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The initial value problem for viscous incompressible flow down an inclined plane

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0. Introduction

Let us consider a viscous incompressible fluid which is bounded above by a free surface and below by a fixed plane inclined at an angle α to the horizontal $(0<\alpha<\pi/2)$. The physical situation is described in Figure 1. We choose an orthogonal coordinate system so that $\xi_1\xi_2$-plane $(\xi_3=0)$ coincides with the fixed bottom S_B, and that ξ_1-axis is in the direction of greatest slope down the plane S_B. In this coordinate system the gravity force is given by $(g_1, g_2, g_3)=(g\sin\alpha, 0, -g\cos\alpha)$ where g is the accelation of gravity. The fluid motion due to gravity is governed by the Navier-Stokes equations with appropriate boundary conditions. At the bottom S_B the fluid satisfies the adherence condition and at the free surface, which is not known a priori, satisfies the condition which states the continuity of stress across the free surface. Surface tension is neglected.

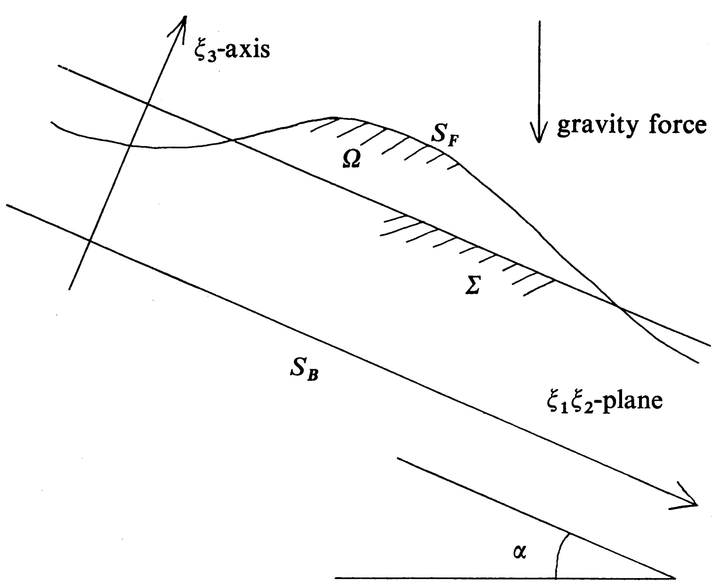

Figure 1

Thus, as in [2], we come to study the following problem: We are given an initial domain $\Omega \subset R^3$ bounded above by a surface S_F and below by the bottom S_B, as well as an initial velocity v_0 in Ω. We wish to find, for each $t \in (0, T)$, a domain $\Omega(t)$ occupied by the fluid at the time t, a velocity field $\bar{v}(\cdot, t)$ and a pressure $\bar{p}(\cdot, t)$ defined on $\Omega(t)$, and a transformation $\eta(\cdot, t)$: $\bar{\Omega} \to R^3$ so that

(0.1) $\Omega(t) = \eta(\Omega, t), \quad \eta(S_B, t) = S_B, \quad 0 < t < T,$

(0.2) $\partial_t \eta(x, t) = (\bar{v} \circ \eta)(x, t), \quad x \in \Omega, \quad 0 < t < T,$

(0.3) $\partial_t \bar{v}_i + (\bar{v}, \nabla_\eta)\bar{v}_i - \nu \Delta_\eta \bar{v}_i + (\partial \bar{p}/\partial \eta_i) = g_i \quad (i=1, 2, 3) \text{ in } \Omega(t), \quad 0 < t < T,$

(0.4) $\nabla_\eta \cdot \bar{v} = \sum_{j=1}^3 (\partial \bar{v}_j/\partial \eta_j) = 0 \quad \text{in} \quad \Omega(t), \quad 0 < t < T,$

(0.5) $\bar{p} n_i - \nu \sum_{j=1}^3 ((\partial/\partial \eta_j)\bar{v}_i + (\partial/\partial \eta_i)\bar{v}_j) n_j = p_0 n_i$

$(i=1, 2, 3) \quad \text{on} \quad \eta(S_F, t), \quad 0 < t < T,$

(0.6) $\bar{v} = 0 \quad \text{on } S_B,$

(0.7) $\bar{v}(x, 0) = v_0(x), \quad x \in \Omega,$

(0.8) $\eta(x, 0) = x, \quad x \in \Omega.$

Here (η_1, η_2, η_3) are the spatial coordinates of $\Omega(t)$; $\nabla_\eta = (\partial/\partial \eta_1, \partial/\partial \eta_2, \partial/\partial \eta_3)$ and $\Delta_\eta = \sum_{j=1}^3 (\partial/\partial \eta_j)^2$; ∂_t means $\partial/\partial t$. The constant ν is the kinematic viscosity of the fluid, and p_0 is the atmospheric pressure assumed to be constant. $n = (n_1, n_2, n_3)$ denotes the outward normal at each point of the free surface $\eta(S_F, t)$.

Among such flows described by (0.1)–(0.8) the simplest case is treated in an exercise in [5, Chap. 2, Sect. 17]: Assume that the fluid region is the inclined slab $\Sigma = \{(\xi_1, \xi_2, \xi_3); 0 < \xi_3 < 1\}$ (see Fig. 1), that is, it does not depend on t and has a constant depth equal to one everywhere. Furthermore, assume that the flow has a velocity component only in the ξ_1-direction and depends only on ξ_3. Then we obtain the flow described by the following velocity field and pressure

$$\text{(0.9)} \qquad \begin{aligned} \bar{w}_S(\xi) &= (\bar{w}_1, \bar{w}_2, \bar{w}_3) = (\xi_3(2-\xi_3) g \sin\alpha/(2\nu), 0, 0), \\ \bar{p}_S(\xi) &= p_0 - (\xi_3 - 1) g \cos\alpha, \end{aligned}$$

where $\xi \in \Sigma$. We call (0.9) the unperturbed flow down the inclined plane S_B.

In this paper we discuss the solvability of the nonstationary problem (0.1)–(0.8) when the initial domain Ω is the image of Σ under a differomorphism. We now rewrite the problem in the Lagrangian formulation to fix the domain of the unknowns upon Ω (see [2], [7], [8]): Let $v(x, t)$ be the velocity at the time t of the fluid particle, which is located at $x \in \Omega$ initially. Then define the transformation $\eta(\cdot, t)$: $\Omega \to R^3$ by the relation

$$\eta(x, t) = x + \int_0^t v(x, \tau)d\tau. \tag{0.10}$$

If $\eta(\cdot, t)\colon \Omega \to R^3$ is a regular diffeomorphism, the upper free surface of the fluid region $\Omega(t)$ at the time t is $\eta(S_F, t)$, and the relation between v and $\bar{v}$ is that $v(x, t) = \bar{v}(\eta(x, t), t)$, $x \in \Omega$, $0 < t < T$. Further, if we set $p(x, t) = \bar{p}(\eta(x, t), t)$, then (0.2)–(0.8) become

$$\partial_t \eta(x, t) = v(x, t) \quad \text{in} \quad \Omega \times (0, T), \tag{0.11}$$

$$\partial_t v_i - \nu \zeta_{kj} \partial_k(\zeta_{lj} \partial_l v_i) + \zeta_{ki} \partial_k p = g_i \qquad (i=1, 2, 3) \text{ in } \Omega \times (0, T), \tag{0.12}$$

$$\zeta_{kj} \partial_k v_j = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{0.13}$$

$$pN_i - \nu(\zeta_{kj} \partial_k v_i + \zeta_{ki} \partial_k v_j) N_j = p_0 N_i \qquad (i=1, 2, 3) \text{ on } S_F \times (0, T), \tag{0.14}$$

$$v = 0 \quad \text{on} \quad S_B \times (0, T), \tag{0.15}$$

$$v(x, 0) = v_0(x) \quad \text{in} \quad \Omega, \tag{0.16}$$

$$\eta(x, 0) = x \quad \text{in} \quad \Omega. \tag{0.17}$$

Here and hereafter we use summation convention; sum over repeated indices. ∂_k means $\partial/\partial x_k$ $(k=1, 2, 3)$. The coefficients ζ_{ij} are the (i, j) entries of the matrix $(D\eta)^{-1} = (\partial_j \eta_i)^{-1}$. $N(x, t)$ is the normal to $\eta(S_F, t)$ at $\eta(x, t)$ $(x \in S_F)$, i.e., $N = n \circ \eta$.

As stated above, Ω is assumed to be the image of the inclined slab Σ under a mapping $I\colon \Sigma \to \Omega$ of the form $I(\xi) = \xi + \phi(\xi)$, $\phi \in C^5(\bar{\Sigma}; R^3)$. The conditions on $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ are as follows:

$$|(\partial/\partial\xi)^\gamma \phi_i| \longrightarrow 0 \quad \text{as} \quad |\xi| \longrightarrow \infty \quad \text{and} \quad (\partial/\partial\xi)^\gamma \phi_i \in L^2(\Sigma) \qquad (i=1, 2, 3) \tag{0.18}$$

for any multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 + \gamma_2 + \gamma_3 \leq 5$,

$$|\partial\phi_i/\partial\xi_j| \leq 1/5 \quad \text{for} \quad \xi \in \bar{\Sigma} \quad (i, j=1, 2, 3), \tag{0.19}$$

$$\phi|_{S_B} = (0, 0, 0). \tag{0.20}$$

An elementary calculation shows that $I\colon \bar{\Sigma} \to \bar{\Omega}$ is a diffeomorphism of class C^5. We denote each point in Ω by $x = (x_1, x_2, x_3)$. Notice that $S_B = \{x_3 = 0\}$ and $S_F = I(\{\xi_3 = 1\})$.

We seek a solution of the form $v(x, t) = u(x, t) + w_S(x)$, $p(x, t) = q(x, t) + p_S(x)$ where $w_S(x) = \bar{w}(I^{-1}(x))$ and $p_S(x) = \bar{p}_S(I^{-1}(x))$. The relation (0.10) becomes

$$\eta(x, t) = x + tw_S(x) + \int_0^t u(x, \tau)d\tau. \tag{0.21}$$

Substituting $v=u+w_S$ and $p=q+p_S$ in (0.11)–(0.17), we obtain the problem for u and q:

(0.22) $$\partial_t\eta(x, t) = u(x, t) + w_S(x) \quad \text{in} \quad \Omega \times (0, T),$$

(0.23) $$\partial_t u_i - \nu\zeta_{kj}\partial_k(\zeta_{lj}\partial_l u_i) + \zeta_{ki}\partial_k q = \nu\zeta_{kj}\partial_k(\zeta_{lj}\partial_l w_i) - \zeta_{ki}\partial_k p_S + g_i \quad (i=1, 2, 3) \quad \text{in} \quad \Omega \times (0, T),$$

(0.24) $$\zeta_{kj}\partial_k u_j = -\zeta_{k1}\partial_k w_1 \quad \text{in} \quad \Omega \times (0, T),$$

(0.25) $$qN_i - \nu(\zeta_{kj}\partial_k u_i + \zeta_{ki}\partial_k u_j)N_j = 0 \quad (i=1, 2, 3) \quad S_F \times (0, T),$$

