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Uniform simplification in a full neighborhood of a turning point

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§1. Introduction

In this paper we shall consider the system of linear ordinary differential equations with a parameter

$$\varepsilon \frac{dX}{dt} = A(t, \varepsilon)X, \tag{1.1}$$

where ε is a complex parameter, t is a complex variable and X is an unknown vector function of t and ε. Let t_0, ε_0 and θ_0 be positive constants. We shall introduce the following assumptions.

(i) $A(t, \varepsilon)$ is an n by n matrix function of t and ε which is holomorphic in the domain:

$$D(t_0, \varepsilon_0, \theta_0) = \{(t, \varepsilon) \mid |t| \leqq t_0,\ 0 < |\varepsilon| \leqq \varepsilon_0,\ |\arg \varepsilon| \leqq \varepsilon_0\};$$

(ii) $A(t, \varepsilon)$ admits an asymptotic expansion:

$$A(t, \varepsilon) \simeq \textstyle\sum_{i=0}^{\infty} A_i(t)\varepsilon^i$$

uniformly for $|t| < t_0$, as ε tends to zero in the sector

$$0 < |\varepsilon| \leqq \varepsilon_0,\quad |\arg \varepsilon| \leqq \theta_0, \tag{1.2}$$

where each $A_i(t)$ is holomorphic in the closed disk $|t| \leqq t_0$;

(iii) the function $A_0(t)$ has the form

$$A_0(t) = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots 0 \\ 0 & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \vdots \ \ddots \\ \vdots & \vdots & \vdots \quad \ddots\ 1 \\ t^q & 0 & 0 \cdots\cdots 0 \end{pmatrix},$$

where q is a positive integer.

Assumption (iii) means that $t = 0$ is a turning point of order q of the differential equation (1.1) and there is no other turning point in the closed disk $|t| \leqq t_0$. In order to investigate the asymptotic behavior of solutions of the system (1.1) in a full neighborhood of the turning point $t=0$, we usually try to find a matrix $Q(t, \varepsilon)$, which is holomorphic in the domain $D(t_1, \varepsilon_1, \theta_1)$ $(0<t_1<t_0,\ 0<\varepsilon_1<\varepsilon_0,\ 0<\theta_1<\theta_0)$ and admits an asymptotic expansion of the form

$$Q(t, \varepsilon) \simeq \sum_{i=0}^{\infty} P_i(t)\varepsilon^i$$

as $\varepsilon\to 0$ in the sector (1.2), where the coefficients $P_i(t)$ $(i=0, 1,\dots)$ are holomorphic in the closed disk $|t|\leqq t_1$, such that the transformation $Y=Q(t, \varepsilon)X$ reduces the system (1.1) to a system of linear differential equations for Y, whose asymptotic behavior can then be in an easy way analyzed in the closed disk $|t|\leqq t_1$.

W. Wasow [8] solved such a problem to seek a simplifying transformation $Q(t, \varepsilon)$ which admits the uniform asymptotic expansion in a full neighborhood of a turning point for the case in which $n=2$ and $q=1$, by utilizing the properties of Airy's integral $Ai(t)$. Wasow's result was subsequently generalized to the case $q=2$ by R. Y. Lee [5], using the Whittaker's parabolic cylinder functions. Y. Sibuya [7] solved such a problem for the general case, by utilizing the subdominant solutions of the differential equation

$$y'' - P(t)y = 0,\ P(t) = t^q + a_1t^{q-1} + \cdots + a_{q-1}t + a_q.$$

In [4], by utilizing the extended Airy function of the first kind defined by M. Kohno [3], we have solved such a problem for the case in which n is any integer and $q=1$. In this paper we shall make use of the method of Sibuya [7].

THEOREM 1.1. (*See* [4]). *Let us denote the matrix* $A_1(t)=(a_{ik}(t);\ i, k=1, 2,\dots, n)$. *Under the assumption that* $a_{ik}(t)=O(t^{q-1})$ $(k<i;\ i, k=1, 2,\dots, n)$, *we can find a formal transformation*

$$Y = P(t, \varepsilon)X = \left(\sum_{i=0}^{\infty} P_i(t)\varepsilon^i\right)X,$$

where the coefficient matrices $P_i(t)$ $(i=0, 1,\dots)$ *are holomorphic for* $|t|<t_0$ *and in paticular,* $P_0(0)=I$ *(identity matrix), which reduces* $\varepsilon\frac{dX}{dt}=A(t, \varepsilon)X$ *to* $\varepsilon\frac{dY}{dt}=B(t, \varepsilon)Y$ *with*

$$B(t, \varepsilon) = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots\cdots 0 & 0 \\ 0 & 0 & 1 \cdots\cdots\cdots 0 & 0 \\ \vdots & \vdots & \vdots \qquad\qquad \vdots & \vdots \\ 0 & 0 & 0 \cdots\cdots\cdots 0 & 1 \\ b_1(t, \varepsilon) & b_2(t, \varepsilon) & b_3(t, \varepsilon)\cdots\cdots b_{n-1}(t, \varepsilon) & 0 \end{pmatrix},$$

where

$$b_1(t, \varepsilon) = t^q + \varepsilon^2\cdot\sum_{m=0}^{q-2}\beta_1^m(\varepsilon)t^m,\quad b_k(t, \varepsilon) = \varepsilon^2\cdot\sum_{m=0}^{q-2}\beta_k^m(\varepsilon)t^m$$

$$\beta_j^m(\varepsilon) = \sum_{i=0}^{\infty}\beta_j^m(i)\varepsilon^i \qquad (k=2, 3,\dots, n-1;\ j=1, 2,\dots, n-1).$$

Considering the first component of the column vector Y and putting $x = t\varepsilon^{-n/(n+q)}$ (if n is odd, applying the change of the independent variable $x = t\varepsilon^{-n/(n+q)}\omega^{-1}$ ($\omega = \exp(\pi i/(n+q))$), we easily see that the reduced system of linear differential equations becomes the single linear differential equation of the form

$$(1.3)\qquad \frac{d^n y}{dx^n} + \left(\sum_{k=1}^{n-2}\sum_{m=0}^{q-2} d_{n-k,m}x^m \frac{d^k y}{dx^k}\right) + ((-1)^{n+1}x^q + \sum_{m=0}^{q-2} b_{n,m}x^m)y = 0,$$

where

$$(1.4)\qquad b_{n-k,m} = \begin{cases} \mu^{(n+q)(n-2)-n^2-(n+q)k+nk-nm}\beta_{k+1}^m(\varepsilon) & (n;\ \text{even}) \\ \mu^{(n+q)(n-2)-n^2-(n+q)k+nk-nm}\beta_{k+1}^m(\varepsilon)\omega^{n-k} & (n;\ \text{odd}), \end{cases}$$

$$(m = 0, 1, \dots, q-2;\ k = 0, 1, \dots, n-2)$$

and $\mu^{n+q} = \varepsilon^{-1}$. *If $q(n-2) < 2n$, we can see for each p, r,*

$$(1.6)\qquad b_{p,r} = O(1) \quad \text{as } \varepsilon \text{ tends to zero in the sector (1.2).}$$

This fact is important in the following analysis.

In §3, we shall derive the following theorem.

THEOREM 1.2 (*Uniform Simplification in a sector*). *For each integer k ($k = 0, 1, \dots, n+q-1$), there exists an n by n matrix function $Q_k(t, \varepsilon)$ such that*

(i) *the components of $Q_k(t, \varepsilon)$ are holomorphic for $D(t_1, \varepsilon_1, \theta_1)$;*

(ii) *$Q_k(t, \varepsilon)$ admits an asymptotic expansion*

$$(1.7)\qquad Q_k(t, \varepsilon) \simeq P(t, \varepsilon)^{-1}$$

uniformly for $t \in \hat{S}_k$ and as μ tends to infinity in the sector

$$(1.8)\qquad |\arg \mu| \leqq \delta/(n+q), \quad |\mu| \geqq M,$$

where

$$(1.9)\qquad \hat{S}_k = \bigcap_{m=k}^{k+n-1} S_m, \quad S_k\colon |t| \leqq t_2 \ \text{ and } \ |\arg t - 2k\pi/(n+q)| \leqq (n+1)\pi/(n+q) - \delta,$$

and δ, t_2 are sufficiently small positive numbers and M is a sufficiently large positive number;

(iii) *the transformation*

$$(1.20)\qquad X = Q_k(t, \varepsilon)Y$$

changes the system of differential equations $\varepsilon\dfrac{dX}{dt} = A(t, \varepsilon)X$ into $\varepsilon\dfrac{dY}{dt} = \hat{B}(t, \varepsilon)Y$, where

$$\hat{B}(t, \varepsilon) = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots\cdots 0 & 0 \\ 0 & 0 & 1 \cdots\cdots\cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad\quad \ddots \vdots & \vdots \\ 0 & 0 & 0 \cdots\cdots\cdots 0 & 1 \\ \hat{b}_1(t, \varepsilon) & \hat{b}_2(t, \varepsilon) & \hat{b}_3(t, \varepsilon) \cdots \hat{b}_{n-1}(t, \varepsilon) & 0 \end{pmatrix}$$

and

$$\hat{b}_j(t, \varepsilon) \simeq b_j(t, \varepsilon) \qquad (j = 1, 2,\dots, n-1).$$

as μ tends to infinity in (1.8).

