parabolic Riemann surface. Then, there exist a subset*

B_H of $\Delta_1(R)$ and a sequence $\{A_n^b\}_n$ in H for every $b \in B_H$ such that the following are valid:

(*a*) $\chi(\Delta(R)-B_H)=0$,

(*b*) $\lim_{n\to\infty} A_n^b(a)=b$ *in* R^* *for all* $a \in R$, *and*

(*c*) *if* $u \in HP(R)$, *then* $\lim_{n\to\infty} u(A_n^b(a))=\hat{u}(b)$ *for almost all* $b \in B_H$ *and for all* $a \in R$.

PROOF. We shall use the same notation as in Theorem B. Let Γ be the Fuchsian model of H via π, and set $B_H=\hat{\pi}(\mathscr{B} \cap \Lambda_c(\Gamma))$. Denoting by 1_E the characteristic function of a set E, we see that

$$1_{B_H}\circ\hat{\pi} = 1_{\hat{\pi}(\mathscr{B}\cap\Lambda_c(\Gamma))}\circ\hat{\pi} = 1_{\mathscr{B}\cap\Lambda_c(\Gamma)} = 1$$

a.e. on ∂U, by Theorem A and Theorem B(a). Hence $\chi(\Delta-B_H)=0$ by Theorem B(b).

Let $b \in B_H$. There are some $e^{i\theta(b)} \in \mathscr{B} \cap \Lambda_c(\Gamma) \cap \hat{\pi}^{-1}(b)$ and a sequence $\{S_n^b\}_n$ in Γ such that, for each $\zeta \in U$, $\lim_{n\to\infty} S_n^b(\zeta)=e^{i\theta(b)}$ in some Stolz domain with vertex $e^{i\theta(b)}$. Set $A_n^b=\rho(S_n^b)$ $(n=1, 2,\dots)$, where $\rho\colon \Gamma\to H$ is the canonical surjection. If $a=\pi(\zeta)\in R$, then it follows from [6; Satz 19.2] that

$$\lim_{n\to\infty} A_n^b(a) = \lim_{n\to\infty} \rho\,(S_n^b)\circ\pi(\zeta) = \lim_{n\to\infty} \pi\circ S_n^b(\zeta) = \hat{\pi}(e^{i\theta(b)}) = b$$

in R^*. We set

$$\mathscr{B}_0 = \{T(e^{i\theta(b)})| b \in B_H \text{ and } T \in K\}.$$

It is clear that $\mathscr{B}_0$ is a K-invariant subset of $\mathscr{B} \cap \Lambda_c(\Gamma)$ and $\hat{\pi}(\mathscr{B}_0)=B_H$. Since $1_{\mathscr{B}_0}=1_{B_H}\circ\hat{\pi}=1$ a.e. on ∂U, the set $\mathscr{B}_0$ is Lebesgue measurable and has a full measure in ∂U.

Finally, let $u \in HP(R)$. Denote by u_q the quasibounded part of u, and by u_s the singular part. Note that $u_q\circ\pi$ is also quasibounded and $u_s\circ\pi$ is singular in $HP(U)$ (cf. Hasumi [10; Theorem 6B in Chapter III]). Let E be the set of all $e^{i\theta}$ in $\mathscr{B}_0$ such that

(i) the non-tangential limit and the fine limit of $u_q\circ\pi$ at $e^{i\theta}$ exist, coincide and are finite,

(ii) the non-tangential limit and the fine limit of $u_s\circ\pi$ at $e^{i\theta}$ exist and are zero,

(iii) $\hat{\pi}(e^{i\theta}) \in \mathscr{F}(u_q) \cap \mathscr{F}(u_s)$,

(iv) $\widehat{u_q\circ\pi}(e^{i\theta})=\hat{u}_q\circ\hat{\pi}(e^{i\theta})$, and

(v) $\hat{u}_s\circ\hat{\pi}(e^{i\theta})=0$.

It is easy to see that E is K-invariant and $m(E)=1$, and hence $\chi(\Delta(R)-\hat{\pi}(E))=0$. If $b \in \hat{\pi}(E)$, then $e^{i\theta(b)} \in E$ and so, for each $a=\pi(\zeta)\in R$,

$$\hat{u}(b) = \hat{u}_q(b) = \hat{u}_q \circ \hat{\pi}(e^{i\theta(b)}) = \widehat{u_q \circ \pi}(e^{i\theta(b)}) = \lim_{n\to\infty} u_q \circ \pi \circ S_n^b(\zeta)$$
$$= \lim_{n\to\infty} u \circ \pi \circ S_n^b(\zeta) = \lim_{n\to\infty} u(A_n^b(a)) .$$

This completes the proof.