(0.26) $$u = 0 \quad \text{on} \quad S_B \times (0, T),$$

(0.27) $$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega,$$

(0.28) $$\eta(x, 0) = x \quad \text{in} \quad \Omega,$$

where $w_S=(w_1, w_2, w_3)$ and $u_0=v_0-w_S$. Here we have used the fact that

$$w_S(x_1, x_2, 0) = \bar{w}_S(I^{-1}(x_1, x_2, 0)) = (0, 0, 0),$$

$$\partial_j w_i(x) = (\partial\xi_3/\partial x_j)(\partial\bar{w}_i/\partial\xi_3)(I^{-1}(x)) = 0 \quad \text{for} \quad x \in S_F,$$

$$p_S(x) = \bar{p}_S(I^{-1}(x)) = p_0 \quad \text{for} \quad x \in S_F.$$

These follow immediately from (0.9). The purpose of this paper is to show

THEOREM. *Let Ω be as above. Suppose $3<r<7/2$. Let $u_0=(u_{0,1}, u_{0,2}, u_{0,3}) \in H^{r-1}(\Omega; R^3)$ satisfy the compatibility conditions*

$$u_0 = 0 \quad \text{on} \quad S_B,$$

(0.29) $$\{\textstyle\sum_{j=1}^3 (\partial_j u_{0,i} + \partial_i u_{0,j})n_j\}_{\tan} = 0 \quad \text{on} \quad S_F,$$

(0.30) $$\operatorname{div} u_0 = -\partial_1 w_1 \quad \text{in} \quad \Omega.$$

Then, there is a $T>0$, depending on Ω and the norm $|u_0|_{r-1}$, such that (0.23)–(0.27) *has a solution (u, q) with $u \in K^r(\Omega \times (0, T))$, $\nabla q \in K^{r-2}(\Omega \times (0, T))$ and $q|_{S_F} \in K^{r-3/2}(S_F \times (0, T))$.*

Here $n(x)=(n_1(x), n_2(x), n_3(x))$ is the unit outward normal to S_F at point $x \in S_F$. $\{\cdot\}_{\tan}$ means the tangential component of the vector in brackets. $H^{r-1}(\Omega)$ denotes the usual Sobolev space. $K^r(\Omega \times (0, T)) = H^{r/2}(0, T; H^0(\Omega)) \cap H^0(0, T; H^r(\Omega))$ etc. are the function spaces introduced in [6, Chap. 4] to study parabolic problems. (See Section 1 for precise definitions.) These function spaces are effectively used by Beale in [2]. There he considered the Navier-Stokes flow of fluid occupying a region which approaches, at infinity, to a horizontal slab vertical to the gravity force, i.e., the case that the inclination approaches to zero at infinity. Hence, he could choose $w=0$ as an unperturbed flow and linearize the problem at $w=0$, and then showed the existence of a solution, local in time, by applying the contraction mapping principle. We also use the same method as in [2] in showing the existence of a solution. We, however, linearize the problem (0.23)–(0.27) at the unperturbed flow (0.9) which has a non-zero component in the velocity. Therefore, our linearized problem is a little more complicated than the one in [2], and estimates for the linearized problem must be carried out more carefully.

Besides the work by Beale [2], there have been several investigations of the motion of viscous fluid with free boundary. In [3] Beale considered the incompressible flow near equilibrium under the effect of surface tension at free surface, and obtained a regular solution, global in time, for sufficiently small initial data. Solonnikov considered in [8] the fluid which is bounded entirely by a free surface, and proved the existence of a solution, local in time, in a suitable Hölder class. His method relies on the Schauder-type estimates and is rather involved compared with [2]. For other results, both for incompressible and compressible fluids, see [7] and the references in [2, 3] and [7].

We begin with introduction and statements of properties of some function spaces in Section 1. We also introduce some notations which are used in the later sections. In Section 2 we study an auxiliary linear problem. We regard our linearized problem as a perturbed problem from the one in [2, Sect. 4], and then solve it by using the results in [2]. Our main concern in this section is to find a "good" estimate for the solution of the linearized problem. Based on the results in Section 2 we shall prove our theorem in Section 3 using the contraction mapping principle.

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1. Preliminaries and basic notations

Throughout this paper Ω denotes the slab-like domain defined in the Introduction. If $s\geq 0$ is an integer, $H^s(\Omega)$ is the usual Sobolev space of functions whose distributional derivatives up to order s belong to $L^2(\Omega)$. For non-integer $s>0$, $H^s(\Omega)$ is the "*fractional order*" Sobolev space defined in the usual way (see [6, Chap. 1]); we denote its norm by $|\cdot|_s$. If X is a Hilbert space, $H^0(0, T; X)$ and $H^1(0, T; X)$ are the spaces of X-valued L^2 and H^1 functions on the interval $(0, T)$ respectively. Their norms are denoted by $|\cdot|_{0;T}$ and $|\cdot|_{1;T}$

respectively. We next introduce fractional order spaces of X-valued functions on $(0, T)$, following [7, Sect. 1]. If $0<r<1$, we set for an X-valued function v on $(0, T)$

$$|v|_{r;T} \equiv \left(\int_0^T \int_0^T |v(t_1) - v(t_2)|_X^2/|t_1 - t_2|^{1+2r} dt_1 dt_2\right)^{1/2}.$$

We say that v belongs to $H^r(0, T; X)$ $(0<r<2, r\neq 1)$, if the quantity

$$|v|_{r;T} \equiv |v|_{[r];T} + |\partial_t^{[r]} v|_{r-[r];T}$$

is finite. Set $Q_T=\Omega\times(0, T)$. As in [2, 3] and [7] we use the space

$$K^r(Q_T) = H^0(0, T; H^r(\Omega)) \cap H^{r/2}(0, T; H^0(\Omega))$$

which is a Hilbert space with the norm

$$|v|_{K^r} \equiv \left(\int_0^T |v(t)|_r^2 dt + |v|_{r/2;T}^2\right)^{1/2}.$$

Similarly we define $K^r(\partial\Omega\times(0, T))$, $K^r(S_F\times(0, T))$ and $K^r(S_B\times(0, T))$. If necessary, we shall write $K^r(Q_T; R)$ or $K^r(Q_T; R^3)$ to indicate real-valued or R^3-valued functions; usually the distinction should be clear from the context.

REMARK 1.1. In [2, Sect. 2] the fractional order space $H^r(0, T; X)$ is defined as the domain of $(r/2)$-th power of the positive operator $1-\partial_t^2$ with Neumann boundary condition in $H^0(0, T; X)$. Here we have followed [7, Sect. 1]. When $X=R$, the equivalence of two definitions was shown in [1, Chap. 7]. We can show that our fractional order space coincides with the one in [2] by the same argument as in [1, Chap. 7]. (See also [6, Chap. 4].) Hence, by the interpolation theory, $H^s(0, T; X)=[H^m(0, T; X), H^0(0, T; X)]_\theta$, where $(1-\theta)m=s$ and m is a positive integer $\geq s$.

We now state some properties of the function spaces introduced above.

LEMMA 1.2. *Assume* $1<r<4$.

(i) *If an integer* j *satisfies* $0\leq j<r-1/2$, *we can define the mapping* $u\to\partial_n^j u$, *which is a bounded operator*: $K^r(Q_T)\to K^{r_j}(\partial\Omega\times(0, T))$ $(r_j=r-j-1/2)$. *Here* ∂_n *is the normal derivative on* $\partial\Omega$.

(ii) *If an integer* k *satisfies* $0\leq 2k<r-1$, *we can define the mapping* $u\to\partial_t^k u(x, 0)$ *which is a bounded operator*: $K^r(Q_T)\to H^{r-2k-1}(\Omega)$.

For a proof of this Lemma, see [6, Chap. 4]. The following three lemmas are variants of [7, Lemmas 3.7–3.9 and Corollary 3.10]; see also [2, Sect. 2].

LEMMA 1.3. *Suppose* $3<r<7/2$. *Let* $u\in K^r(Q_T)$. *If* $u(0)=\partial_t u(0)=0$, *then* u *can be extended to a function* $\tilde{u}\in K^r(\Omega\times(0, \infty))$ *such that*

$$|\tilde{u}|_{K^r(\Omega\times(0,\infty))} \le C|u|_{K^r(Q_T)},$$

where C does not depend on T.

PROOF. It was shown in [7, Lemma 3.7] that $u\in H^0(0, T; H^r(\Omega))\cap H^s(0, T; H^0(\Omega))$, $s=1$ or 2, with $\partial_t^j u(0)=0$ $(0\le j<s-1/2)$ can be extended to $\tilde{u}\in H^0(0, \infty; H^r(\Omega))\cap H^s(0, \infty; H^0(\Omega))$ such that

$$\begin{aligned}|\tilde{u}|_{H^0(0,\infty;H^r(\Omega))} &+ |\tilde{u}|_{H^s(0,\infty;H^0(\Omega))} \\ &\le C(|u|_{H^0(0,T;H^r(\Omega))} + |u|_{H^s(0,T;H^0(\Omega))})\end{aligned}$$

where C does not depend on T. Hence, by interpolation we can show our case.

LEMMA 1.4. *Suppose $3<r<7/2$. Let $u\in K^r(Q_T)$. Then there is a $\tilde{u}\in K^r(\Omega\times(0, \infty))$ such that*

$$|\tilde{u}|_{K^r(\Omega\times(0,\infty))} \le C(|u(0)|_{r-1} + |\partial_t u(0)|_{r-3} + |u|_{K^r(\Omega\times(0,T))}), \tag{1.1}$$

where C is independent of T.

PROOF. By [6, Chap. 4, Theorem 2.3] we can find a function $U\in K^r(\Omega\times(0, \infty))$ such that $U(0)=u(0)$ and $\partial_t U(0)=\partial_t u(0)$ with

$$|U|_{K^r(\Omega\times(0,\infty))} \le C(|u(0)|_{r-1} + |\partial_t u(0)|_{r-3}).$$

Applying Lemma 1.3 to $u-U$, we obtain the extension $(u-U)^{\sim}$ of $u-U$ in $K^r(\Omega\times(0, \infty))$. Then, setting $\tilde{u}=(u-U)^{\sim}+U$, we obtain the desired extension.

LEMMA 1.5. *Let r and u be as above. If $0\le 2p\le r$, u belongs to $H^p(0, T; H^{r-2p}(\Omega))$ and satisfies*

$$|u|_{H^p(0,T;H^{r-2p}(\Omega))} \le C(|u(0)|_{r-1} + |\partial_t u(0)|_{r-3} + |u|_{K^r(\Omega\times(0,T))}), \tag{1.2}$$

where C does not depend on T.

PROOF. Let $\tilde{u}$ be the extension of u obtained in Lemma 1.4. By [6, Chap. 4, Proposition 2.3], u belongs to $H^p(0, \infty; H^{r-2p}(\Omega))$. The estimate (1.2) follows from (1.1) immediately.

Following [2] we write ${}^0H^r(\Omega)$ or ${}_0H^r(\Omega)$ to denote the subspace of $H^r(\Omega)$ consisting of functions which vanish on S_F or S_B, respectively. ${}^0K^r(\Omega_T)$ or ${}_0K^r(Q_T)$ is the subspace of $K^r(Q_T)$ consisting of functions which vanish on S_F or S_B. The dual space of ${}^0H^1(\Omega)$ is denoted by ${}_0H^{-1}(\Omega)$. We set

$$\tilde{K}^r(\Omega_T) = H^0(0, T; H^{r-1}(\Omega)) \cap H^{r/2}(0, T; {}_0H^{-1}(\Omega)).$$

See [2, Sect. 4] to observe how one comes to consider the spaces ${}_0H^{-1}(\Omega)$ and

$\tilde{K}^r(\Omega_T)$. We need the following two lemmas to see that the multiplications in (0.23)–(0.25) make sense in the function spaces introduced above. For their proofs see [2, Lemmas 2.5–2.6].

LEMMA 1.6. (i) *Suppose* $r>3/2$ *and* $r\geq s\geq 0$. *If* $u\in H^r(\Omega)$ *and* $v\in H^s(\Omega)$, *then* $uv\in H^s(\Omega)$ *and* $|uv|_s\leq C|u|_r|v|_s$.