Sectors $\hat{S}_k$ for $k=0, 1,\dots, n+q-1$ cover a neighborhood of the turning point $t=0$ completely. However, since the transformation (1.20) depends on k, in order to establish a uniform simplification of (1.1) in a full neighborhood of $t=0$, we shall choose $\beta_j^m(\varepsilon)$ by the aid of the implicit function theorem, so that the matrices $Q_k(t, \varepsilon)$ become independent of k. To do this, we must investigate the so-called connection formulas and Stockes multipliers for solutions of the reduced differential equation (1.3). In §4, applying the Mellin transformation to (1.3), we shall show that the recessive solution of the differential equation (1.3) corresponds to the principal solution of a difference equation. In §5, and 6, utilizing the form of the principal solution, we derive that the partial derivatives of Stockes multipliers on the coefficients of $b_{p,r}$ $(pq-nr\neq n+q)$ do not vanish. In §7, we shall prepare some lemmas on relations between Stockes multipliers. In §8 and 9, we shall complete the proof of the following main theorem on uniform simplification of this paper.

THEOREM 1.3. *We assume that*

$$a_{ik}(t) = O(t^{q-1}) \qquad (k < i) \quad \text{and} \quad q(n-2) < 2\text{n},$$

then there exists an n by n matrix $Q(t, \varepsilon)$ such that

(i) *the components of $Q(t, \varepsilon)$ are holomorphic for $D(t_1, \varepsilon_1, \theta_1)$;*

(ii) *$Q(t, \varepsilon)$ admits an asymptotic expansion*

$$Q(t, \varepsilon)^{-1} \simeq \textstyle\sum_{i=0}^{\infty} P_i(t)\varepsilon^i$$

uniformly for $|t|\leqq t_1$, and as ε tends to zero in the sector

$$|\varepsilon| \leqq \varepsilon_1 \quad \text{and} \quad |\arg\varepsilon| \leqq \varepsilon_1,$$

where the components of the n by n matrices $P_i(t)$ are holomorphic for $|t|\leqq t_0$ and $P_0(0)=I$ (identity matrix);

(iii) *the transformation $X=Q(t, \varepsilon)Y$ changes the system of differential equation $\varepsilon\dfrac{dX}{dt}=A(t, \varepsilon)X$ into a system $\varepsilon\dfrac{dY}{dt}=\hat{B}(t, \varepsilon)Y$.*

The major contribution of this uniform simplification theorem is the study of the case in which $n \geqq 3$ and $q \geqq 2$. (cf. [4] and [7]) Therefore, in the following analysis, we may assume that $n \geqq 3$ and $q \geqq 2$. Furthermore, from the condition $q(n-2)<2n$, we shall consider the following cases;

case (I): $n>2$ and $q=2$, case (II): $n=3$ and $q=3$,
case (III): $n=3$ and $q=4$, case (IV): $n=3$ and $q=5$,
case (V): $n=4$ and $q=3$, case (VI): $n=5$ and $q=3$.

§2. Recessive solutions of the reduced equation

We consider the single linear ordinary differential equation of the form

$$\frac{d^n y}{dx^n} + \sum_{j=1}^{n} a_j(x) \frac{d^{n-j} y}{dx^{n-j}} = 0, \tag{2.1}$$

where $a_j(x)$ is a polynomial of degree m_j ($j=1, 2,\dots, n$). We assume that

$$m_j/j < m_n/n \qquad (j=1, 2,\dots, n-1) \tag{2.2}$$

and that the leading coefficient of $a_n(x)$ is $(-1)^{n+1}$. Under this condition, B. L. J. Braaksma [2] obtained the following results.

THEOREM 2.1. *The differential equations* (2.1) *has a solution* $\tilde{y}(x, a)=\tilde{y}(x; a_1, a_2,\dots, a_n)$ *which is an entire function of* x *and the coefficients of the polynomials* $a_1(x), a_2(x),\dots, a_{n-1}(x)$ *and* $a_n(x)$ *and which has an asymptotic representation*:

$$y(x, a) \simeq x^{\alpha_{n+m_n}(a)-(n-1)m_n/2n} \exp\left[\sum_{j=0}^{n+m_n-1} \frac{n\alpha_j(a)}{n+m_n-j} x^{(n+m_n-j)/n}\right] \times \sum_{i=0}^{\infty} A_i x^{-i/n} \tag{2.3}$$

as $x \to \infty$ *uniformly on* $|\arg x| \leqq (n+1)\pi/(n+m_n)-\sigma$ *for any positive constant* $\sigma<(n+1)\pi/(n+m_n)$ *and the coefficients of the polynomials* $a_j(x)$ *on compact set. Here* $\alpha_0(a), \alpha_1(a),\dots, \alpha_{n+m_n}(a)$ *are defined by*

$$[x^{m_n/n} \sum_{j=0}^{\infty} \alpha_j(a) x^{-j/n}]^n + \sum_{k=1}^{n} a_k(x)[x^{m_n/n} \sum_{j=0}^{\infty} \alpha_j(a) x^{-j/n}]^{n-k} = 0 \tag{2.4}$$

and

$$\alpha_0(a) = -1. \tag{2.5}$$

The coefficients A_i ($i=0, 1, 2,\dots$) *are functions of the coefficients of the polynomials* $a_j(x)$ ($j=1, 2,\dots, n$) *and* $A_0=1$. *Furthermore, the solution* $\tilde{y}(x, a)$ *is recessive on* $|\arg x|<n\pi/2(n+m_n)$ *and therefore it is uniquely determined.*

If we assume that $q(n-2)<2n$, it is easy to see that the equation (1.3) satisfies the assumption (2.2). Therefore, applying Theorem 2.1 to the reduced differential equation (1.3), we can obtain the following lemma.

LEMMA 2.2. *Let δ be a small positive number. There exists a solution $\tilde{y}(x; b)=\tilde{y}(x; b_{2,0},\dots, b_{2,q-2},\dots, b_{p,r},\dots, b_{n,0},\dots, b_{n,q-2})$ of the differential equation* (1.3) *such that*

(i) *$\tilde{y}(x; b)$ is an entire function of x and $b_{p,r}$ $(p=2, 3,\dots, n;\ r=0, 1, 2,\dots, q-2)$;*

(ii) *$\tilde{y}(x; b)$ admits an asymptotic representation:*

$$(2.6)\qquad \tilde{y}(x;b) \simeq x^{\alpha_{n+q}(b)-(n-1)q/2n}\exp\left[\sum_{j=0}^{n+q-1}\frac{n\alpha_j(b)}{n+q-j}x^{(n+q-j)/n}\right] \times [1+0(x^{-1/n})]$$

as $x\to\infty$ uniformly in the sector $|\arg x|\leqq(n+1)\pi/(n+q)-\delta$ and for $b_{p,r}$ $(p=2, 3,\dots, n;\ r=0, 1,\dots, q-2)$ on compact sets. The quantities $\alpha_j(b)$ $(j=0, 1,\dots, n+q)$ are determined by the following equalities:

$$(2.7)\qquad \alpha_0(b) = -1,\quad Z = x^{q/n}\sum_{j=0}^{\infty}\alpha_j(b)x^{-j/n},$$
$$Z^n + \sum_{j=0}^{n-1}\sum_{m=0}^{q-2} b_{j,m}x^m Z^{n-j} + [(-1)^{n+1}x^q + \sum_{m=0}^{q-2} b_{n,m}x^m x^m] = 0.$$

In the equation (1.3), by the change of the independent variable

$$x = \omega\xi,\quad \omega = \exp[2\pi i/(n+q)],$$

we can easily find that the equation (1.3) becomes

$$(2.8)\qquad \frac{d^n y}{d\xi^n} + \left[\sum_{k=1}^{n-2}\sum_{m=0}^{q-2} b_{n-k,m}\omega^{n+m-k}\xi^m\frac{d^k y}{d\xi^k}\right] + [(-1)^{n+1}\xi^q + \sum_{m=0}^{q-2} b_{n,m}\omega^{n+m}\xi^m]y = 0.$$

Therefore, for each integer k $(k=0, 1, 2,\dots)$, if we put

$$(2.9)\qquad y_k(x; b) = \tilde{y}(\omega^{-k}x; G^k(b)),$$
$$G^k(b) = (\dots, b_{n-j,m}\omega^{(n-j+m)k},\dots),$$

then $y_k(x; b)$ are solutions of the differential equation (1.3). At the same time, the quantities $\alpha_j(G^k(b))$ $(j=0, 1, 2,\dots, n+q;\ k=0, 1, 2,\dots, n+q-1)$ are determined by the following equalities:

$$\alpha_0(G^k(b)) = -1,$$
$$(2.10)\qquad Z_k = \xi^{q/n}\sum_{j=0}^{\infty}\alpha_j(G^k(b))\xi^{-j/n},$$
$$(Z_k)^n + \sum_{j=0}^{n-1}\sum_{m=0}^{q-2} b_{j,m}\omega^{(m+j)k}\xi^m(Z_k)^{n-j} + (-1)^{n+1}\xi^q + \sum_{m=0}^{q-2} b_{n,m}\omega^{(m+n)k}\xi^m = 0,$$

Hence, by an easy calculation, we can derive the following lemma.