The next theorem is a corollary to Lemma 5.

THEOREM 5. *Let G be a non-elementary Kleinian group with $\infty \in \Omega(G)$, and let D be a G-invariant connected open subset of $\Omega(G)$. Suppose that D/G is a parabolic Riemann surface. If $1 \leq p \leq \infty$ and $F \in E_{1-q}^p(D, G) \cap H^p(D)$, then for almost every $b \in \Delta(D)$ there exists a sequence $\{A_n\}$ in G such that*

$$\hat{F}(b) = \lim_{n\to\infty} ((\mathrm{pd}\, F)(A_n))(z) \cdot A_n'(z)^{q-1}$$

for all $z \in D - \{\infty\}$.

PROOF. Since both $\mathrm{Re}\, F$ and $\mathrm{Im}\, F$ are quasibounded in Parreau's sense, it follows from Lemma 5 that for almost every b in $\Delta(D)$ there exists a sequence $\{A_n\}$ in G such that

$$\hat{F}(b) = \lim_{n\to\infty} F(A_n(z))$$
$$= \lim_{n\to\infty} \{((\mathrm{pd}\, F)(A_n))(z) \cdot A_n'(z)^{q-1} + F(z) \cdot A_n'(z)^{q-1}\}$$

for all $z \in D - \{\infty\}$. Since $\infty \in \Omega(G)$, we have $\lim_{n\to\infty} A_n'(z) = 0$. Thus, we obtain the conclusion.

REMARK. Theorem 5 says that the boundary function $\hat{F}$ is determined by the period of F. In particular, if D is bounded, this means that the period map is injective, and we have obtained another proof of Theorem 4 in this case.

§ 4. Fuchsian groups

Let $\mathrm{M\ddot{o}b}(U)$ denote the group of all Möbius transformations that fix the unit disk U. Let $dm = \frac{1}{2\pi} d\theta$ be the normalized Lebesgue linear measure on the unit circle ∂U. We set $P(\theta, \zeta) = \mathrm{Re}\, \{(e^{i\theta} + \zeta)/(e^{i\theta} - \zeta)\}$ for $\theta \in [0, 2\pi]$ and $\zeta \in U$. Recall that q is an integer greater than one.

LEMMA 6. *Let $A \in \mathrm{M\ddot{o}b}(U)$. If $f \in L^1(dm)$, then*

$$A'(0)^{q-1} \int_0^{2\pi} (A_{1-q}^* f)(e^{i\theta})\, dm(\theta)$$
$$= \int_0^{2\pi} f(e^{i\theta}) \{(1 - |A(0)|^2)/(1 - \overline{A(0)}\, e^{i\theta})\}^{2q-2} P(\theta, A(0))\, dm(\theta) .$$

In particular,

$$\int_0^{2\pi} \{(1-|A(0)|^2)/(1-\overline{A(0)}\,e^{i\theta})\}^{2q-2}P(\theta, A(0))\,dm(\theta) = 1.$$

PROOF. We can set $B(\zeta)=A^{-1}(\zeta)=e^{i\alpha}(\zeta-a)/(1-\bar{a}\zeta)$ with $a\in U$. Then, $B'(\zeta)=e^{i\alpha}(1-|a|^2)/(1-\bar{a}\zeta)^2$, $A(0)=a$, and $A'(0)=B'(a)^{-1}=e^{-i\alpha}(1-|a|^2)$. Therefore, we have

$$\begin{aligned}
&\int_0^{2\pi}(A_{1-q}^{*}f)(e^{i\theta})P(\theta,\zeta)\,dm(\theta)\\
&=\int_0^{2\pi} f(A(e^{i\theta}))A'(e^{i\theta})^{1-q}P(\theta,\zeta)\,dm(\theta)\\
&=\int_0^{2\pi} f(e^{i\theta})A'(B(e^{i\theta}))^{1-q}P(\theta,A(\zeta))\,dm(\theta)\\
&=\int_0^{2\pi} f(e^{i\theta})B'(e^{i\theta})^{q-1}P(\theta,A(\zeta))\,dm(\theta)\\
&=\int_0^{2\pi} f(e^{i\theta})\{e^{i\alpha}(1-|a|^2)\}^{q-1}(1-\bar{a}\,e^{i\theta})^{2-2q}P(\theta,A(\zeta))\,dm(\theta)\\
&=A'(0)^{1-q}\int_0^{2\pi} f(e^{i\theta})\{(1-|a|^2)/(1-\bar{a}\,e^{i\theta})\}^{2q-2}P(\theta,A(\zeta))\,dm(\theta).
\end{aligned}$$

Substituting zero for ζ, we have the first equality. Taking $f=1$, we see the second equality follows from the first.