(ii) *Let r and u be as above. Let* $\sigma\in {}_0H^{-1}(\Omega)$. *If we regard multiplication by u on* ${}_0H^{-1}(\Omega)$ *as the adjoint of multiplication on* ${}^0H^1(\Omega)$, *then* $u\sigma\in {}_0H^{-1}(\Omega)$ *and* $|u\sigma|_{-1}\leq C|u|_r|\sigma|_{-1}$.

(iii) *If* $u\in H^1(\Omega)$ *and* $v\in H^0(\Omega)$, *then uv is defined in* ${}_0H^{-1}(\Omega)$ *and* $|uv|_{-1}\leq C|u|_1|v|_0$.

LEMMA 1.7. *Suppose that X, Y and Z are Hilbert spaces, and that there is a bounded bilinear map* $M\colon X\times Y\to Z$. *Let* $1/2<s<2$, $s\neq 3/2$. *Let* $u\in H^s(0, T; X)$ *and* $v\in H^s(0, T; Y)$. *Then the Z-valued function uv defined by* $(uv)(t)=M(u(t), v(t))$ *belongs to* $H^s(0, T; Z)$ *and satisfies*

$$|uv|_{H^s(0,T;Z)}\leq C(|u|_{H^s}+|u(0)|_X)(|v|_{H^s}+|v(0)|_Y)$$

if $1/2<s<3/2$, *and*

$$|uv|_{H^s(0,T;Z)}\leq C(|u|_{H^s}+|u(0)|_X+|\partial_t u(0)|_X)(|v|_{H^s}+|v(0)|_Y+|\partial_t v(0)|_Y)$$

if $3/2<s<2$. *The constant C on the right hand side does not depend on T.*

To deal with $\tilde{K}^r(\Omega_T)$ we need

LEMMA 1.8. *Let* $\sigma_1, \sigma_2\in {}_0H^{-1}(\Omega)$. *Then there is* $\sigma\in\tilde{K}^r(\Omega\times(0, \infty))$ *such that* $\sigma(0)=\sigma_1$ *and* $\partial_t\sigma(0)=\sigma_2$ *with*

$$|\sigma|_{\tilde{K}^r(\Omega\times(0,\infty))}\leq C(|\sigma_1|_{-1}+|\sigma_2|_{-1}).$$

PROOF. Let $({}^0H^1(\Omega))^\perp$ be the orthogonal complement of ${}^0H^1(\Omega)$ in $H^1(\Omega)$. Setting $(\sigma_j, \phi)=0$ for $\phi\in({}^0H^1(\Omega))^\perp$ $(j=1, 2)$, we can regard σ_j $(j=1, 2)$ as an element of $(H^1(\Omega))'$, the dual of $H^1(\Omega)$. As characterized in [6, Chap. 1, Sect. 12], $(H^1(\Omega))'$ consists of elements of $H^{-1}(R^3)$ with support in $\bar{\Omega}$. Hence, $\tilde{\sigma}_j$, the Fourier transfrorm of σ_j $(j=1, 2)$, satisfies $(1+|y|^2)^{-1/2}\tilde{\sigma}_j\in H^0(R^3)$ $(y\in R^3)$. (i) We first assume $\sigma_1=0$. Take a real-valued function $\rho(t)\in C^2([0, \infty))\cap H^0(0, \infty)$ such that $\rho(0)=0$, $\rho'(0)=1$, and then set $\sigma(t)=\mathscr{F}^{-1}(\tilde{\sigma}_2\rho(t)\cdot\exp(-|y|^2t))$. It is easily checked that $\sigma(t)$ is the desired function. (ii) We next assume that $\sigma_2=0$. The choice of $\rho(t)$ is now made so that $\rho(0)=1$, $\rho'(0)=0$. The desired function is obtained by setting $\sigma(t)=\mathscr{F}^{-1}(\tilde{\sigma}_1\rho(t)\exp(-|y|^2t^2))$. By combining (i) and (ii) we can prove the general case.

By the same argument as in the proof of Lemma 1.4, we obtain

LEMMA 1.9. *Let* $\sigma \in \tilde{K}^r(Q_T)$ $(3<r<7/2)$. *Then we can extend* σ *to* $\tilde{\sigma} \in \tilde{K}^r(\Omega \times (0, \infty))$ *satisfying*

$$(1.3) \qquad |\tilde{\sigma}|_{\tilde{K}^r(\Omega\times(0,\infty))} \leq C(|\sigma|_{\tilde{K}^r(\Omega\times(0,T))} + |\sigma(0)|_{-1} + |\partial_t\sigma(0)|_{-1}),$$

where C is independent of T.

The lemma below will be used in calculating $(\zeta_{ij})=(\delta_{ij}+t\partial_j w_i+\int_0^t \partial_j u_i d\tau)^{-1}$, and in obtaining crucial estimates in the following two sections. For a proof see [2, Lemma 2.4].

LEMMA 1.10. *Let X be a Hilbert space. Let* $T_0>0$ *be arbitrary, and choose* $T\leq T_0$. *For* $u\in H^0(0, T; X)$ *define* $U\in H^1(0, T; X)$ *by*

$$U(t) = \int_0^t u(\tau)d\tau.$$

Then $u\to U$ *is a bounded operator from* $H^0(0, T; X)$ *to* $H^{1-\delta}(0, T; X)$ $(0<\delta<1)$, *and*

$$|U|_{H^{1-\delta}(0,T;X)} \leq CT^\delta |u|_{H^0(0,T;X)}.$$

Furthermore, if $0<\delta<1/4$, *then* $u\to U$ *is a bounded operator from* $H^{2\delta}(0, T; X)$ *to* $H^{1+\delta}(0, T; X)$, *and*

$$|U|_{H^{1+\delta}(0,T;X)} \leq CT^\delta |u|_{H^{2\delta}(0,T;X)}.$$

The constants on the right hand sides do not depend on T for $0<T<T_0$.

Let us introduce some notations to rewrite (0.22)–(0.28) in a more convenient way. From now on the constant r is fixed so that $3<r<7/2$ and write $r=3+2\delta$, $0<\delta<1/4$. First we set for $u\in K^r(Q_T; R^3)$

$$(1.4) \qquad \eta[u](x, t) \equiv x + tw_S(x) + \int_0^t u(x, \tau)d\tau, \quad x\in\Omega, \quad t>0.$$

$(D\eta[u](x, t))$ denotes the Jacobian matrix $(\partial_j\eta_i[u])$ of $\eta[u]$. If $(D\eta[u](x, t))^{-1}$ exists, we set $Z[u](x, t)\equiv(\zeta_{ij}[u](x, t))\equiv(D\eta[u](x, t))^{-1}$. In particular, for $u\equiv 0$ we set $\eta^0(x, t)=\eta[0](x, t)$, and $Z^0(x, t)\equiv(\zeta^0_{ij}(x, t))\equiv(D\eta^0)^{-1}=(\delta_{ij}+t\partial_j w_i)^{-1}$. By an easy calculation

$$\det(D\eta^0(x, t)) = 1 + t\partial_1 w_1(x).$$

Hence, from the assumption on the mapping $I: \Sigma\to\Omega$ we can choose $T_0>0$ so that

$$\min\{1+t\partial_1 w_1(x); (x, t)\in\Omega\times[0, T_0]\} > 0.$$

Consequently, on the interval $[0, T_0]$ $(\zeta^0_{ij}(x, t))$ exists. Taking into account of the form of $w_S(x)=\bar{w}_S(I^{-1}(x))$, we see that

$$(1.5)\quad \zeta^0_{11} = 1/(1+t\partial_1 w_1),\ \zeta^0_{12} = -\, t\partial_2 w_1/(1+t\partial_1 w_1),\ \zeta^0_{13} = -\, t\partial_3 w_1/(1+t\partial_1 w_1),$$
$$\zeta^0_{22} = \zeta^0_{33} = 1,\ \zeta^0_{21} = \zeta^0_{31} = \zeta^0_{32} = \zeta^0_{23} = 0,$$

and that their derivatives up to the fourth order are continuous and bounded on Q_{T_0}. In the sequel T_0 always denotes the constant chosen above.

For a while we assume that $u \in {}_0K^{3+2\delta}(Q_T)$ $(T \le T_0)$ is so small that each entry of the series of the matrix

$$(1.6)\qquad \sum_{k=1}^{\infty}(-1)^k\left(\left(\int_0^t Du d\tau\right)Z^0\right)^k$$

is convergent in $H^{1+\delta}(0, T; H^{2-2\delta}(\Omega)) \cap H^{1-\delta}(0, T; H^{2+2\delta}(\Omega))$. Note that this is possible by virtue of Lemmas 1.5, 1.6 and 1.7. Then, one can see that $Z[u]=(\zeta_{ij}[u])$ exists and is given by

$$(1.7)\qquad Z[u](x, t) = Z^0 + Z^0 \sum_{k=1}^{\infty}(-1)^k\left(\left(\int_0^t Du d\tau\right)Z^0\right)^k$$

for $(x, t) \in \Omega \times [0, T]$. Therefore, $\eta[u](\cdot, t)\colon \Omega \to \eta[u](\Omega, t)$ is a diffeomorphism for each $t \in [0, T]$. The normal $N=(N_1, N_2, N_3)$ to $\eta[u](S_F, t)$ appeared in (0.25) may be described as follows: By the assumption on Ω we can take on S_F a pair of vector fields τ_1, τ_2 of class C^4, which span the tangent space of S_F at each point. The unit normal $N[u](x, t)$ to $\eta[u](S_F, t)$ at $\eta[u](x, t)$, $x \in S_F$, can be written as

$$(1.8)\qquad N[u](x, t) = (\tau_1^* \times \tau_2^*)(x, t)/|(\tau_1^* \times \tau_2^*)(x, t)|$$

where $\tau_j^*(x, t)=(D\eta[u](x, t))\tau_j(x)$, $(x, t) \in S_F \times [0, T]$, $j=1, 2$.

Let us set for $T \le T_0$

$$X_T = \{(v, q);\ v \in {}_0K^{3+2\delta}(Q_T; R^3),\ \nabla q \in K^{1+2\delta}(Q_T; R^3),\ \text{and}\ \ q|_{S_F} \in K^{3/2+2\delta}(S_F \times (0, T))\}.$$

With $N[u]$ and $(\zeta_{ij}[u])$ introduced above we put for $(v, q) \in X_T$

$$(1.9)\qquad S_i[u](v, q)(x, t) = qN_i[u] - \nu(\zeta_{kj}[u]\partial_k v_i + \zeta_{ki}[u]\partial_k v_j)N_j[u],$$
$$i = 1, 2, 3,\ (x, t) \in S_F \times [0, T].$$

Let $S_{\tan}[u](v)$ denote the tangential part of $S[u](v, q)$, that is,

$$S_{\tan}[u](v) \equiv S[u](v, q) - (S_i[u](v, q)N_i[u])N[u].$$

Note that the right hand side does not depend on q. Also note that $S_{\tan}[u](v)$ $(x, 0)$ does not depend on u, since $\zeta_{ij}[u](x, 0)=\delta_{ij}$ $(x \in \bar{\Omega})$ and $N[u](x, 0)=$

$(\tau_1\times\tau_2)(x)/|(\tau_1\times\tau_2)(x)|=n(x)$, i.e., the normal to S_F at $x\in S_F$. Therefore, we write $S_{\tan}(v(\cdot, 0))(x)$ for $S_{\tan}[u](v)(x, 0)$, $x\in S_F$.