LEMMA 2.3. *Let us put*

$$(2.11)\qquad S_k:\ -(n+1)\pi/(n+q)+2k\pi/(n+q)+\delta \leqq \arg x \leqq (n+1)\pi/(n+q)+2k\pi/(n+q)-\delta$$

and

$$(2.12)\qquad E_k(x;b)=\sum_{j=0}^{n+q-1}\exp[-2k\pi i/n]\frac{n\alpha_j(G^k(b))}{n+q-j}\omega^{jk/n}x^{(n+q-j)/n}$$

where δ *is a small positive number. Suppose that*

$$(2.13)\qquad y_k(x;b)=\tilde{y}(\omega^{-k}x;G^k(b))$$

and

$$(2.14)\qquad y_k(x;b)=y_h(x;b)\quad \text{if}\quad k=h\pmod{n+q}.$$

Then each $y_k(x;b)$ *is a solution of the reduced differential equation* (1.3) *which is an entire function of* x *and* $b_{p,r}$ $(p=2, 3,\dots, n;\ r=0, 1,\dots, q-2)$ *and admits an asymptotic representation*:

$$(2.15)\qquad y_k(x;b)\simeq \omega^{-k(\alpha_{n+p}(G^k(b))-(n-1)q/2h)}x^{\alpha_{n+q}(G^k(b))-(n-1)q/2n}\times\exp[E_k(x;b)][1+O(x^{-1/n})],$$

as $x\to\infty$ *uniformly for* $x\in S_k$ *and for* $b_{p,r}$ *on compact sets, where*

$$(2.16)\qquad \alpha_j(G^k(b))\omega^{jk/n}=\alpha_j(b)\quad (j=0, 1,\dots, n+q;\ k=0, 1,\dots, n+q-1).$$

Furthermore, it holds that for each k $(k=0, 1,\dots, n+q-1)$

$$(2.17)\qquad \mathrm{Wron}[y_k(x;b), y_{k+1}(x;b),\dots, y_{k+n-1}(x;b)] = \omega^{-\sum_{h=k}^{k+n-1}(h\alpha_{n+q}(G^h(b))-hq(n-1)/2n}\det|(\exp[-2hm\pi i/n])|$$
$$h=k, k+1,\dots, k+n-1$$
$$m=0, 1,\dots, n-1.$$

We shall now prove the following lemma.

LEMMA 2.4. *Let* $\hat{r}$, R *and* M *be arbitrary but positive numbers, and let* δ *be a sufficiently small positive number. Suppose that* $\psi_{p,r}(\mu)$ $(p=2, 3,\dots, n;\ r=0, 1,\dots, q-2)$ *are given functions of* μ *which are holomorphic in the sector*

$$(2.18)\qquad |\arg\mu|\leqq\delta/(n+1),\ |\mu|\geqq M$$

and

(2.19) $$\psi_{p,r}(\mu) \simeq 0 \quad (p=2, 3,\dots, n;\ r=0, 1,\dots, q-2)$$

as μ tends to infinity in (2.18). *Then, for* $m=0, 1,\dots, n-1$,

(2.20) $$\left(\frac{d^m}{dt^m} y_k(\mu^n t, b+\psi) - \frac{d^m}{dt^m} y_k(\mu^n t; b)\right)\exp[-E_k(\mu^n t; b)] \simeq 0$$

uniformly for

(2.21) $$|t| \leqq \hat{r},\ |\arg t - 2k\pi/(n+q)| \leqq (n+1)/(n+q) - \delta,\ \textstyle\sum_{p=2}^{n} \sum_{r=0}^{q-2} |b_{p,r}| \leqq R,$$

as μ tends to infinity in (2.18).

PROOF. Put $x=\mu^n t$. For a given positive constant $\tilde{R}$, there exists a positive constant c_1 such that

$$|y_k^{(m)}(x; b+\psi) - y_k^{(m)}(x; b)| \leqq c_1 \textstyle\sum_{p=2}^{n} \sum_{r=0}^{q-2} |\psi_{p,r}(\mu)| \quad (m = 0, 1,\dots, n-1)$$

for

$$|x| \leqq \tilde{R},\ \textstyle\sum_{p=2}^{n} \sum_{r=0}^{q-2} |b_{p,r}| \leqq R,\ |\arg \mu| \leqq \delta/(n+1),\ |\mu| \geqq M,$$

where c_1 depends on $\tilde{R}$, R, M and $\psi_{p,r}(\mu)$. On the other hand, the function $\exp[-E_k(x, b)]$ is bounded in the domain

$$|x| \geqq \tilde{R},\ \textstyle\sum_{p=2}^{n} \sum_{r=0}^{q-2} |b_{p,r}| \leqq R.$$

Therefore, in order to complete the proof of the lemma, it is sufficient to consider (t, μ) in the domain

(2.22) $$\begin{aligned} &|x| \geqq \tilde{R},\ |t| \leqq \hat{r},\ |\arg t - 2k\pi/(n+q)| \leqq (n+1)\pi/(n+q) - \delta, \\ &\textstyle\sum_{p=2}^{n} \sum_{r=0}^{q-2} |b_{p,r}| \leqq R,\ |\arg \mu| \leqq \delta/(n+1),\ |\mu| \geqq M. \end{aligned}$$

Let us put

$$\varepsilon(\mu, \theta) = (\dots, |\mu|^{-(n+q-2)} \exp[i\theta_{p,r}],\dots),$$

where $\theta_{p,r}$ $(p=2, 3,\dots, n;\ r=0, 1,\dots, q-2)$ are real variables. We shall now prove that

(2.23) $$|y_k^{(m)}(x; b+\varepsilon) \exp[-E_k(x; b)]| \leqq c_2 |\mu|^{\tilde{p}} \quad (m=0, 1,\dots, n-1)$$

in the domain (2.22) uniformly in $\theta_{p,r}$, where c_2 is a positive constant and $\tilde{p}$ is a non-negative constant. Since we have

$$|\arg x - 2k\pi/(n+q)| \leqq (n+1)\pi/(n+q) - \delta/(n+1)$$

in the domain (2.22), we can use asymptotic representations (2.15). In fact, we get

$$|y_k^{(m)}(x;\, b+\varepsilon) \exp[-E_k(x;\, b)]|$$
$$\leqq c_3 |x^{\tilde{r}(m,k)}| |\exp[E_k(x;\, b+\varepsilon) - E_k(x;\, b)]|$$

in the domain (2.22) uniformly in $\theta_{p,r}$, where c_3 is a positive constant and

$$\tilde{r}(m,\, k) = \alpha_{n+q}(G^k(b)) - (n-1)/n + qm/n$$
$$(m = 0, \ldots, n-1;\; k = 0, 1, \ldots, n+q-1).$$

Observe that the function $\tilde{r}(m, k)$ is bounded for

$$\sum_{p=2}^{n} \sum_{r=0}^{q-2} |b_{p,r}| \leqq R \quad \text{and} \quad |\mu| \geqq M,$$

uniformly in $\theta_{p,r}$. We have also

$$\tilde{R} \leqq |x| = |\mu|^n |t| \leqq \hat{r} |\mu|^n.$$

Hence, we get

$$|\log x| \leqq c_4[1 + \log |\mu|]$$

for (2.22), where c_4 is a positive constant. Thus we get

$$|x^{\tilde{r}(m,k)}| \leqq c_5 |\mu|^{p^*} \qquad (m=0, 1, \ldots, n-1;\; k=0, 1, \ldots, n+q-1)$$

in the domain (2.22) uniformly in $\theta_{p,r}$, where c_5 is a positive constant and p^* is a non-negative constant. Observe next that

$$E_k(x;\, b+\varepsilon) - E_k(x;\, b)$$
$$= \sum_{j=0}^{n+q-1} \frac{n}{n+q-j} [\alpha_j(G^k(b+\varepsilon) - \alpha_j(G^k(b))] \exp[-qk\pi i/n] \omega^{jk/n} x^{(n+q-j)/n}$$
$$|\alpha_j(G^k(b+\varepsilon)) - \alpha_j(G^k(b))| \leqq c_6 |\mu|^{-(n+q-2)}$$

and

$$|x| \leqq \hat{r} |\mu|^n$$

in domain (2.22) uniformly in $\theta_{p,r}$, where c_6 is a positive constant. Hence, the function $E_k(x;\, b+\varepsilon) - E_k(x;\, b)$ is bounded in domain (2.22) uniformly in $\theta_{p,r}$. Thus we proved (2.23).