COROLLARY. *If $A\in \text{Möb}(U)$, then A_{1-q}^{*} defines a bounded linear operator $L^p(dm)\to L^p(dm)$ for $1\le p\le\infty$.*

THEOREM 6. *Let Γ be a Fuchsian group that fixes a disk or a half plane D.*
(a) If D/Γ is a parabolic Riemann surface and if $1\le p\le\infty$, then

$$\text{pd}\,(E_{1-q}^{p}(D,\Gamma))\cap\text{pd}\,(E_{1-q}^{p}(\hat{C}-\bar{D},\Gamma)) = B^1(\Gamma,\textstyle\prod_{2q-2}).$$

(b) If Γ is of the second kind and if $1\le p<\infty$, then either

$$\text{pd}\,(E_{1-q}^{p}(D,\Gamma))\cap\text{pd}\,(E_{1-q}^{p}(\hat{C}-\bar{D},\Gamma)) \supsetneqq B^1(\Gamma,\textstyle\prod_{2q-2})$$

or the period map $\text{pd}\colon E_{1-q}^{p}(D,\Gamma)\to Z^1(\Gamma,\prod_{2q-2})$ *is not injective.*

PROOF. We may assume that D is the unit disk U, by Lemma 1.

(*a*) Let $F_1\in E_{1-q}^{p}(U,\Gamma)$ and $F_2\in E_{1-q}^{p}(\hat{C}-\bar{U},\Gamma)$, and suppose that $\text{pd}\,F_1=\text{pd}\,F_2$. Theorem 1 implies that F_j admits radial boundary values $\hat{F}_j(e^{i\theta})=\lim_{r\to1}F_j(r\,e^{i\theta})$ for almost all $e^{i\theta}\in\partial U$ ($j=1, 2$). Set $f=\hat{F}_1-\hat{F}_2$, and note that $f\in L^p(dm)$. We shall show that $f=0$.

Let E be the set of all points $e^{i\alpha}\in A_c(\Gamma)$ such that

$$\lim_{t\to0}\frac{1}{t}\int_0^t |f(e^{i(\theta+\alpha)})-f(e^{i\alpha})|\,d\theta = 0.$$

It follows from Theorem A and Lebesgue's theorem that $m(E)=1$.

Let $e^{i\alpha}\in E$. There exists a sequence $\{A_n\}$ in Γ such that $\lim_{n\to\infty} A_n(0)=e^{i\alpha}$ in some Stolz domain with vertex $e^{i\alpha}$. By Lemma 6, we have

$$\left|A_n'(0)^{q-1}\int_0^{2\pi}((A_n)_{1-q}^* f)(e^{i\theta})dm(\theta)-f(e^{i\alpha})\right|$$

$$=\left|\int_0^{2\pi}\{f(e^{i\theta})-f(e^{i\alpha})\}\{(1-|A_n(0)|^2)/(1-\overline{A_n(0)}e^{i\theta})\}^{2q-2}P(\theta, A_n(0))\,dm(\theta)\right|$$

$$\leq 4^{q-1}\int_0^{2\pi}|f(e^{i\theta})-f(e^{i\alpha})|P(\theta, A_n(0))dm(\theta).$$

The last member tends to zero as $n\to\infty$, by Fatou's theorem. Thus,

$$\lim_{n\to\infty} A_n'(0)^{q-1}\int_0^{2\pi}((A_n)_{1-q}^* f)(e^{i\theta})\,dm(\theta)=f(e^{i\alpha}).$$

On the other hand, since $A_{1-q}^* f=f$ for each $A\in\Gamma$, we see that

$$\lim_{n\to\infty} A_n'(0)^{q-1}\int_0^{2\pi}((A_n)_{1-q}^* f)(e^{i\theta})\,dm(\theta)$$

$$=\lim_{n\to\infty} A_n'(0)^{q-1}\int_0^{2\pi}f(e^{i\theta})\,dm(\theta)=0.$$

Hence, we have $f(e^{i\alpha})=0$, and thus $f=0$ a.e. on ∂U.

Now, every negative Fourier coefficient of $\hat{F}_1$ is zero, and the n-th Fourier coefficient of $\hat{F}_2$ is zero if $n>2q-2$. Thus, it must be that $\hat{F}_1$ $(=\hat{F}_2)$ is the restriction of some polynomial in $\prod_{2q-2}$ to ∂U, and so pd F_1 is a coboundary.

(*b*) We have only to prove the assertion under the assumption that $1<p<\infty$.