Let Y_T be the space of $(f, \sigma, a, v_0)\in K^{1+2\delta}(\Omega_T; R^3)\times\tilde{K}^{3+2\delta}(Q_T)\times K^{3/2+2\delta}(S_F\times(0, T); R^3)\times{}_0H^{2+2\delta}(\Omega; R^3)$ such that

$$(1.10)\qquad \nabla\cdot v_0=\sigma(0)\ \text{ in } \Omega,\quad S_{\tan}(v_0)=a_{\tan}(0)\ \text{ on } S_F.$$

Here $a_{\tan}(0)$ means the tangential part of $a(0)$. For $u\in{}_0K^{3+2\delta}(Q_T; R^3)$ such that the series (1.6) converges, we define the operator $\Lambda[u]\colon X_T\to Y_T$ by

$$(1.11)\qquad \Lambda[u](v, q)\equiv((\partial_t v_i-\nu\zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l v_i) + \zeta_{ki}[u]\partial_k q)_i,\ \zeta_{kj}[u]\partial_k v_j,\ S[u](v, q),\ v(0)).$$

We have to check that the right hand side of (1.11) belongs to Y_T under the assumption that (1.6) converges in $H^{1-\delta}(0, T; H^{2+2\delta}(\Omega))\cap H^{1+\delta}(0, T; H^{2-2\delta}(\Omega))$. First we consider the first component in the right hand side of (1.11). Since $v\in K^{3+2\delta}(Q_T; R^3)$ belongs to $H^{1/2+\delta}(0, T; H^2(\Omega))$ by Lemma 1.5, $\partial_l v_i\in H^{1/2+\delta}(0, T; H^1(\Omega))$. Taking account of the form (1.5) of $Z^0=(\zeta^0_{ij})$ and the fact that each entry of $Z[u]-Z^0$ belongs to $H^{1+\delta}(0, T; H^{2-2\delta}(\Omega))$, and by using Lemmas 1.6–1.7, we can deduce that $\zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l v_i)\in H^{1/2+\delta}(0, T; H^0(\Omega))$. Similar argument gives that $\partial_k(\zeta_{lj}[u]\partial_l v_i)\in H^0(0, T; H^{1-2\delta}(\Omega))$. By the Sobolev imbedding theorem, $H^{1+\delta}(0, T; H^{2+2\delta}(\Omega))\subset C([0, T]; H^{2+2\delta}(\Omega))$. From this and the form (1.5), it follows that $\zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l v_i)\in H^0(0, T; H^{1+2\delta}(\Omega))$. Hence, $\zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l v_i)\in K^{1+2\delta}(Q_T)$. By applying Lemmas 1.5–1.7 we can see more easily that $\partial_t v_i$ and $\zeta_{ki}[u]\partial_k q$ belong to $K^{1+2\delta}(Q_T)$. We next consider the divergence term (i.e., the second component) of the right hand side of (1.11). Since $\partial_k v_j\in H^0(0, T; H^{2+2\delta}(\Omega))$, it follows from (1.5)–(1.7) and Lemmas 1.6–1.7 that $\zeta_{kj}[u]\partial_k v_j\in H^0(0, T; H^{2+2\delta}(\Omega))$. To see $\zeta_{kj}[u]\partial_k v_j\in H^{3/2+\delta}(0, T; {}_0H^{-1}(\Omega))$, we first notice that $\partial_k v_j\in H^{3/2+\delta}(0, T; {}_0H^{-1}(\Omega))$ (see the beginning of [2, Sect. 4]). Then, by Lemma 1.6 (ii) and Lemma 1.7, $\zeta_{kj}[u]\partial_k v_j\in H^1(0, T; {}_0H^{-1}(\Omega))$. As noticed above, $\partial_t(\partial_k v_j)\in H^{1/2+\delta}(0, T; {}_0H^{-1}(\Omega))$. By direct calculation we have

$$\partial_t Z[u](x, t) = -Z[u](D(w+u))Z[u]$$

and

$$\partial_t(\zeta_{kj}[u]\partial_k v_j)=(\partial_t\zeta_{kj}[u])\partial_k v_j+\zeta_{kj}[u]\partial_t(\partial_k v_j).$$

Taking account of the form of w_S (1.5) and the fact that $\zeta_{ij}[u]-\zeta^0_{ij}\in H^{1+\delta}(0, T; H^{2+2\delta}(\Omega))$, we see that $\partial_t(\zeta_{kj}[u]\partial_k v_j)\in H^{1/2+\delta}(0, T; {}_0H^{-1}(\Omega))$, by Lemmas 1.6 and 1.7. Thus, we see that $\zeta_{kj}[u]\partial_k v_j\in\tilde{K}^{3+2\delta}(Q_T)$. As to the boundary term we observe as follows: Regard τ_j $(j=1, 2)$ as the restriction of a C^4 vector field in Ω. Then by an argument similar to the above we can see that $S_i[u](v, q)\in$

$K^{2+2\delta}(Q_T)$. By Lemma 1.2 (i) we have that $S_i[u](v, q) \in K^{3/2+2\delta}(S_F \times (0, T))$. The condition (1.10) is trivial.

Our problem (0.23)–(0.27) is now written as follows: Find $(u, q) \in X_T$ such that (1.6) converges in $H^{1+\delta}(0, T; H^{2-2\delta}(\Omega)) \cap H^{1-\delta}(0, T; H^{2+2\delta}(\Omega))$ so that $Z[u]=(\zeta_{ij}[u])=(D\eta[u])^{-1}$ exists, and (u, q) satisfies

$$(1.12) \qquad \Lambda[u](u, q) = (\Phi[u], -\zeta_{k1}[u]\partial_k w_1, 0, u_0)$$

where

$$(1.13) \quad \Phi_i[u](x, t) \equiv \nu\zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l w_i) - \zeta_{ki}[u]\partial_k p_S + g_i, \quad i = 1, 2, 3.$$

REMARK 1.11. Let u be as above. We have to check that the right hand side of (1.12) belongs to Y_T. We first note that, by (0.18)–(0.19), the inverse mapping of I is of the form, $I^{-1}(x)=x+\Xi(x)(x \in \bar{\Omega})$, where $\Xi \in H^5(\Omega; R^3) \cap C^5(\bar{\Omega}; R^3)$. Hence, from (0, 9) and (0.18) it follows that

$$(1.14) \quad \begin{aligned} &\partial_j w_1(x) = (\partial\xi_3/\partial x_j)(\partial\bar{w}_1/\partial\xi_3)(I^{-1}(x)) \in H^4(\Omega) \cap C^4(\bar{\Omega}), \\ &\partial_j p_S(x) = (\partial\xi_3/\partial x_j)(\partial\bar{p}_S/\partial\xi_3)(I^{-1}(x)) \in H^4(\Omega) \cap C^4(\bar{\Omega}), j = 1, 2,. \end{aligned}$$

Similarly we can see that

$$(1.15) \quad \partial_k\partial_j w_1 \text{ and } \partial_3\partial_j w_1 \text{ belong to } H^3(\Omega) \cap C^3(\bar{\Omega}) \qquad (k, j=1, 2).$$

By direct calculation, we have

$$\begin{aligned} \Phi_1[u] = {} & g_1(1-(\partial\xi_3/\partial x_3)^2) + (\partial^2\xi_3/\partial x_3^2)(\partial\bar{w}_1/\partial\xi_3) + \nu((\zeta_{33}[u])^2-1)\partial_3^2 w_1 \\ & + \nu\zeta_{33}[u](\partial_3\zeta_{33}[u])\partial_3 w_1 + \nu\textstyle\sum_{j=1}^2 \zeta_{3j}[u]\partial_3(\zeta_{3j}[u]\partial_3 w_1) \\ & + \nu\textstyle\sum_{k\neq 3 \, or \, l\neq 3} \zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l w_1) - \zeta_{kl}[u]\partial_k p_S. \end{aligned}$$

Then, taking account of the form of I^{-1} and the fact that $\zeta_{ij}[u]-\zeta_{ij}^0 \in H^{1+\delta}(0, T; H^{2-2\delta}(\Omega) \cap H^{1-\delta}(0, T; H^{2+2\delta}(\Omega))$, and using (1.5) and (1.14)–(1.15), we can see that $\Phi_1[u] \in K^{1+2\delta}(Q_T)$. By similar reasoning, one can see that $\Phi_j[u]$ $(j=2, 3)$ belong to $K^{1+2\delta}(Q_T)$, and that $-\zeta_{k1}[u]\partial_k w_1 \in \tilde{K}^{3+2\delta}(Q_T)$. The condition (1.10) follows from the compatibility conditions on u_0, (0.29)–(0.30). Consequently the right hand side of (1.12) belongs to Y_T.

2. Auxiliary linear problem

In this section we study the following linear problem: For an arbitrarily given $(f, \sigma, a, u_0) \in Y_T$ $(T \leq T_0)$ find $(u, q) \in X_T$ such that

$$(2.1) \qquad \Lambda[0](u, q) = (f, \sigma, a, u_0).$$

Regarding (2.1) as a perturbed problem of the one in [2, Sect. 4], we rewrite this as follows

$$(2.2)\quad \partial_t u_i - \nu\Delta u_i + \nabla_i q = f_i + \nu(\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})\partial_k\partial_l u_i + \nu\zeta^0_{kj}\partial_k\zeta^0_{lj}\partial_l u_i + (\delta_{ki} - \zeta^0_{ki})\partial_k q \quad (i=1, 2, 3) \quad \text{in} \quad \Omega\times(0, T),$$

$$(2.3)\quad \nabla\cdot u = \sigma + (\delta_{kj} - \zeta^0_{kj})\partial_k u_j \quad \text{in} \quad \Omega\times(0, T),$$

$$(2.4)\quad S_i(u, q) = a_i \quad (i=1, 2, 3) \quad \text{on} \quad S_F\times(0, T),$$

$$(2.5)\quad u(x, 0) = u_0(x) \quad \text{in} \quad \Omega.$$

Here $S_i(u, q) = qn_i - \nu(\partial_j u_i + \partial_i u_j)n_j$ with the unit normal $n(x) = (n_1, n_2, n_3)$ to S_F at x. Note that $\zeta^0_{ij}|_{S_F} = \delta_{ij}$, because $\partial_j w_i = 0$ on S_F. The main result of this section is the following proposition.

PROPOSITION 2.1. *There is a positive number $T_1 \le T_0$ such that, for an arbitrarily given $(f, \sigma, a, u_0) \in Y_{T_1}$, there exists a unique solution $(u, q) \in X_{T_1}$ of (2.2)–(2.5) with*

$$(2.6)\quad |u|_{K^{3+2\delta}(\Omega\times(0,T_1))} + |\nabla q|_{K^{1+2\delta}(\Omega\times(0,T_1))} + |q|_{K^{3/2+2\delta}(S_F\times(0,T_1))} \le C(|f|_{K^{1+2\delta}(\Omega\times(0,T_1))} + |\sigma|_{\bar{K}^{3+2\delta}(\Omega\times(0,T_1))} + |a|_{K^{3/2+2\delta}(S_F\times(0,T_1))} + |u_0|_{2+2\delta} + |f(0)|_{2\delta} + |\sigma(0)|_{-1} + |\partial_t\sigma(0)|_{-1} + |a(0)|_{H^{1/2+2\delta}(S_F)}).$$

The constant C does not depend on T_1.