We shall now estimate the function

$$[y_k^{(m)}(\mu^n t;\, b+\psi) - y_k^{(m)}(\mu^n t;\, b)] \exp[-E_k(\mu^n t;\, b)]$$

in (2.22). Note first that

$$y_k^{(m)}(x;\, b) = (2\pi)^{-(n+q-2)} \int_0^{2\pi} \cdots \int_0^{2\pi} y_k^{(m)}(x;\, b+\varepsilon) d\theta_{2,0} \ldots d\theta_{p,r} \ldots d\theta_{n,q-2},$$

and that

$$(2.24)\qquad |\psi_{p,r}(\mu)| \leqq \frac{1}{2}|\mu|^{-(n+q-2)} \quad (p=2, 3,\dots, n;\ r=0, 1,\dots, q-2)$$

if $|\mu|$ is sufficiently large. Since $\tilde{R}\leqq|x|\leqq\hat{r}|\mu|^n$ in (2.22), (2.24) holds in (2.22) if $\tilde{R}$ is sufficiently large. Then, by virtue of Cauchy's integral representation theorem we get

$$y_k^{(m)}(x;\, b+\psi(\mu))$$
$$= (2\pi)^{-(n+q-2)}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{y_k^{(m)}(x;\, b+\varepsilon(\mu,\theta))}{\prod_{p=0}^{n}\prod_{r=0}^{q-2}(1-\psi_{p,r}(\mu)|\mu|^{n+q-2}\exp[-i\theta_{p,r}]}$$
$$\times\, d\theta_{2,0}\cdots d\theta_{p,r}\cdots d\theta_{n,q-2}.$$

Therefore,

$$y_k^{(m)}(x;\, b+\psi) - y_k^{(m)}(x;\, b)$$
$$= (2\pi)^{-(n+q-2)}\int_0^{2\pi}\cdots\int_0^{2\pi} G(\mu,\theta)\, y_k^{(m)}(x;\, b+\varepsilon)\, d\theta_{2,\theta}\cdots d\theta_{n,q-2},$$

where

$$G(\mu,\theta) = \frac{1}{\prod_{p=2}^{n}\prod_{r=0}^{q-2}(1+\psi_{p,r}(\mu)|\mu|^{n+q-2}\exp[-i\theta_{p,r}]} - 1.$$

Hence the estimates of (2.23) yield that

$$|[y_k^{(m)}(\mu^n t,\, b+\psi(\mu)) - y_k^{(m)}(\mu^n t;\, b)]\exp[-E_k(\mu^n t;\, b]]|$$
$$\leqq c|\mu|^{n+q-2+p^*}\sum_{p=2}^{n}\sum_{r=0}^{q-2}|\psi_{p,r}(\mu)|$$

in (2.22), where c is a positive constant. Thus we have proved Lemma 2.4. (cf. [7] Lemma 3.1.)

§3. Existence theorem in the sector

Hereafter we shall consider the system of linear differential equations (1.1). Let $\tilde{P}(t,\varepsilon)$ be an n by n matrix satisfying the following conditions:

(i) the components of $\tilde{P}(t,\varepsilon)$ are holomolphic in domain $D(t_0, \varepsilon_0, \theta_0)$,

(ii) $\frac{d^m}{dt^m}\tilde{P}(t,\varepsilon)=\tilde{P}^{(m)}(t,\varepsilon)$ admit the asymptotic expansions

$$(3.1)\qquad \tilde{P}^{(m)}(t,\varepsilon) \simeq \sum_{i=0}^{\infty} P_i^{(m)}(t)\varepsilon^i \qquad (m=0, 1,\dots, n-1),$$

uniformly for $|t|\leqq t_0$, as $\varepsilon\to0$ in the sector (1.2), where

$$P(t,\varepsilon) = \sum_{i=0}^{\infty} P_i(t)\varepsilon^i$$

is the formal matrix given by the formal reduction Theorem 1.1. The existence of such a matrix $\tilde{P}(t, \varepsilon)$ is a consequence of the theorem of J. F. Ritt. (See [8] §9.) Since $P_0(0)=I$ (identity matrix), the inverse matrix $P(t, \varepsilon)^{-1}$ of the formal matrix is well defined. The components of the inverse matrix $P(t, \varepsilon)^{-1}$ are formal power series in ε whose coefficients are holomorphic in the closed disk $|t| \leqq t_1 \leqq t_0$. The inverse matrix $\tilde{P}(t, \varepsilon)^{-1}$ also exists in the domain $D(t_1, \varepsilon_1, \theta_1)$, where $t_1 \leqq t_0$, $\varepsilon_1 \leqq \varepsilon_0$ and $\theta_1 \leqq \theta_0$. The components of $\tilde{P}(t, \varepsilon)^{-1}$ are holomorphic in the closed disk $|t| \leqq t_1$ and in (1.2) and it holds that

$$\tilde{P}(t, \varepsilon)^{-1} \simeq P(t, \varepsilon)^{-1} \tag{3.2}$$

as $\varepsilon \to 0$ in the sector (1.2), uniformly for $|t| \leqq t_1$.

Let

$$\varepsilon \frac{dY}{dt} = [B(t, \varepsilon) + E(t, \varepsilon)]Y \tag{3.3}$$

be the system to which the system of linear differential equations (1.1) is reduced by the transformation

$$Y = \tilde{P}(t, \varepsilon)X. \tag{3.4}$$

This means that $E(t, \varepsilon)$ must satisfy the relation

$$E(t, \varepsilon) = \tilde{P}(t, \varepsilon)A(t, \varepsilon)\tilde{P}(t, \varepsilon)^{-1} + \tilde{P}(t, \varepsilon)^{-1}\tilde{P}'(t, \varepsilon) - B(t, \varepsilon).$$

On the other hand, from Theorem 1.1, $P(t, \varepsilon)$ formally satisfies the following relation

$$B(t, \varepsilon) = P(t, \varepsilon)A(t, \varepsilon)P(t, \varepsilon)^{-1} + \varepsilon P(t, \varepsilon)^{-1}P'(t, \varepsilon).$$

Therefore, according to (3.1) and (3.2), we can easily get

$$E(t, \varepsilon) \simeq 0 \tag{3.5}$$

uniformly for $|t| \leqq t_1$, as $\varepsilon \to 0$ in (1.2)

We shall apply to the system (3.3) the transformation

$$Y = \exp\left[\frac{1}{n}\varepsilon^{-1}\int_0^t \text{trace}\,[E(s, \varepsilon)\right]ds]Z. \tag{3.6}$$

Then, the system (3.3) becomes

$$\varepsilon \frac{dZ}{dt} = [B(t, \varepsilon) + F(t, \varepsilon)]Z, \tag{3.7}$$

where

$$F(t, \varepsilon) = E(t, \varepsilon) - \frac{1}{n}\text{trace}\,[E(t, \varepsilon)]. \tag{3.8}$$

The components of the matrix function $F(t, \varepsilon)$ are holomorphic in the closed disk $|t| \leqq t_1$ and it holds that

$$F(t, \varepsilon) \simeq 0 \tag{3.9}$$

uniformly for $|t| \leqq t_1$, as $\varepsilon \to 0$ in (1.2). Furthermore, we can easily see that

$$\operatorname{trace}[B(t, \varepsilon) + F(t, \varepsilon)] = \operatorname{trace} B(t, \varepsilon) + \operatorname{trace} F(t, \varepsilon) = 0 \tag{3.10}$$

and that

$$\exp\left[\frac{1}{n}\varepsilon^{-1}\int_0^t \operatorname{trace}[E(s, \varepsilon)]ds\right] \simeq 1, \tag{3.11}$$

uniformly for $|t| \leqq t_1$ as $\varepsilon \to 0$ in (1.2).

In order to investigate the property of solutions of the system (3.7), we shall now compare the system (3.7) with the simpler system

$$\varepsilon \frac{dZ}{dt} = B(t, \varepsilon)Z. \tag{3.12}$$

The system (3.12) is equivalent to the single linear differential equation (1.3). Therefore, according to Lemmas 2.2 and 2.3, the system (3.12) admits solutions

$$\tilde{z}_k(t, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \mu^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \mu^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu^{-n+1} \end{pmatrix} \begin{pmatrix} y_k(x, b) \\ y'_k(x, b) \\ y''_k(x, b) \\ \vdots \\ y_k^{(n-1)}(x; b) \end{pmatrix} \tag{3.13}$$

$$(k=0, 1, \dots, n+q-1),$$

where

$$x = \mu^n t \quad \text{and} \quad \mu = \varepsilon^{-1/(n+q)}. \tag{3.14}$$

Let us put n by n matrices $\Phi_k(t, \varepsilon)$ and $D_k(t, \varepsilon)$ $(k=0, 1, 2, \dots, n+q-1)$ as follows;

$$\Phi_k(t, \varepsilon) = (\tilde{z}_k(t, \varepsilon), \tilde{z}_{k+1}(t, \varepsilon), \dots, \tilde{z}_{k+n-1}(t, \varepsilon)), \tag{3.15}$$

$$D_k(t, \varepsilon) = \operatorname{diag}(\exp[E_k(x; b)], \exp[E_{k+1}(x; b)], \dots, \exp[E_{k+n-1}(x; b)]). \tag{3.16}$$