It follows from the hypothesis that we can choose a circular arc I in $\partial U\cap\Omega(\Gamma)$ such that $A(I)\cap I=\phi$ if $A\in\Gamma-\{\mathrm{id}\}$. Take f_0 in $L^p(dm)-\{0\}$ for which $f_0=0$ on $\partial U-I$, and set $f=\sum_{A\in\Gamma}A_{1-q}^* f_0$. Since $p(q-1)+1\geq 2$, the series $\sum_{A\in\Gamma}|A'(e^{i\theta})|^{p(q-1)+1}$ converges uniformly on any compact subset of $\partial U\cap\Omega(\Gamma)$ (cf. Kra [16; Lemma 9.2 in Chapter III]). Hence, $f\in L^p(dm)$, for

$$\int_0^{2\pi}|f(e^{i\theta})|^p(\theta)\ dm(\theta)$$

$$=\sum_{A\in\Gamma}\int_{A^{-1}(I)}|f_0(A(e^{i\theta}))|^p|A'(e^{i\theta})|^{p(1-q)}dm(\theta)$$

$$=\sum_{A\in\Gamma}\int_I|f_0(e^{i\theta})|^p|A'(A^{-1}(e^{i\theta}))|^{p(1-q)-1}dm(\theta)$$

$$=\sum_{A\in\Gamma}\int_I|f_0(e^{i\theta})|^p|(A^{-1})'(e^{i\theta})|^{p(q-1)+1}\ dm(\theta)$$

$$=\int_I|f_0(e^{i\theta})|^p\sum_{A\in\Gamma}|A'(e^{i\theta})|^{p(q-1)+1}\ dm(\theta)<+\infty.$$

Let H^p denote the space of all functions in $L^p(dm)$ whose negative Fourier coefficients vanish, and $\bar{H}_0^p$ the space of all functions in $L^p(dm)$ whose non-negative Fourier coefficients vanish. M. Riesz's theorem (cf. Heins [12; Theorem 3 in Chapter IV] or Hoffman [13; pp. 151–152]) assures us that f is uniquely decomposed into the sum of $Pf \in H^p$ and $Qf \in \bar{H}_0^p$. If the arc I is so short that f vanishes on an open arc, then neither Pf nor Qf is the zero function (cf. [13; the second Corollary in p. 52]). Since $A_{1-q}^* f = f$ for each $A \in \Gamma$, it follows that $A_{1-q}^*(Pf) - Pf$ $(= Qf - A_{1-q}^*(Qf))$ is the restriction of some polynomial in $\prod_{2q-2}$ to ∂U. Thus Pf and Qf determine $F_1 \in E_{1-q}^p(U, \Gamma)$ and $F_2 \in E_{1-q}^p(\hat{C} - \bar{U}, \Gamma)$ with $F_2(\infty) = 0$, respectively. Since F_2 is not a polynomial, we conclude from Painlevé's theorem that F_1 is not a polynomial, either. It is clear that $\mathrm{pd}\, F_1 = \mathrm{pd}(-F_2)$. If $\mathrm{pd}\, F_1$ is a coboundary, then the period map $\mathrm{pd}\colon E_{1-q}^p(U, \Gamma) \to Z^1(\Gamma, \prod_{2q-2})$ is not injective since F_1 is not a polynomial. Thus we have the conclusion.

REMARK. (*a*) If Γ in Theorem 6(*b*) is trivial, then $\mathrm{pd}(E_{1-q}^p(D, \Gamma)) \cap \mathrm{pd}(E_{1-q}^p(\hat{C} - \bar{D}, \Gamma)) = B^1(\Gamma, \prod_{2q-2}) = \{0\}$.

(*b*) Let Γ be a finitely generated Fuchsian group of the second kind acting on a disk or a half plane D. If $1 \leq p < \infty$, then the period map $\mathrm{pd}\colon E_{1-q}^p(D, \Gamma) \to Z^1(\Gamma, \prod_{2q-2})$ is not injective.

We assume that D is the unit disk U, and use the same notation as in the proof of Theorem 6(*b*). It follows from the proof of Theorem 6(*b*) that there exists an infinitely dimensional vector subspace K of $L^p(dm)$ such that if $f \in K - \{0\}$, then $A_{1-q}^* f = f$ for every $A \in \Gamma$, $Pf \neq 0$, and $Qf \neq 0$. Either $P(K)$, the image of K under P, or $Q(K)$ is of infinite dimension. On the other hand, since Γ is finitely generated, the vector space $Z^1(\Gamma, \prod_{2q-2})$ is of finite dimension. Hence, there is some $f_0 \in K - \{0\}$ such that $A_{1-q}^*(Pf_0) = Pf_0$ for every $A \in \Gamma$. The Poisson integral of Pf_0 is a non-zero element in $E_{1-q}^p(U, \Gamma)$, whose period is zero.

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