We shall solve (2.2)–(2.5) by successive approximations: For $(u, q) \in X_T$ put

$$(2.7)\quad \begin{aligned} F_i[u, q] &= \nu(\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})\partial_k\partial_l u_i \\ &\quad + \nu\zeta^0_{kj}\partial_k\zeta^0_{lj}\partial_l u_i + (\delta_{ki} - \zeta^0_{ki})\partial_k q, \quad i = 1, 2, 3, \\ G[u] &= (\delta_{kj} - \zeta^0_{kj})\partial_k u_j. \end{aligned}$$

Take $u^{(0)} \in K^{3+2\delta}(\Omega\times(0, \infty))$ so that $u^{(0)}(0) = u_0$ and $\partial_t u^{(0)}(0) = 0$ with

$$|u^{(0)}|_{K^{3+2\delta}(\Omega\times(0,\infty))} \le C|u_0|_{2+2\delta}.$$

This is possible by [6, Chap. 4, Theorem 2.3]. Set $q^{(0)} \equiv 0$. We take as the n-th approximation $(u^{(n)}, q^{(n)}) \in X_T$, $n = 1, 2, 3, \ldots$, the solution of the initial value problem

$$(2.8)\quad \partial_t u_i^{(n)} - \nu\Delta u_i^{(n)} + \partial_i q^{(n)} = f_i + F_i[u^{(n-1)}, q^{(n-1)}] \; (i=1, 2, 3) \quad \text{in} \quad \Omega\times(0, T),$$

$$(2.9)\quad u^{(n)} = \sigma + G[u^{(n-1)}] \quad \text{in} \quad \Omega\times(0, T),$$

$$S_i(u^{(n)}, q^{(n)}) = a_i \quad (i=1, 2, 3) \quad \text{on} \quad S_F \times (0, T), \tag{2.10}$$

$$u^{(n)} = 0 \quad \text{on} \quad S_B \times (0, T), \tag{2.11}$$

$$u^{(n)}(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \tag{2.12}$$

To show that $(u^{(n)}, q^{(n)}) \in X_T$ can be determined from the known $(n-1)$–th approximation, we need to show the solvability of the problem

$$\partial_t u - \nu \Delta u + \nabla q = \psi \quad \text{in} \quad \Omega \times (0, T), \tag{2.13}$$

$$\nabla \cdot u = \omega \quad \text{in} \quad \Omega \times (0, T), \tag{2.14}$$

$$S(u, q) = b \quad \text{on} \quad S_F \times (0, T), \tag{2.15}$$

$$u = 0 \quad \text{on} \quad S_B \times (0, T), \tag{2.16}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega, \tag{2.17}$$

where $(\psi, \omega, b, u_0) \in Y_T$. Furthermore, for the convergence of $\{(u^{(n)}, q^{(n)})\}_{n=0}^{\infty}$ to the solution in X_T, we need a priori estimates for the solution of (2.13)–(2.17). Though this problem was investigated in detail in [2, Sections 3, 4], we have to check the construction of its solution carefully to verify (2.6) with a constant C which has the stated property. Therefore, we review here the arguments of the construction carried out in [2, Sections 3, 4] to observe how the solution of (2.13)–(2.17) depends on the given data.

(I) First we state the result in [2, Sect. 3] for the problem

$$\partial_t u - \nu \Delta u + \nabla q = f \quad \text{in} \quad \Omega \times (0, T), \tag{2.18}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{2.19}$$

$$S(u, q) = 0 \quad \text{on} \quad S_F \times (0, T), \tag{2.20}$$

$$u = 0 \quad \text{on} \quad S_B \times (0, T), \tag{2.21}$$

$$u(x, 0) = 0 \quad \text{in} \quad \Omega. \tag{2.22}$$

Set $H_d = \{\nabla \phi;\ \phi \in {}^0H^1(\Omega; R)\}$, and let H_σ be the L^2-orthogonal complement of H_d in $L^2(\Omega; R^3)$. Let P be the orthogonal projection of $L^2(\Omega; R^3)$ onto H_σ, which is used to eliminate the unknown q from (2.18). (See [2, 3]. Also see [4] and [9] for the problem in a fixed domain.) Beale [2, Theorem 3.2] obtained the following

LEMMA 2.2. *Given $f \in K^{1+2\delta}(Q_T; R^3)$ with $Pf(0)=0$, there is a unique (u, q) satisfying* (2.18)–(2.22) *with*

$$(2.23)\qquad |u|_{K^{3+2\delta}(\Omega\times(0,T))} + |\nabla q|_{K^{1+2\delta}(\Omega\times(0,T))} + |q|_{K^{3/2+2\delta}(S_F\times(0,T))} \le C|f|_{K^{1+2\delta}(\Omega\times(0,T))},$$

where C does not depend on T.

(II) We next treat (2.13)–(2.17). As in [2, Sect. 4] we consider an operator $L: X_T \to Y_T$ defined by the left hand side of (2.13)–(2.15), (2.17). The problem is rewritten in a form: For an arbitrarily given $(\phi, \omega, b, u_0) \in Y_T$, find $(u, q) \in X_T$ such that

$$(2.24)\qquad L(u, q) = (\phi, \omega, b, u_0).$$

The purpose of this subsection is to show

LEMMA 2.3. *Let (ϕ, ω, b, u_0) be as above. Then there is a unique solution $(u, q) \in X_T$ of (2.24) satisfying*

$$(2.25)\qquad |u|_{K^{3+2\delta}(\Omega\times(0,T))} + |\nabla q|_{K^{1+2\delta}(\Omega\times(0,T))} + |q|_{K^{3/2+2\delta}(S_F\times(0,T))} \le C\{|\phi|_{K^{1+2\delta}(\Omega\times(0,T))} + |\omega|_{K^{3+2\delta}(\Omega\times(0,T))} + |b|_{K^{3/2+2\delta}(S_F\times(0,T))} + |u_0|_{2+2\delta} + |\phi(0)|_{2\delta} + |\omega(0)|_{-1} + |\partial_t\omega(0)|_{-1} + |b(0)|_{H^{1/2+2\delta}(S_F)}\},$$

where C does not depend on T.

In [2, Sect. 4] the existence of the solution is proved, but the estimate like (2.25) is not shown. To see that (2.25) holds, we review the argument in the proof of [2, Theorem 4.1].

(i) The first step is the reduction of (2.24) to the case of zero initial data. Define $q_0^1 \in H^{1/2+2\delta}(S_F)$ by the equation $S(u_0, q_0^1)\cdot n = b(0)\cdot n$, and then extend q_0^1 to $H^{1+2\delta}(\Omega)$. Then we choose $q^1 \in K^{1+2\delta}(\Omega\times(0,\infty))$ such that $q^1(0) = q_0^1$ in $H^{1+2\delta}(\Omega)$ with

$$(2.26)\qquad |q^1|_{K^{2+2\delta}(\Omega\times(0,\infty))} \le C(|u_0|_{2+2\delta} + |b(0)|_{H^{1/2+2\delta}(S_F)}).$$

We choose $u^1 \in K^{3+2\delta}(\Omega\times(0,\infty))$ such that

$$u^1(0) = u_0 \quad \text{in} \quad H^{2+2\delta}(\Omega),$$
$$\partial_t u^1(0) = \nu\Delta u_0 - \nabla q_0^1 + \phi(0) \quad \text{in} \quad H^{2\delta}(\Omega),$$
$$u^1 = 0 \quad \text{on} \quad S_B \times (0,\infty),$$

with

$$(2.27)\qquad |u^1|_{K^{3+2\delta}(\Omega\times(0,\infty))} \le C(|u_0|_{2+2\delta} + |\nabla q_0^1|_{2\delta} + |\phi(0)|_{2\delta}) \le C(|u_0|_{2+2\delta} + |b(0)|_{H^{1/2+2\delta}(S_F)} + |\phi(0)|_{2\delta}).$$

The choice of (u^1, q^1) is possible by [6, Chap. 4, Theorem 2.3].

(ii) The second step is to adjust the divergence term. Set $\omega^1 = \nabla \cdot u^1$. By Lemma 1.8 and (2.27), we can extend $\omega - \omega^1$ to an element $\tilde{\omega}$ in $\tilde{K}^{3+2\delta}(\Omega \times (0, \infty))$ in such a way that $\tilde{\omega}$ satisfies

$$(2.28)\quad |\tilde{\omega}|_{\tilde{K}^{3+2\delta}(\Omega\times(0,\infty))} \le C(|\omega|_{\tilde{K}^{3+2\delta}(\Omega\times(0,T))} + |\omega(0)|_{-1} + |\partial_t\omega(0)|_{-1} + |u_0|_{2+2\delta} + |b(0)|_{H^{1/2+2\delta}(S_F)} + |\phi(0)|_{2\delta}),$$

where C does not depend on T. Then, for each $t \in (0, \infty)$, define $\theta(t) \in H^{4+2\delta}(\Omega)$ by

$$\Delta\theta(t) = \tilde{\omega}(t) \quad \text{in} \quad \Omega,$$
$$\theta(t) = 0 \quad \text{on} \quad S_F, \quad \partial_3\theta(t) = 0 \quad \text{on} \quad S_B.$$

By [2, Lemma 2.8] we have

$$(2.29)\qquad |\theta(t)|_{4+2\delta} \le C|\tilde{\omega}(t)|_{2+2\delta} \quad \text{for} \quad t \in (0, \infty).$$

Here C is independent of t. As in [2, page 376], we obtain

$$(2.30)\qquad |\nabla\theta(t_1) - \nabla\theta(t_2)|_0 \le C|\tilde{\omega}(t_1) - \tilde{\omega}(t_2)|_{-1} \quad \text{for} \quad t_1, t_2 > 0,$$

where C is independent of t_1, t_2. From (2.28)–(2.30) we obtain

$$(2.31)\quad |\nabla\theta|_{K^{3+2\delta}(\Omega\times(0,\infty))} \le C|\tilde{\omega}|_{\tilde{K}^{3+2\delta}(\Omega\times(0,\infty))}$$
$$\le C(|\omega|_{\tilde{K}^{3+2\delta}(\Omega\times(0,T))} + |\omega(0)|_{-1} + |\partial_t\omega(0)|_{-1} + |u_0|_{2+2\delta} + |b(0)|_{H^{1/2+2\delta}(S_F)} + |\phi(0)|_{2\delta}),$$

where C does not depend on T. We set $u^2 = u^1 + \nabla\theta$.

(iii) We next adjust the tangential boundary condition without changing the divergence term. As in the proof of Lemma 1.4, we can extend b to an element $\tilde{b} \in K^{3/2+2\delta}(S_F \times (0, \infty))$ satisfying

$$(2.32)\qquad |\tilde{b}|_{K^{3/2+2\delta}(S_F\times(0,\infty))} \le C(|b(0)|_{H^{1/2+2\delta}(S_F)} + |b|_{K^{3/2+2\delta}(S_F\times(0,T))}),$$

where C is independent of T. Applying [2, Lemma 4.2] with $\alpha = \{\tilde{b} - S(u^2, q^1)\}_{\tan}$ and $\beta = -\nabla\theta|_{S_B\times(0,\infty)}$, we find $u' \in K^{3+2\delta}(\Omega \times (0, \infty))$ such that $u'(0) = \partial_t u'(0) = 0$, $\nabla \cdot u'(0) = 0$, $S(u', q^1)_{\tan} = \alpha$ on $S_F \times (0, \infty)$ and $u' = -\nabla\theta$ on $S_B \times (0, \infty)$. By the construction of u' in the proof of [2, Lemma 4.2], one can see

$$|u'|_{K^{3+2\delta}(\Omega\times(0,\infty))} \le C(|\tilde{b} - S(u^2, q^1)|_{K^{3/2+2\delta}(S_F\times(0,\infty))} + |\nabla\theta|_{K^{5/2+2\delta}(S_B\times(0,\infty))}).$$

Hence, from (2.26), (2.27), (2.31) and (2.32), we can deduce

$$(2.33)\quad |u'|_{K^{3+2\delta}(\Omega\times(0,\infty))}$$
$$\le C(|\omega|_{\tilde{K}^{3+2\delta}(\Omega\times(0,T))} + |\omega(0)|_{-1} + |\partial_t\omega(0)|_{-1} + |u_0|_{2+2\delta} + |b(0)|_{H^{1/2+2\delta}(S_F)} + |b|_{K^{3/2+2\delta}(S_F\times(0,T))} + |\phi(0)|_{2\delta}).$$

Here C is independent of T. We set $u^3 = u^2 + u'$.