Then, for each k $(k=0, 1, \dots, n+q-1)$, $\Phi_k(t, \varepsilon)$ is a fundamental matrix solution of the system (3.12). In fact, according to the fact that $\operatorname{trace}[B(t, \varepsilon)]=0$, $\det \Phi_k(t, \varepsilon)$ is independent of t. Therefore, we have only to show that

$$\det \Phi_k(t, \varepsilon) \neq 0 \qquad \text{for} \quad x \in \cap_{m=k}^{k+n-1} S_m.$$

It is obvious from (3.13) and Lemma 2.3 that

$$\text{(3.17)} \qquad \det \Phi_k(t, \varepsilon) = \mu^{-n(n-1)/2} \omega^{\sum_{h=k}^{k+n-1} (h\alpha_{n+q}(G^h(b)) - hq(n-1)/2n)} \times \det |(\exp[-qhm\pi i/n])| \begin{matrix} h = k, k+1, \ldots, k+n-1 \\ m = 0, 1, \ldots, n-1 \end{matrix} \neq 0.$$

If we put

$$\text{(3.18)} \qquad \Phi_k(t, \varepsilon) = \hat{\Phi}_k(t, \varepsilon) D_k(t, \varepsilon) \qquad (k = 0, 1, \ldots, n+q-1),$$

then, each $\hat{\Phi}_k(t, \varepsilon)$ satisfies the condition

$$\text{(3.19)} \qquad \|\hat{\Phi}_k(t, \varepsilon)\| \leqq c|\mu|^{q^*}$$

in the domain

$$\text{(3.20)} \qquad |t| \leqq t_1, \quad x \in \cap_{m=k}^{k+n-1} S_m, \quad |\mu| \geqq M,$$

where c is a positive constant, q^* is a non-negative constant and M is a sufficiently large positive number. We here defined the norm $\|A\|$ for a matrix $A = (a_{ij})$ $(i, j = 1, 2, \ldots, n)$ by

$$\|A\| = \max_{i,j} \{|a_{ij}|\}.$$

Moreover, inverse matrices $\hat{\Phi}_k(t, \varepsilon)^{-1}$ of $\hat{\Phi}_k(t, \varepsilon)$ exist and then

$$\hat{\Phi}_k(t, \varepsilon)^{-1} = D_k(t, \varepsilon) \Phi_k(t, \varepsilon)^{-1} = \frac{1}{\det D_k(t, \varepsilon)^{-1} \det \Phi_k(t, \varepsilon)} \Delta(\Phi_k(t, \varepsilon)),$$

where $\Delta(\Phi_k(t, \varepsilon))$ is a cofactor matrix of $\Phi_k(t, \varepsilon)$. Therefore, utilizing (2.16), (3.17) and (3.19), we can obtain

$$\text{(3.21)} \qquad \|\hat{\Phi}_k(t, \varepsilon)^{-1}\| \leqq c|\mu|^{q^*},$$

where c is a positive constant and q^* is a non-negative constant.

Let $\Phi(t, s, \varepsilon)$ be the n by n matrix function such that

$$\text{(3.22)} \qquad \frac{\partial \Phi(t, s, \varepsilon)}{\partial t} = B(t, \varepsilon) \Phi(t, s, \varepsilon), \quad \Phi(t, t, \varepsilon) = I.$$

Then the uniqueness of solutions shows that

$$\text{(3.23)} \qquad \Phi(t, s, \varepsilon) = \Phi_k(t, \varepsilon) \Phi_k(s, \varepsilon)^{-1} = \hat{\Phi}_k(t, \varepsilon) D_k(t, \varepsilon) D_k(s, \varepsilon)^{-1} \hat{\Phi}_k(s, \varepsilon)^{-1} \quad (k = 0, 1, \ldots, n+q-1).$$

It follows from (3.16) that

$$D_k(t, \varepsilon)D_k(s, \varepsilon)^{-1} = \operatorname{diag}[\exp[E_k(x; b)-E_k(\xi; b)],\dots, \exp[E_{k+n-1}(x; b)-E_{k+n-1}(\xi; b)]], \tag{3.24}$$

where

$$\xi = \mu^n s. \tag{3.25}$$

Hence, according to (3.18), (3.21), (3.23) and (3.24), we have

$$\begin{aligned}\|\Phi(t, s, \varepsilon)\| &\leqq \|\hat{\Phi}_k(t, \varepsilon)\|\cdot\|D_k(t, \varepsilon)D_k(s, \varepsilon)^{-1}\|\cdot\|\hat{\Phi}_k(s, \varepsilon)^{-1}\| \\ &\leqq c^2|\mu|^{2q^*}\textstyle\sum_{j=k}^{k+n-1}|\exp[E_j(x; b)-E_j(\xi; b)]|,\end{aligned}$$

in the domain (3.20).

We are now in a position to state the following theorem.

THEOREM 3.1. *The system of differential equations* (3.7) *admits a solution* $z=z_k(t, \varepsilon)$ *such that*

(i) *the components of* $z_k(t, \varepsilon)$ *are holomorphic in the domain* $D(t_1, \varepsilon_1, \theta_1)$;

(ii) $z_k(t, \varepsilon)$ *satisfies the asymptotic condition*

$$\exp[-E_k(x; b)][z_k(t, \varepsilon)-\tilde{z}_k(t, \varepsilon)] \simeq 0 \tag{3.27}$$

uniformly for

$$|t| \leqq t_2, \quad |\arg t - 2k\pi/(n+q)| \leqq (n+1)\pi/(n+q) - \delta, \tag{3.28}$$

as μ *tends to infinity in the sector*

$$|\arg\mu| \leqq \delta/(n+q), \quad |\mu| \geqq M, \tag{3.29}$$

where δ *and* t_2 *are sufficiently small positive numbers and* M *is a sufficiently large positive number.*

PROOF. Let us reduce the system (3.7) to the integral equation

$$z(t, \varepsilon) = \tilde{z}_k(t, \varepsilon) + \varepsilon^{-1}\int_{t_0}^{t}\Phi(t, s, \varepsilon)F(s, \varepsilon)z(s, \varepsilon)ds, \tag{3.30}$$

and put as follows, assuming that N is a sufficiently large positive number,

$$\begin{aligned}z(t, \varepsilon) &= \mu^N\exp[E_k(x; b)]\zeta(t, \varepsilon),\\ z_k(t, \varepsilon) &= \mu^N\exp[E_k(x; b)]\zeta_k(t, \varepsilon),\\ \tilde{\Phi}(t, s, \varepsilon) &= \exp[-E_k(x; b)]\Phi(t, s, \varepsilon)\exp[E_k(\xi, b)],\end{aligned}$$

where

$$\xi = \mu^n s.$$

Then, the above integral equation (3.30) becomes

$$\zeta(t, \varepsilon) = \zeta_k(t, \varepsilon) + \varepsilon^{-1}\int_{t_1}^{t} \tilde{\Phi}(t, s, \varepsilon)F(s, \varepsilon)\zeta(s, \varepsilon)ds. \tag{3.31}$$

The norm of the matrix $\tilde{\Phi}(t, s, \varepsilon)$ is estimated as follows;

$$\begin{aligned}\|\tilde{\Phi}(t, s, \varepsilon)\| &= \|\hat{\Phi}_k(t, \varepsilon)\exp[-E_k(x; b)]D_k(t, \varepsilon)D_k(s, \varepsilon)^{-1} \\ &\quad \times \exp[E_k(\xi, b)]\hat{\Phi}_k(s, \varepsilon)^{-1}\| \\ &\leqq c^2|\mu|^{2q^*}\textstyle\sum_{j=k}^{k+n-1}|\exp[E_j(x; b)-E_k(x; b)-(E_j(\xi; b)-E_k(\xi; b))]|.\end{aligned}$$

If N is sufficiently large, $\zeta_k(t, \varepsilon)$ is bounded for (3.28).

We shall construct a bounded solution of the integral equation (3.31). To do this, we shall fix t_1 and a path of integration $\gamma(t)$ so that the function $\|\varepsilon^{-1}\tilde{\Phi}(t, s, \varepsilon)F(s, \varepsilon)\|$ is sufficiently small along the path $\gamma(t)$. Let us consider the mapping

$$\hat{t} = \frac{n}{n+q}\exp[-2k\pi i/n]t^{(n+q)/n} \tag{3.33}$$

in the domain

$$|t| \leqq t_1, \quad |\arg t - 2k\pi/(n+q)| \leqq (n+1)\pi/(n+q) - \delta. \tag{3.34}$$

The image of this domain under the mapping (3.33) is given by

$$\begin{cases} |\hat{t}| \leqq \dfrac{n}{n+q}t_1^{(n+q)/n} = \hat{t}_1, \\ |\arg\hat{t}| \leqq (n+1)\pi/n - (n+1)\delta/n = (n+1)\pi/n - \hat{\delta}. \end{cases} \tag{3.35}$$

Put

$$\hat{D}\colon |\hat{t}| \leqq \hat{t}_2 \quad \text{and} \quad |\arg\hat{t}| \leqq (n+1)\pi/n - \hat{\delta}, \qquad \text{(See fig 1,)}, \tag{3.36}$$

where $\hat{t}_2$ is a sufficiently small positive number such that

$$\sin\alpha = \hat{t}_2/\hat{t}_1 \leq \sin\left[\frac{1}{n}\pi - \hat{\delta}\right]. \tag{3.37}$$

Then every point $\hat{t}$ in $\hat{D}$ can be joined to $\hat{t}_1$ either by a line segment:

$$\text{(i)} \qquad \hat{\tau} = \hat{t}_1 + s\cdot\exp[i\cdot\arg(\hat{t}-\hat{t}_1)] \qquad (0 \leqq s \leqq |\hat{t}-\hat{t}_1|),$$

where

$$|\arg(\hat{t}-\hat{t}_1)-\pi| \leqq \pi/n - \hat{\delta},$$

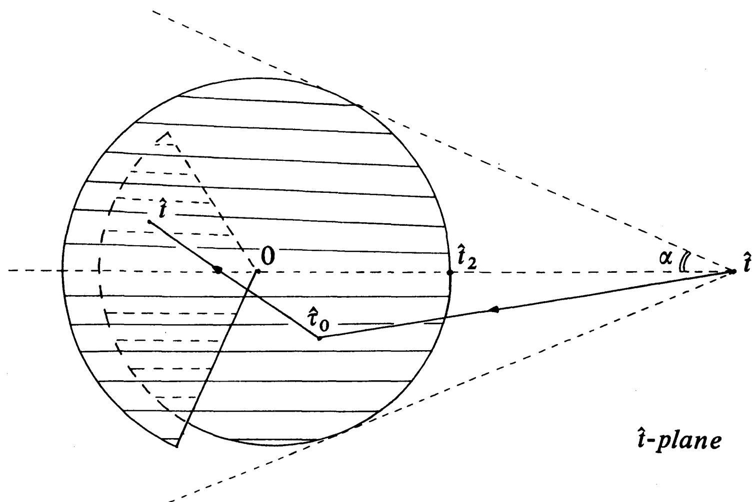

Fig. 1.

(ii) $$\hat{\tau} = \hat{t}_1 + s\cdot\exp[i\cdot\arg(\hat{\tau}_0-\hat{t}_1)] \qquad (0\leqq s\leqq|\hat{\tau}_0-\hat{t}_1|)$$

and

$$\hat{t} = \hat{\tau}_0 + s\cdot\exp[i\cdot\arg(\hat{t}-\hat{\tau}_0)] \qquad (0\leqq s\leqq|\hat{t}-\hat{\tau}_0|),$$

where

$$|\arg(\hat{\tau}_0-\hat{t}_0)-\pi| \leqq \pi/n - \hat{\delta} \quad \text{and} \quad |\arg(\hat{t}-\hat{\tau}_0)-\pi| \leqq \pi/n - \hat{\delta}.$$

For every point in $\hat{D}$, we shall denote this path by $\hat{\gamma}(\hat{t})$. Let $\gamma(t)$ be the path which is mapped onto $\hat{\gamma}(\hat{t})$ by the mapping (3.33). Then this path $\gamma(t)$ is a desirable one. Now we consider an arc which is defined by

(3.38) $$\frac{n}{n+q}\exp[-2k\pi i/n]\cdot t(s)^{(n+q)/n} = \hat{\tau}_0 + s\cdot e^{i\theta},$$

where $\hat{\tau}_0$ is a point in D, and θ is a fixed real number such that

(3.39) $$|\theta-\pi| \leqq \pi/n - \hat{\delta}.$$

Taking account of (3.38), (3.39) and

$$\exp[-2k\pi i][t(s)]^{q/n}\frac{dt(s)}{ds} = e^{i\theta},$$

we observe that

$$\frac{d}{ds}[E_m(x;b) - E_k(x;b)]$$

$$= \textstyle\sum_{j=0}^{n+q-1}\mu^{n+q-j}[t(s)]^{(q-j)/n}\frac{dt(s)}{ds}[\exp(-2m\pi i/n)\alpha_j(G^m(b))\omega^{jm/n}$$

$$- \exp(-2k\pi i/n)\alpha_j(G^k(b))\omega^{jk/n}]$$
$$= \mu^{n+q}e^{i\theta}\{(1-\exp[2(k-m)\pi i/n]) + \textstyle\sum_{j=1}^{n+q-1} \mu^{-j}[t(s)]^{-j/n} \times$$
$$\times [\exp[2(k-m)\pi i/n]\alpha_j(G^m(b))\omega^{jm/n} - \alpha_j(G^k(b))\omega^{jk/n})]\}.$$

Since

$$\mathrm{Re}\,[e^{i\theta}(1-\exp[2(k-m)\pi i/n])] < 0 \qquad (k\neq m;\ m=k+1,\dots, k+n+q-1),$$

we get

$$\mathrm{Re}\left[\frac{d}{ds}[E_m(\mu^n t(s);\ b) - E_k(\mu^n t(s);\ b)]\right] < 0,$$

provided that $|\mu^n t(s)| \geqq N^*$, where N^* is a sufficiently large positive number. If $|\mu^n t(s)| \leqq N^*$, it is obvious that $E_m(\mu^n t(s);\ b)$ are bounded.
Therefore,

$$\mathrm{Re}\,[[E_m(x;\ b)-E_k(x;\ b)] - [E_m(\xi;\ b)-E_k(\xi;\ b)]]$$

admit uniform upper bounds along the path $\gamma(t)$ for every t in D.

Let us consider the domain (3.28) and (3.29), where M is a sufficiently large positive number. Denote by B_M the set of all n-dimensional vector $\zeta(t, \varepsilon)$ whose components are bounded and continuous in this domain and holomorphic in the interior of this domain. The set B_M becomes a Banach space if we define a norm of $\zeta(t, \varepsilon)$ by $\|\zeta\| = \sup_{(t,\varepsilon)} |\zeta(t, \varepsilon)|$. Define a linear transformation $L[\zeta]$ by

(3.40) $$L[\zeta] = \varepsilon^{-1}\int_{\gamma(t)} \tilde{\Phi}(t, s, \varepsilon)F(s, \varepsilon)\zeta(s, \varepsilon)ds.$$

The definition of the integral path $\gamma(t)$, the estimate of $\tilde{\Phi}(t, s, \varepsilon)$ and the asymptotic property (3.9) of $F(t, \varepsilon)$ imply that

(a) $L[\zeta] \in B_M$ for every ζ in B_M;

(b) $L[\zeta](t, \varepsilon) \simeq 0$ uniformly for (3.28) as $\varepsilon \to 0$ in (3.29);

(c) the norm of L is bounded by 1/2 if M is sufficiently large. Since $\zeta_k(t, \varepsilon)$ is in B_M, we can define an n-demensional vector $\zeta(t, \varepsilon)$ by

$$\zeta(t, \varepsilon) = \zeta_k(t, \varepsilon) + \textstyle\sum_{n=1}^{\infty} L^n[\zeta_k](t, \varepsilon),$$

and this vector is in B_M if M is sufficiently large. Furthermore, we have

$$\zeta(t, \varepsilon) = \zeta_k(t, \varepsilon) + L[\zeta](t, \varepsilon).$$

Now if we put

$$z = z_k(t, \varepsilon) = \mu^N \exp[E_k(x;\ b)]\zeta(t, \varepsilon),$$

then $z_k(t, \varepsilon)$, for each k, is a solution of the system (3.7) which satisfies all of the requirement given in Theorem 3.1. Furthermore, $z_k(t, \varepsilon)$ is holomorphic in the domain $D(t_1, \varepsilon_1, \theta_1)$, since there is no singular point of the system (3.7) with respect to t in the closed disk $|t| \leqq t_1$. This completes the proof of Theorem 3.1.