(iv) The final step is to find $q' \in K^{2+2\delta}(\Omega \times (0. \infty))$ so that $q'(0) = 0$ and

$$q' = \tilde{b} \cdot n - S(u^3, q^1) \cdot n \qquad \text{on} \quad S_F \times (0, \infty).$$

This is possible by [6, Chap. 4, Theorem 2.3]. Moreover, q' can be estimated as

$$(2.34) \quad |q'|_{K^{2+2\delta}(\Omega \times (0,\infty))} \leq C(|\tilde{b}|_{K^{3/2+2\delta}(S_F \times (0,\infty))} + |S(u^3, q^1)|_{K^{3/2+2\delta}(S_F \times (0,\infty))}).$$

If we set $q^3 = q^1 + q'$, we have

$$L(u^3, q^3) = (\phi^3, \omega, b, u_0),$$

where $\phi^3 = \partial_t u^3 - \nu \Delta u^3 + \nabla q^3$. Thus, for $\tilde{u} \equiv u - u^3$, $\tilde{q} \equiv q - q^3$ our problem is reduced to

$$(2.35) \qquad L(\tilde{u}, \tilde{q}) = (\phi - \phi^3, 0, 0, 0).$$

By the construction of (u^3, q^3), ϕ^3 satisfies $P\phi^3(0) = P\phi(0)$, and the estimates (2.26–27), (2.31) and (2.33–34) imply

$$\begin{aligned} |\phi^3|_{K^{1+2\delta}(\Omega \times (0,\infty))} &\leq C(|u^3|_{K^{3+2\delta}(\Omega \times (0,\infty))} + |q^3|_{K^{2+2\delta}(\Omega \times (0,\infty))}) \\ &\leq C(|\phi(0)|_{2\delta} + |u_0|_{2+2\delta} + |b(0)|_{H^{1/2+2\delta}(S_F)} + |\omega(0)|_{-1} \\ &\quad + |\partial_t \omega(0)|_{-1} + |b|_{K^{3/2+2\delta}(S_F \times (0,T))} + |\omega|_{\bar{K}^{3/2+2\delta}(\Omega \times (0,T))}), \end{aligned}$$

where C does not depend on T. Then, applying Lemma 2.2 to (2.35), we obtain (2.25).

Hereafter we denote the left hand side of (2.25) by $|(u, q)|_{X_T}$, and denote by $|(\phi, \omega, b, u_0)|_{Y_T}$ the quantity in braces in the right hand side of (2.25).

Proof of Proposition 2.1: Since $(\zeta^0_{ij})|_{t=0} = (\delta_{ij})$, $G[u](0) = 0$ for any $(u, q) \in X_T$. Hence, the right hand side of (2.2)–(2.4), (2.6) belongs to Y_T. From this and Lemma 2.3, it follows that $\{(u^{(n)}, q^{(n)})\}_{n=1}^{\infty}$ can be determined successively by the scheme (2.8)–(2.12). To see the convergence, set $U^{(n)} = u^{(n+1)} - u^{(n)}$, $Q^{(n)} = q^{(n+1)} - q^{(n)}$, $n = 0, 1, 2, \ldots$ From (2.8)–(2.12), we have

$$\begin{aligned} \partial_t U^{(n)} - \nu \Delta U^{(n)} + \nabla Q^{(n)} &= F[U^{(n-1)}, Q^{(n-1)}] \qquad \text{in} \quad \Omega \times (0, T), \\ \nabla \cdot U^{(n)} &= G[U^{(n-1)}] \qquad \text{in} \quad \Omega \times (0, T), \\ S(U^{(n)}, Q^{(n)}) &= 0 \qquad \text{on} \quad S_F \times (0, T), \\ U^{(n)}(x, 0) &= 0 \qquad \text{in} \quad \Omega, \end{aligned}$$

for $n = 1, 2, 3, \ldots$. By Lemma 2.3 we obtain for $n = 1, 2, 3, \ldots,$,

$$(2.36) \qquad |(U^{(n)}, Q^{(n)})|_{X_T} \leq C|(F[U^{(n-1)}, Q^{(n-1)}], G[U^{(n-1)}], 0, 0)|_{Y_T}.$$

We proceed to estimate the norms of $F[U^{(n)}, Q^{(n)}]$ and $G[U^{(n)}]$ in $K^{1+2\delta}(Q_T)$ and $\tilde{K}^{3+2\delta}(Q_T)$, respectively. Note that, from (1.5), we have

$$|(\zeta^0_{kj}\zeta^0_{lj})(x, t) - (\zeta^0_{kj}\zeta^0_{lj})(x, \tau)| \le C|t-\tau|, \quad x \in \Omega,\ 0 \le t, \tau \le T_0.$$

Also note that, by Lemma 1.5 and the Sobolev imbedding theorem, $U^{(n)} \in H^{1/2+\delta}(0, T; H^2(\Omega)) \subset C([0, T]; H^2(\Omega))$. Using these, we estimate the seminorm $|(\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})\partial_k\partial_l U_i^{(n)}|_{1/2+\delta;T}$ in $H^{1/2+\delta}(0, T; H^0(\Omega))$ as follows

$$\begin{aligned}
&\int_0^T\int_0^T |t_1-t_2|^{-2-2\delta}|(\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})\partial_k\partial_l U_i^{(n)}(t_1) \\
&\qquad\qquad - (\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})\partial_k\partial_l U_i^{(n)}(t_2)|_0^2 dt_1 dt_2 \\
&\le C\left\{\int_0^T\int_0^T |t_1-t_2|^{-2\delta}dt_1dt_2\right\} \sup\nolimits_{0<t<T}|\partial_k\partial_l U_i^{(n)}(t)|_0^2 \\
&\quad + 2\left\{\int_0^T\int_0^T |t_1-t_2|^{-2-2\delta}|\partial_k\partial_l U_i^{(n)}(t_1) - \partial_k\partial_l U_i^{(n)}(t_2)|_0^2 dt_1dt_2\right\} \\
&\qquad\qquad \times \sup\nolimits_{Q_{T_0}}|(\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})|^2 \\
&\le CT^{2-2\delta}|U_i^{(n)}|^2_{K^{3+2\delta}} + CT^2|U_i^{(n)}|^2_{K^{3+2\delta}}.
\end{aligned}$$

The norm of this term in $H^0(0, T; H^0(\Omega))$ and $H^0(0, T; H^{1+2\delta}(\Omega))$ can be estimated more easily;

$$\begin{aligned}
&|(\zeta^0_{kj}\zeta^0_{lj} - \delta_{kj}\delta_{lj})\partial_k\partial_l U_i^{(n)}|_{H^0(0,T;H^s(\Omega))} \\
&\qquad \le CT|U_i^{(n)}|_{H^0(0,T;H^{s+2}(\Omega))}, \quad s = 0,\ 1+2\delta.
\end{aligned}$$

Similarly we can estimate the norms of other two terms in $F_i[U^{(n)}, Q^{(n)}]$ in $K^{1+2\delta}(Q_T)$. We next estimate $G[U^{(n)}]$ in $\tilde{K}^{3+2\delta}(Q_T)$. By (1.5) we have

$$\begin{aligned}
|G[U^{(n)}]|_{H^0(0,T;H^{2+2\delta}(\Omega))} &= |(\delta_{kj} - \zeta^0_{kj})\partial_k U_j^{(n)}|_{H^0(0,T;H^{2+2\delta}(\Omega))} \\
&\le CT|U^{(n)}|_{H^0(0,T;H^{3+2\delta}(\Omega))}.
\end{aligned}$$

To estimate its norm in $H^{3/2+\delta}(0, T; {}_0H^{-1}(\Omega))$, we need the fact that $\partial_j U_i^{(n)} \in H^1(0, T; H^{2\delta}(\Omega))$, which follows from Lemma 1.5. From this and the equality

$$\partial_j U_i^{(n)}(t) = \int_0^t \partial_t(\partial_j U_i^{(n)})(\tau)d\tau,$$

by applying Lemma 1.10 we obtain

$$|\partial_j U_i^{(n)}|_{H^{1/2+\delta}(0,T;H^{2\delta}(\Omega))} \le CT^{1/2-\delta}|\partial_j U_i^{(n)}|_{H^1(0,T;H^{2\delta}(\Omega))}. \tag{2.37}$$

Direct differentiation of $G[U^{(n)}]$ in t gives

$$\partial_t G[U^{(n)}] = (\partial_t(\delta_{kj} - \zeta^0_{kj}))\partial_k U_j^{(n)} + (\delta_{kj} - \zeta^0_{kj})\partial_t\partial_k U_j^{(n)}.$$

Since $U^{(n)}(0)=0$ for $n=0, 1, 2,\ldots$, we see that $\partial_t G[U^{(n)}](0)=0$. Also note that $F[U^{(n)}, Q^{(n)}](0)=0$. Then, by [2, Theorem 4.3], we see that $\partial_t U^{(n)}(0)=0$ and $Q^{(n)}(0)=0$. By the argument in [2, page 375], we obtain

$$|\partial_t\partial_k U_j^{(n)}|_{H^{1/2+\delta}(0,T;{}_0H^{-1}(\Omega))} \leq C|U_j^{(n)}|_{K^{3+2\delta}(Q_T)},$$

where C is independent of T. Hence, by (1.5),

$$|(\delta_{kj}-\zeta_{kj}^0)\partial_t\partial_k U_j^{(n)}|_{H^{1/2+\delta}(0,T;{}_0H^{-1}(\Omega))} \leq CT|U^{(n)}|_{K^{3+2\delta}(Q_T)}.$$

Since we can regard $H^{2\delta}(\Omega)$ as the subspace of ${}_0H^{-1}(\Omega)$ by the L^2-inner product, we obtain from (2.37)

$$|(\partial_t(\delta_{kj}-\zeta_{kj}^0))\partial_k U_j^{(n)}|_{H^{1/2+\delta}(0,T;{}_0H^{-1}(\Omega))} \leq CT^{1/2-\delta}|U^{(n)}|_{K^{3+2\delta}(Q_T)}.$$

Therefore, we obtain

$$|\partial_t G[U^{(n)}]|_{H^{1/2+\delta}(0,T;{}_0H^{-1}(\Omega))} \leq CT^{1/2-\delta}|U^{(n)}|_{K^{3+2\delta}(Q_T)}.$$

Similarly and more easily, one can see that $|G[U^{(n)}]|_{H^0(0,T;{}_0H^{-1}(\Omega))}\leq CT|U^{(n)}|_{K^{3+2\delta}(Q_T)}$. Collecting these, we obtain

$$\begin{aligned}|F[U^{(n)}, Q^{(n)}]|_{K^{1+2\delta}(Q_T)} &+ |G[U^{(n)}]|_{\bar{K}^{3+2\delta}(Q_T)}\\ &\leq CT^{1/2-\delta}(|U^{(n)}|_{K^{3+2\delta}(Q_T)} + |\nabla Q^{(n)}|_{K^{1+2\delta}(Q_T)}),\end{aligned}$$

where C is independent of T. Hence, we can choose $T_1>0$ so that for some $0<\gamma<1$

$$|(U^{(n+1)}, Q^{(n+1)})|_{X_{T_1}} \leq \gamma|(U^{(n)}, Q^{(n)})|_{X_{T_1}} \qquad \text{for} \quad n=0, 1, 2,\ldots.$$