Let us consider now the system of differential equations

$$\varepsilon \frac{d\hat{z}}{dt} = \hat{B}(t, \varepsilon)\hat{z} \tag{3.41}$$

with

$$\hat{B}(t, \varepsilon) = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots\cdots 0 & 0 \\ 0 & 0 & 1 \cdots\cdots\cdots 0 & 0 \\ \vdots & \vdots & \vdots \qquad\quad \vdots & \vdots \\ \hat{b}_1(t, \varepsilon) & \hat{b}_2(t, \varepsilon) & \hat{b}_3(t, \varepsilon) \cdots \hat{b}_{n-1}(t, \varepsilon) & 0 \end{pmatrix},$$

where

$$\begin{aligned} \hat{b}_1(t, \varepsilon) &= t^q + \varepsilon^2 \textstyle\sum_{m=0}^{q-2} [\beta_1^m(\varepsilon) + \delta_1^m(\varepsilon)] t^m, \\ \hat{b}_j(t, \varepsilon) &= \varepsilon^2 \textstyle\sum_{m=0}^{q-2} [\beta_j^m(\varepsilon) + \delta_j^m(\varepsilon)] t^m \qquad (j=2, 3, \dots, n-1) \end{aligned} \tag{3.42}$$

and

$$\delta_k^m(\varepsilon) \simeq 0 \quad (k=1, 2, \dots, n-1) \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad (1.2). \tag{3.43}$$

Applying the change of the independent variable, we easily see that the system (3.41) becomes as follows; (cf. (1.3) and (1.4))

$$\frac{d^n \hat{z}_1}{dx^n} + \Big[\textstyle\sum_{k=0}^{n-2} \sum_{m=0}^{q-2} \hat{b}_{n-k,m} x^m \frac{d^k \hat{z}_1}{dx^k} + (-1)^{n+1} x^q \hat{z}_1 = 0, \tag{3.44}$$

where

$$x = \mu^n t, \quad \mu = \varepsilon^{-1/(n+q)},$$

$$\hat{b}_{p,r}(\varepsilon) = b_{p,r}(\varepsilon) + \psi_{p,r}(\varepsilon) \qquad (p=2, 3, \dots, n;\ r=0, 1, \dots, q-2) \tag{3.45}$$

and $\hat{z}_1$ is the first component of the n-dimensional vector $\hat{z}$. Therefore, it holds from (3.43) that

$$\psi_{p,r}(\varepsilon) \simeq 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad (1.2). \tag{3.46}$$

Furthermore, from (3.13), the system (3.41) admits solutions

$$z_k(t, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & \mu^{-1} & 0 \cdots & 0 \\ 0 & 0 & \mu^{-2} \cdots & 0 \\ \vdots & \vdots & \vdots \ \ddots & \vdots \\ 0 & 0 & 0 \cdots & \mu^{-n+1} \end{pmatrix} \begin{pmatrix} y_k(x; b+\psi) \\ y_k'(x; b+\psi) \\ y_k''(x; b+\psi) \\ \vdots \\ y_k^{(n-1)}(x; b+\psi) \end{pmatrix} \tag{3.47}$$

$$(k=0, 1, \dots, n+q-1).$$

From Lemma 2.4, we can easily derive that

$$\exp[-E_k(x;\, b)][z_k(t, \varepsilon)-\hat{z}_k(t, \varepsilon)] \simeq 0 \tag{3.48}$$

uniformly for (3.28), as μ tends to infinity in (3.29).

In order to show the existence of analytic transformations in the sector, we make some preparation for it. We now define

$$\hat{S}_k = \cap_{m=k}^{k+n-1} S_m. \qquad \text{(cf. (2.11))} \tag{3.49}$$

Then $\hat{S}_k \cap \hat{S}_{k+1} \neq \emptyset$ $(k=0, 1,\dots, n+q-1)$ and $n+q$ sectorial domains $\hat{S}_0, \hat{S}_1,\dots, \hat{S}_{n+q-1}$ cover a full neighborhood of the turning point $t=0$.

As we defined $\Phi_k(t, \varepsilon)$ which is a fundamental matrix solution of the system (3.12), using Theorem 3.1 and (3.47), we here define n by n matrix functions $\Psi_k(t, \varepsilon)$ and $\hat{\Psi}_k(t, \varepsilon)$ as follows;

$$\begin{aligned} \Psi_k(t, \varepsilon) &= [z_k(t, \varepsilon), z_{k+1}(t, \varepsilon),\dots, z_{k+n-1}(t, \varepsilon)], \\ \hat{\Psi}_k(t, \varepsilon) &= [\hat{z}_k(t, \varepsilon), \hat{z}_{k+1}(t, \varepsilon),\dots, \tilde{z}_{k+n-1}(t, \varepsilon)] \quad (k=0, 1,\dots, n+q-1). \end{aligned} \tag{3.50}$$

Then it is easily seen from (3.27) and (3.48) that $\Psi_k(t, \varepsilon)$ and $\hat{\Psi}_k(t, \varepsilon)$ are fundamental sets of solutions of the systems (3.7) and (3.41), respectively, and satisfy the asymptotic conditions

$$\begin{aligned} &[\Psi_k(t, \varepsilon)-\Phi_k(t, \varepsilon)]D_k(t, \varepsilon) \simeq 0, \\ &[\hat{\Psi}_k(t, \varepsilon)-\Phi_k(t, \varepsilon)]D_k(t, \varepsilon) \simeq 0 \end{aligned} \tag{3.51}$$

uniformly for (3.49), as $\varepsilon \to 0$ in (1.2). Moreover, if we put

$$T_k(t, \varepsilon) = \Psi_k(t, \varepsilon)\hat{\Psi}_k(t, \varepsilon)^{-1} \qquad (k=0, 1,\dots, n+q-1), \tag{3.52}$$

the components of the n by n matrices $T_k(t, \varepsilon)$ are holomorphic in the domain $D(t_1, \varepsilon_1, \theta_1)$ and $T_k(t, \varepsilon)$ satisfy the asymptotic condition

$$\begin{aligned} T_k(t, \varepsilon) &= \Psi_k(t, \varepsilon)D_k(t, \varepsilon)D_k(t, \varepsilon)^{-1}\hat{\Psi}_k(t, \varepsilon)^{-1} \\ &\simeq \Phi_k(t, \varepsilon)D_k(t, \varepsilon)D_k(t, \varepsilon)^{-1}\Phi_k(t, \varepsilon)^{-1} \\ &\simeq I \text{ (identity matrix)} \end{aligned} \tag{3.53}$$

uniformly for (3.49) as $\varepsilon \to 0$ in (1.2). Here we used (3.27) and (3.48). Hence, if we put

$$Q_k(t, \varepsilon) = \tilde{P}(t, \varepsilon) \exp\left[\frac{1}{n}\varepsilon^{-1}\int_0^t \text{trace}\,[E(s, \varepsilon)]ds\right]T_k(t, \varepsilon), \tag{3.54}$$

we have obtained Theorem 1.2.

§4. Difference equation and Stokes multipliers (I)

The $n+1$ solutions $y_j(x; b)$ $(j=k, k+1,\dots, k+n)$ of the differential equation (1.3) are linearly independent. Therefore, there are $C_j^k(b)$ $(k=0, 1,\dots, n+q-1;$ $j=1, 2,\dots, n)$, which are independent of x, such that

$$(4.1) \qquad y_k(x; b) = \sum_{j=1}^{n} C_j^k(b) y_{k+j}(x; b).$$

Relation (4.1) is a connection formula for $y_k(x; b)$ and the coefficients $C_j^k(b)$ are the Stokes multipliers for $y_k(x; b)$ with respect to $y_{k+j}(x; b)$. In this section we shall consider the Stokes multipliers as functions of $b_{p,r}$, utilizing the solutions of difference equations which are obtained by the Mellin transformation.

We now represent $\tilde{y}(x; b)$ as a power series of $b_{p,r}$ $(p=2, 3,\dots, n;\ r=0,$ $1,\dots, q-2)$ with coefficients that are entire functions of x as follows;

$$(4.2) \qquad \begin{aligned} \tilde{y}(x; b) &= \tilde{y}(x,\dots, b_{p,r},\dots) \\ &= \eta_0(x) + \sum_{p=2}^{n} \sum_{r=0}^{q-2} \eta_{p,r}(x) b_{p,r} + [\text{higher order}]. \end{aligned}$$

This series is uniformly and absolutely convergent on each compact set of the $(x; b)$-space, so we can differentiate (4.2) termwise. Inserting (4.2) into (1.3), we get the following differential equations for the coefficient functions $\eta_0(x)$ and $\eta_{p,r}(x)$.

$$(4.3) \qquad \begin{aligned} &\eta_0^{(n)}(x) + (-1)^{n+1} x^q \eta_0(x) = 0, \\ &\eta_{p,r}^{(n)}(x) + (-1)^{n+1} x^q \eta_{p,r}(x) + x^r \eta_0^{(n-p)}(x) = 0 \\ &\qquad\qquad (p=2,\dots, n;\ r=0,\dots, q-2). \end{aligned}$$

Since

$$(4.4) \qquad \eta_{p,r}(x) = \frac{\partial}{\partial b_{p,r}} \tilde{y}(x; b)|_{b=0} \qquad (p=2, 3,\dots, n;\ r=0, 1,\dots, q-2),$$

applying a theorem on differentiation of asymptotic expansions with parameters to the representation (2.3), we can obtain the asymptotic expansion as follows;

(4.5)

$$\begin{aligned} \eta_{p,r}(x) \simeq & \left[x^{-(n-1)q/2n} \exp\left[-\frac{n}{n+q} x^{(n+q)/n} \right] \sum_{j=0}^{\infty} \frac{\partial}{\partial b_{p,r}} A_j \Big|_{b=0} x^{-j/n} \right] + \\ & \frac{\partial \alpha_{n+q}(b)}{\partial b_{p,r}} \Big|_{b=0} (\log x) x^{-(n-1)q/2n} \exp\left[-\frac{n}{n+q} x^{(n+q)/n} \right] \sum_{j=0}^{\infty} A_j x^{-j/n} \\ & + x^{-(n-1)q/2n} \exp\left[\sum_{j=0}^{n+q-1} \frac{n}{n+q-j} \frac{\partial \alpha_j(b)}{\partial b_{p,r}} \Big|_{b=0} x^{(n+q-j)/n} \right] \sum_{j=0}^{\infty} A_j x^{-j/n} \\ & \times \left[\sum_{j=0}^{n+q-1} \frac{n}{n+q-j} \frac{\partial \alpha_j(b)}{\partial b_{p,r}} \Big|_{b=0} x^{(n+q-j)/n} \right] \end{aligned}$$

as x tends to infinity uniformly for

$$|\arg x| \leqq (n+1)\pi/(n+q) - \delta, \tag{4.6}$$

where δ is a small positive number. Here we used the condition $\alpha_j(b)|_{b=0}=0$ $(j=1, 2,\dots, n+q)$, which is easily derived from (2.7).