From this, we can deduce that $(u^{(n)}, q^{(n)})$ converges to some $(u, q)\in X_{T_1}$ which is a unique solution of (2.1). Also we can deduce

$$\begin{aligned}(2.38)\qquad |(u, q)|_{X_{T_1}} &\leq |(u^{(0)}, q^{(0)})|_{X_{T_1}} + \textstyle\sum_{n=0}^{\infty}|(U^{(n)}, Q^{(n)})|_{X_{T_1}}\\ &\leq (1-\gamma)^{-1}((2-\gamma)|(u^{(0)}, 0)|_{X_{T_1}} + |(u^{(1)}, q^{(1)})|_{X_{T_1}}).\end{aligned}$$

Since $(u^{(1)}, q^{(1)})$ satisfies (2.8)–(2.12) with $n=1$, by Lemma 2.3 we obtain

$$(2.39)\qquad |(U^{(1)}, q^{(1)})|_{X_{T_1}} \leq C|(f+F[u^{(0)}, 0], \sigma + G[u^{(0)}], a, u_0)|_{Y_{T_1}}$$

Taking account of (2.7) and the choice of $u^{(0)}$, we have

$$|(F[u^{(0)}, 0], G[u^{(0)}], 0, 0)|_{Y_{T_1}} \leq C|u^{(0)}|_{K^{3+2\delta}(Q_{T_1})} \leq C|u_0|_{H^{2+2\delta}(\Omega)},$$

where C is independent of T_1. From this, (2.38) and (2.39), the estimate (2.6) follows, which completes the proof of Proposition 2.1.

3. Proof of Theorem

We now proceed to solve the nonlinear problem (1.12). Let $u_0=(u_{0,1}, u_{0,2}, u_{0,3})\in {}_0H^{2+2\delta}(\Omega; R^3)$ be as in the Theorem. If a solution (u, q) is known, then

$$\partial_t\zeta_{ij}[u]|_{t=0} = -(\partial_j w_i+\partial_j u_{0,i}).$$

Hence, (u, q) must satisfy

$$\partial_t(\zeta_{ij}[u]\partial_i w_j)|_{t=0} = -(\partial_1 w_1)^2-(\partial_i w_1)(\partial_1 u_{0,i}). \tag{3.1}$$

By Lemma 1.8 we can choose $\sigma_0\in\tilde{K}^{3+2\delta}(Q_{T_1})$ so that

$$\sigma_0(0) = -\partial_1 w_1, \quad \partial_t\sigma_0(0) = (\partial_1 w_1)^2+(\partial_i w_1)(\partial_1 u_{0,i}).$$

Here T_1 is the constant given in Proposition 2.1. As the first approximation (u^0, q^0) to the solution (u, q), we take the solution of the problem

$$\Lambda[0](u^0, q^0) = (0, \sigma_0, 0, u_0). \tag{3.2}$$

Since $\nabla\cdot u_0=-\partial_1 w_1$ and $S_{\tan}(u_0)=0$ by the assumptions on u_0, $(0, \sigma_0, 0, u_0)$ belongs to Y_{T_1}. The existence of $(u^0, q^0)\in X_{T_1}$ is assured by Proposition 2.1. As the second approximation (u^1, q^1) we take the solution of the problem

$$\Lambda[0](u^1, q^1) = (\Phi[u^0], -\zeta_{ij}[u^0]\partial_i w_j-\sigma_0, 0, 0). \tag{3.3}$$

(See (1.13) for the definition of Φ.) To do so, we note that, by virtue of Lemma 1.10, if $T(\leq T_1)$ is small, $Z[u_0]=(\zeta_{ij}[u^0])=(\partial_j\eta_i[u^0])^{-1}$ exists and each entry of $Z[u^1]-Z^0$ belongs to $H^{1+\delta}(0, T; H^{2-2\delta}(\Omega))\cap H^{1-\delta}(0, T; H^{2+2\delta}(\Omega))$. By the choice of σ_0, the right hand side of (3.3) belongs to Y_T for such a T. Hence, there exists (u^1, q^1) by Proposition 2.1. Provided $T>0$ is kept small so that $Z[u^0]$, $Z[u^1]$ and $Z[u^0+u^1]$ exist, we seek a solution of (1.12) in the form $(u, q)=(u^0, q^0)+(u^1, q^1)+(u^2, q^2)$. The problem to find the unknown (u^2, q^2) is now written as follows: Find $(u^2, q^2)\in X_T$ such that

$$\begin{aligned}\Lambda[u^0+u^1+u^2](u^2, q^2) &= -\Lambda[u^0+u^1+u^2](u^0+u^1, q^0+q^1)\\ &\quad+(\Phi[u^0+u^1+u^2], -\zeta_{ij}[u^0+u^1+u^2]\partial_i w_j, 0, u_0).\end{aligned} \tag{3.4}$$

By using (3.2) and (3.3), we can rewrite (3.4) as follows

$$\begin{aligned}\Lambda[0](u^2, q^2) &= (\Phi[u^2+u^1+u^0]-\Phi[u^0], (\zeta_{ij}[u^0]\\ &\qquad -\zeta_{ij}[u^2+u^1+u^0])\partial_i w_j, 0, 0)\\ &\quad+(\Lambda[u^1+u^0]-\Lambda[u^2+u^1+u^0])(u^2+u^1+u^0, q^2+q^1+q^0)\\ &\quad+(\Lambda[0]-\Lambda[u^1+u^0])(u^2, q^2)+(\Lambda[0]-\Lambda[u^1+u^0])(u^1+u^0, q^1+q^0).\end{aligned} \tag{3.5}$$

For convenience we simply write $\Lambda[u]v$ for $\Lambda[u](v, q)$ etc., and set $\Lambda_0=\Lambda[0]$. If we can define on some complete subset B of X_T the mapping $R: B\to B$ by

$$(3.6)\quad \begin{aligned} &R(u^2, q^2)\\ &\equiv \Lambda_0^{-1}(\Phi[u^2+u^1+u^0]-\Phi[u^0], (\zeta_{ij}[u^0]-\zeta_{ij}[u^2+u^1+u^0])\partial_i w_j, 0, 0)\\ &\quad+\Lambda_0^{-1}((\Lambda[u^1+u^0]-\Lambda[u^2+u^1+u^0])(u^2+u^1+u^0))\\ &\quad+\Lambda_0^{-1}((\Lambda_0-\Lambda[u^1+u^0])u^2)+\Lambda_0^{-1}\Psi, \end{aligned}$$

where $\Psi=(\Lambda_0-\Lambda[u^1+u^0])(u^1+u^0, q^1+q^0)$, then our desired solution (u^2, q^2) is a fixed point of $R: B\to B$. To fix a subset B in X_T we need the following

LEMMA 3.1. *Let*

$$X_{T,0} = \{(v, q)\in X_T;\, v(0) = \partial_t v(0) = 0,\, q(0) = 0\},$$
$$Y_{T,0} = \{(f, \sigma, a, 0)\in Y_T;\, f(0) = 0,\, \sigma(0) = \partial_t\sigma(0) = 0,\, a(0) = 0\}.$$

Then, $\Lambda_0: X_{T,0}\to Y_{T,0}$ has a bounded inverse $\Lambda_0^{-1}: Y_{T,0}\to X_{T,0}$. The norms of Λ_0 and Λ_0^{-1} are bounded for $0<T\leq T_1$.

PROOF. We have only to show $\Lambda_0^{-1}(Y_{T,0})\subset X_{T,0}$. It is shown in [2, Theorem 4.3] that $L^{-1}(Y_{T,0})\subset X_{T,0}$ with the norm independent of T, where L is the linear operator defined in Section 2 (II). From this it immediately follows that each $(u^{(n)}, q^{(n)})$ in the iteration scheme (2.8)–(2.12) for a given $(f, \sigma, a, 0)\in Y_{T,0}$ belongs to $X_{T,0}$. Therefore, the limit of $\{(u^{(n)}, q^{(n)})\}_{n=1}^{\infty}$ in X_T also belongs to $X_{T,0}$. The assertion for the norm of Λ_0^{-1} follows from (2.6).

Let (u^0, q^0) and (u^1, q^1) be as above. Let $u^2\in K^{3+2\delta}(Q_T)$ with $u^2(0)=0$. Assume $|u^2|_{K^{3+2\delta}(\Omega\times(0,T))}$ is small enough so that $Z[u^0+u^1+u^2]$ exists. Then, as calculated in the beginning of this section

$$\partial_t\zeta_{ij}[u^0+u^1]\,|_{t=0} = \partial_t\zeta_{ij}[u^0+u^1+u^2]\,|_{t=0} = -\,(\partial_j w_i+\partial_j u_{0,i}).$$

Using this and $\zeta_{ij}[u^0+u^1+u^2](0)=\zeta_{ij}[u^0+u^1](0)=\delta_{ij}$, we can easily see that the first and second terms in the right hand side of (3.5) belong to $Y_{T,0}$. For the divergence term of the third term in (3.5), we have

$$\begin{aligned} \partial_t((\zeta_{ij}^0-\zeta_{ij}[u^0+u^1])\partial_i u_j^2)|_{t=0} &= (\partial_t(\zeta_{ij}^0-\zeta_{ij}[u^0+u^1]))(0)\partial_i u_j^2(0)\\ &\quad+(\zeta_{ij}^0-\zeta_{ij}[u^0+u^1])(0)\partial_t(\partial_j u_i^2)(0) = 0. \end{aligned}$$

So the third term in (3.5) also belongs to $Y_{T,0}$, if $u^2(0)=0$. Consequently, under the assumption on u^2 stated above, $R(u^2, q^2)$ defined by (3.6) satisfies that $R(u^2, q^2)-\Lambda_0^{-1}\Psi\in X_{T,0}$.

From this consideration, we take

$$B = \{(u^2, q^2) \in X_T;\ (u^2, q^2) - \Lambda_0^{-1}\Psi \in X_{T,0},\ |(u^2, q^2) - \Lambda_0^{-1}\Psi|_{X_T} \leq 1\}\ .$$

If we take T small enough, then the series (1.6) converges in $H^{1+\delta}(0, T; H^{2-2\delta}(\Omega)) \cap H^{1-\delta}(0, T; H^{2+2\delta}(\Omega))$ for every $(u^2, q^2) \in B$ by virtue of Lemma 1.10. The rest of this paper is devoted to show that the mapping R defined by (3.6) maps B into itself and is a strict contraction mapping on B, provided that T is sufficiently small. This is carried out by estimating the first three terms in the right hand side of (3.5).

(I) To begin with, we estimate the first component of the second term in (3.5), which is written as

$$\begin{aligned}(3.7)\quad & -\nu\zeta_{kj}[u^1+u^0]\partial_k(\zeta_{lj}[u^1+u^0]\partial_l u_i) + \zeta_{ki}[u^1+u^0]\partial_k q \\ &\qquad + \nu\zeta_{kj}[u]\partial_k(\zeta_{lj}[u]\partial_l u_i) - \zeta_{ki}[u]\partial_k q \\ &= -\nu(\zeta_{kj}[u^1+u^0] - \zeta_{kj}[u])\partial_k(\zeta_{lj}[u^1+u^0]\partial_l u_i) - \nu\zeta_{kj}[u]\partial_k((\zeta_{lj}[u^1+u^0] \\ &\qquad - \zeta_{lj}[u])\partial_l u_i) + (\zeta_{ki}[u^1+u^0] - \zeta_{ki}[u])\partial_k q.\end{aligned}$$

Here we have set $(u, q) = (u^2+u^1+u^0, q^2+q^1+q^0)$ again. Note that for $u \in K^{3+2\delta}(Q_T)$

$$(3.8)\qquad |\zeta_{ij}[u] - \zeta_{ij}^0|_{H^s(0,T;H^{4-2s}(\Omega))} = O(T^\delta),\ s = 1+\delta,\ 1-\delta,$$

as $T \to 0$ by (1.7) and Lemma 1.10. Since $\partial_l u_i \in H^{1/2+\delta}(0, T; H^1(\Omega))$ by Lemma 1.5, $\zeta_{lj}[u^1+u^0]\partial_l u_i \in H^{1/2+\delta}(0, T; H^1(\Omega))$ by Lemmas 1.6 and 1.7. Hence $\partial_k(\zeta_{lj}[u^1+u^0]\partial_l u_i) \in H^{1/2+\delta}(0, T; H^0(\Omega))$. Again by Lemmas 1.6 and 1.7, and using (3.8), we obtain

$$|(\zeta_{kj}[u^1+u^0] - \zeta_{kj}[u])\partial_k(\zeta_{lj}[u^1+u^0]\partial_l u_i)|_{H^{1/2+\delta}(0,T;H^0(\Omega))} = O(T^\delta)$$

uniformly for $(u^2, q^2) \in B$ as T tends to zero. As to the estimate in $H^0(0, T; H^{1+2\delta}(\Omega))$, we first note that $\partial_l u_i \in H^0(0, T; H^{2+2\delta}(\Omega))$. By the Sobolev imbedding theorem, $\zeta_{lj}[u^1+u^0] - \zeta_{lj}^0$ belongs to $C([0, T]; H^{2+2\delta}(\Omega))$. From this and Lemma 1.5, it follows that $\zeta_{lj}[u^1+u^0]\partial_l u_i$ belongs to $H^0(0, T; H^{2+2\delta}(\Omega))$, and hence $\partial_k(\zeta_{lj}[u^1+u^0]\partial_l u_i) \in H^0(0, T; H^{1+2\delta}(\Omega))$. By the same reasoning as above, we see that $(\zeta_{kj}[u^1+u^0] - \zeta_{kj}[u])\partial_k(\zeta_{lj}[u^1+u^0]\partial_l u_i)$ belongs to $H^0(0, T; H^{1+2\delta}(\Omega))$, and that its norm in $H^0(0, T; H^{1+2\delta}(\Omega))$ is of order T^δ as $T \to 0$. By the same argument, we can prove that

$$\begin{aligned}&|-\nu\zeta_{kj}[u]\partial_k((\zeta_{lj}[u^1+u^0] - \zeta_{lj}[u])\partial_l u_i) \\ &\qquad + (\zeta_{ki}[u^1+u^0] - \zeta_{ki}[u])\partial_k q|_{K^{1+2\delta}(Q_T)} \leq CT^\delta,\end{aligned}$$

for every $(u^2, q^2) \in B$. Let $(\tilde{u}^2, \tilde{q}^2)$ also be in B. Set $(\tilde{u}, \tilde{q}) \equiv (\tilde{u}^2+u^1+u^0, \tilde{q}^2+q^1+q^0)$. Note that

$$Z[\tilde{u}](x, t) - Z[u](x, t) = -Z[\tilde{u}]\Big(\int_0^t D(\tilde{u}-u)d\tau\Big)Z[u].$$

Applying Lemmas 1.5, 1.6 and 1.10 to each entry of the right hand side, we obtain

(3.9) $|\zeta_{ij}[\tilde{u}]-\zeta_{ij}[u]|_{H^s(0,T;H^{4-2s}(\Omega))} \le CT^\delta|\tilde{u}-u|_{K^{3+2\delta}(Q_T)}, \; s=1+\delta,\, 1-\delta.$

Using (3.8)–(3.9), by an argument similar to the above, we can show

$$|(\Lambda[u^1+u^0] - \Lambda[\tilde{u}])(\tilde{u}, \tilde{q}))_1 - ((\Lambda[u^1+u^0] - \Lambda[u])(u, q))_1|_{K^{1+2\delta}(Q)}$$
$$\le CT^\delta|(\tilde{u}^2, \tilde{q}^2) - (u^2, q^2)|_{X_T},$$

for $(\tilde{u}^2, \tilde{q}^2)$, $(u^2, q^2) \in B$. Here $(\Lambda[u](u, q))_1$ denotes the first component of $\Lambda[u](u, q)$. Similarly, we can show that

$$|\Phi[u^2+u^1+u^0] - \Phi[u^0]|_{K^{1+2\delta}(Q_T)} \le CT^\delta,$$
$$|\Phi[\tilde{u}^2+u^1+u^0] - \Phi[u^2+u^1+u^0]|_{K^{1+2\delta}(Q_T)} \le CT^\delta|(\tilde{u}^2, \tilde{q}^2) - (u^2, q^2)|_{X_T}$$

and that

$$|((\Lambda[0]-\Lambda[u^1+u^0])\ (u^2, q^2))_1|_{K^{1+2\delta}(Q_T)} \le CT^\delta,$$
$$|((\Lambda[0]-\Lambda[u^1+u^0])\ (\tilde{u}^2, \tilde{q}^2))_1-((\Lambda[0]-\Lambda[u^1+u^0])\ (u^2, q^2))_1|_{K^{1+2\delta}(Q_T)}$$
$$\le CT^\delta|(\tilde{u}^2, \tilde{q}^2) - (u^2, q^2)|_{X_T},$$

for $(\tilde{u}^2, \tilde{q}^2)$, $(u^2, q^2) \in B$.

(II) We next estimate the divergence term (i.e., the second component) in $\tilde{K}^{3+2\delta}(Q_T)$. The estimates in $H^0(0, T; H^{2+2\delta}(\Omega))$ can be carried out in just the same way as in (I). We only have to be concerned with the estimates in $H^{3/2+\delta}(0, T; {}_0H^{-1}(\Omega))$. Since $\partial_k(u^2+u^1+u^0)_j \in H^{1+\delta}(0, T; H^0(\Omega))$ by Lemma 1.5, it follows from Lemmas 1.6 and 1.7 and the estimate (3.8) that

$$|(\zeta_{ki}[u^1+u^0] - \zeta_{kj}[u])\partial_k u_j|_{H^1(0,T;H^0(\Omega))} = O(T^\delta)$$

uniformly for $(u^2, q^2) \in B$ as $T \to 0$. Next we estimate

(3.10) $\partial_t((\zeta_{kj}[u^1+u^0] - \zeta_{kj}[u^2+u^1+u^0])\partial_k u_j)$
$$= (\partial_t\zeta_{kj}[u^1+u^0]-\partial_t\zeta_{kj}[u])\partial_k u_j + (\zeta_{kj}[u^1+u^0]-\zeta_{kj}[u])\partial_t(\partial_k u_j)$$

in $H^{1/2+\delta}(0, T; {}_0H^{-1}(\Omega))$. Since $\partial_k u_j \in H^{3/2+\delta}(0, T; {}_0H^{-1}(\Omega))$, $\partial_t(\partial_k u_j)$ belongs to $H^{1/2+\delta}(0, T; {}_0H^{-1}(\Omega))$. Hence, by Lemmas 1.6 and 1.7, and by (3.8),

$$|(\zeta_{kj}[u^1+u^0] - \zeta_{kj}[u])\partial_t\partial_k u_j|_{H^{1/2+\delta}(0,T;{}_0H^{-1}(\Omega))} = O(T^\delta)$$

uniformly for $(u^2, q^2) \in B$ as $T \to 0$. To estimate the first term on the right hand side of (3.10), we express $\partial_k u_j$ as

$$\partial_k u_j(t) = \partial_k u_j(0) + \int_0^t \partial_\tau(\partial_k u_j) d\tau.$$

Note that, by Lemma 1.5, $\partial_t(\partial_k u_j) \in H^0(0, T; H^{2\delta}(\Omega))$. Then, applying Lemma 1.10 to the second term, we see that its norm in $H^{1/2+\delta}(0, T; H^{2\delta}(\Omega))$ is of order $T^{1/2-\delta}$. Regarding $\partial_k u_j(0)$ as a constant function in $H^{1/2+\delta}(0, T; H^{2\delta}(\Omega))$, we also see that the norm of $\partial_k u_j(0)$ in $H^{1/2+\delta}(0, T; H^{2\delta}(\Omega))$ is of order $T^{1/2-\delta}$. Consequently, we have

$$|\partial_k u_j|_{H^{1/2+\delta}(0,T;H^{2\delta}(\Omega))} \leq CT^{1/2-\delta}.$$

To estimate $\partial_t \zeta_{kj}[u]$, we differentiate the matrices $Z[u]$ and $Z[u^1+u^0]$ in t and obtain

$$\begin{aligned}\partial_t Z[u^1+u^0] - \partial_t Z[u] = &-(Z[u^1+u^0]-Z[u])(Dw_S+D(u^1+u^0))Z[u^1+u^0] \\ &+ Z[u](Du^2)Z[u^1+u^0] - Z[u](Dw_S+Du)(Z[u^1+u^0]-Z[u]).\end{aligned}$$

Taking account of the form of w_S, the fact that $\partial_j u_i^r \in H^{1/2+\delta}(0, T; H^1(\Omega))$ $(r=0, 1, 2)$, and the form of $(\zeta_{ij}[u])$, we can see that $\partial_t \zeta_{kj}[u^1+u^0] - \partial_t \zeta_{kj}[u] \in H^{1/2+\delta}(0, T; H^1(\Omega))$. Hence, by Lemmas 1.6 and 1.7,

$$|(\partial_t \zeta_{kj}[u^1+u^0] - \partial_t \zeta_{kj}[u])\partial_k u_j|_{H^{1/2+\delta}(0,T;{}_0H^{-1}(\Omega))} = O(T^{1/2-\delta})$$

uniformly for $(u^2, q^2) \in B$ as $T \to 0$. Furthermore, we can show in the same way that

$$\begin{aligned}|(\zeta_{kj}[u^1+u^0] - \zeta_{kj}[\tilde{u}])\partial_k \tilde{u}_j - (\zeta_{kj}[u^1+u^0] - \zeta_{kj}[u])\partial_k u_j|_{K^{3+2\delta}(Q_T)} \\ \leq CT^{1/2-\delta}|\tilde{u}^2 - u^2|_{K^{3+2\delta}(Q_T)}\end{aligned}$$

for (u^2, q^2), $(\tilde{u}^2, \tilde{q}^2) \in B$. We can similarly estimate other two divergence terms in (3.5).

(III) To treat the boundary term (i.e., the third component), we regard the vector fields τ_1, τ_2 used for the construction of the unit normal $N[u]$ as the restrictions to S_F of vector fields in Ω of class C^4. Further, we extend $q|_{S_F} \in K^{3/2+\delta}(S_F \times (0, T))$ to $q^* \in K^{2+2\delta}(\Omega \times (0, T))$. Then, we can estimate the extension of the boundary term in $K^{2+2\delta}(Q_T; R^3)$ in just the same way as in (I). After this, we restrict it to $S_F \times (0, T)$ and use Lemma 1.2(i) to obtain estimates similar to those in (I) and (II).

Collecting the results in (I), (II) and (III) and using Proposition 2.1, we see that the mapping R defined by (3.6) maps B into itself and is strictly contractive in $(u^2, q^2) \in B$, provided that $T>0$ is sufficiently small. Thus R has a fixed point in B for small $T>0$. This completes the proof of the Theorem.

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