In order to know the asymptotic expansion (4.5) precisely, we shall prove the following lemma.

LEMMA 4.1.

$$\left.\frac{\partial\alpha_j(b)}{\partial b_{p,r}}\right|_{b=0} = \begin{cases} 0 & (j \neq pq-nr) \\ (-1)^p/n & (j=pq-nr) \end{cases}$$
$$(p=2, 3,\dots, n;\ r=0,\dots, q-2).$$

PROOF. Differentiating (2.7) by $b_{p,r}$, we can get

$$nZ^{n-1}\frac{\partial Z}{\partial b_{p,r}} + x^r Z^{n-p} + \textstyle\sum_{j=2}^{n}\sum_{m=0}^{q-2} b_{p,r}x^m(n-j)Z^{n-j}\frac{\partial Z}{\partial b_{p,r}} = 0. \tag{4.7}$$

If we put $b=0$ ($b_{p,r}=0$ for $p=2, 3,\dots, n$; $r=0, 1,\dots, q-2$) in (4.7), we get

$$nZ^{n-1}\left.\frac{\partial Z}{\partial b_{p,r}}\right|_{b=0} + x^r Z^{n-p}\Big|_{b=0} = 0. \tag{4.8}$$

Since

$$Z|_{b=0} = -x^{q/n},$$

(4.8) becomes

$$x^{q/n}\textstyle\sum_{j=0}^{\infty}\left.\frac{\partial\alpha_j(b)}{\partial b_{p,r}}\right|_{b=0}x^{-j/n} + \frac{(-1)^{1-p}}{n}x^{r+q(1-p)/n} = 0.$$

From this relation, we can easily obtain Lemma 4.1.

Hence, using this lemma and (4.5), we can obtain the following results;

(4.9) $\eta_{p,r}(x)$

$$= \begin{cases} x^{-(n-1)q/2n}\exp\left[-\frac{n}{n+q}x^{(n+q)/n}\right]\sum_{j=0}^{\infty}\left.\frac{\partial A_j}{\partial b_{p,r}}\right|_{b=0}x^{-j/n} & (pq-nr>n+q), \\ \frac{(-1)^p}{n}x^{-(n-1)q/2n}(\log x)\exp\left[-\frac{n}{n+q}x^{(n+q)/n}\right]\sum_{j=0}^{\infty}A_j|_{b=0}x^{-j/n} & \\ & (pq-nr=n+q), \\ \frac{(-1)^p}{n+q-pq+nr}x^{-(n-1)q/(2n)+(n+q-pq+nr)/n}\exp\left[-\frac{n}{n+q}x^{(n+q)/n}\right]\sum_{j=0}^{\infty}A_j|_{b=0}x^{-j/n} & \\ & (pq-nr<n+q) \end{cases}$$

as x tends to infinity uniformly for (4.6).

Let M_{l,l^*} be the class of functions on $(0, \infty)$, which are summable in the sense of Lebesgue on each compact set of $(0, \infty)$ and which, with $l<l^*$, satisfy the two boundary conditions

$$F(t) = O(t^{-l}) \quad (t \to 0) \text{ and } F(t) = O(t^{-l^*}) \quad (t \to +\infty).$$

Then for each $F \in M_{l,l^*}$ the integral

$$f(s) = \int_0^{\infty} F(t)t^{s-1}dt$$

exists in the strip $l<\mathrm{Re}[s]<l^*$ and represents there a holomorphic function. We write

$$f(s) = M[F, s]$$

and call f the Mellin transform of F.

Now the solution $\tilde{y}(x; b)$ of the equation (1.3) is an element of M_{0,l^*} for any $l^*>0$, so its Mellin transform

$$H(s, b) = M[y(x; b), s] \tag{4.10}$$

exists as a holomorphic function in the right half-plane $\mathrm{Re}[s]>0$. Moreover, we have the following lemma. (See Wyrwich [9].)

LEMMA 4.2. *The Mellin transform $H(s; b)$ of $\tilde{y}(x; b)$ has the following properties*:

(i) *It is a meromorphic function of s with at most simple poles in $s=-k$, $k=0, 1,\dots$. The residues of $H(s, b)$ are given by*

$$\mathrm{Res}_{s=-k} H(s, b) = \frac{1}{k!}\tilde{y}^{(k)}(0; b), \quad k = 0, 1,\dots. \tag{4.11}$$

(ii) *It is a solution of the difference equation*

$$(-1)^n s(s+1)\cdots(s+n-1)H(s, b) + (-1)^{n+1}H(s+n+q, b) + \sum_{k=0}^{n-2}\sum_{m=0}^{q-2} b_{n-k,m}(-1)^k s(s+1)\cdots(s+k-1)H(s+n-k+m, b) = 0. \tag{4.12}$$

(iii) *For each complex number s ($s\neq 0, -1, -2,\dots$), $H(s, b)$ is an entire function of the parameters $b_{p,r}$ ($p=2,\dots, n$; $r=0, 1,\dots, q-2$).*

Using this lemma, we represent $H(s, b)$ as a power series of $b_{p,r}$;

$$H(s, b) = W_0(s) + \sum_{p=2}^{n}\sum_{r=0}^{q-2} W_{p,r}(s)b_{p,r} + \cdots. \tag{4.13}$$

For the coefficients $W_0(s)$ and $W_{p,r}(s)$ of this expansion we have

LEMMA 4.3. *The coefficient functions* $W_{p,r}(s)$ *of* (4.13) *are meromorphic functions. They are connected with* $\eta_{p,r}(x)$ *by*

$$W_{p,r}(s) = M[\eta_{p,r}(x), s] \tag{4.14}$$

and

$$\operatorname{Res}_{s=-k} W_{p,r}(s) = \frac{1}{k!}\eta_{p,r}^{(k)}(0), \quad k = 0, 1, \dots, \tag{4.15}$$

and they satisfy the following system of difference equations

$$W_0(s+n+q) = s(s+1)\cdots(s+n-1)W_0(s), \tag{4.16}$$

$$(-1)^n s(s+1)\cdots(s+n-1)W_{p,r}(s) + (-1)^{n+1}W_{p,r}(s+n+q) + (-1)^{n-p}s(s+1)\cdots(s+n-p-1)W_0(s+p+r) = 0. \tag{4.17}$$

In order to know the asymptotic properties of $W_0(s)$ and $W_{p,r}(s)$, we need the following lemma. (See Wyrwich [9] Lemma 18.)

LEMMA 4.4. *Let* $F(t)$ *be summable in the sense of Lebesgue on each compact subset of* $(0, \infty)$ *and satisfy the two conditions*:

(i) $F(t) = O(t^{-c}), \quad t \longrightarrow 0, \; c \in \mathbf{R}$

(ii) $F(t) \simeq \exp[-\alpha t^{\beta}]t^{-\gamma}(\log t)^k, \quad t \longrightarrow \infty, \quad k \in \mathbf{N}, \; \alpha, \beta > 0, \; \gamma \in \mathbf{C}.$

Then the Mellin transform $f(s)$ *of* $F(t)$ *exists in the half-plane* $\operatorname{Re}[s] > c$ *and satifies*

$$f(s) \simeq \frac{1}{\beta}\alpha^{\gamma/\beta}\frac{d^k}{ds^k}\left[\alpha^{-s/\beta}\Gamma\left(\frac{s-\gamma}{\beta}\right)\right], \tag{4.18}$$

as $s \to \infty$ *in any half-strip*

$$\operatorname{Re}[s] > c, \quad |\operatorname{Im} s| < d. \tag{4.19}$$

Now, utilizing (4.9), we can state

LEMMA 4.5. *Putting* $\nu = 1/(n+q)$, *the associated coefficient functions admit the asymptotic representations*:

$$W_0(s) \simeq (n\nu)^{1+(q\nu/2)(n-1)-n\nu s}\sum_{j=0}^{\infty} A_j|_{b=0}(n\nu)^{j\nu} \times \Gamma(n\nu s-(q\nu/2)(n-1)-j\nu), \tag{4.20}$$

$$W_{p,r}(s) \simeq (n\nu)^{1+(q\nu/2)(n-1)-n\nu s}\sum_{j=0}^{\infty} \frac{\partial A_j}{\partial b_{p,r}}\bigg|_{b=0}(n\nu)^{j\nu} \times \Gamma(n\nu s-(q\nu/2)(n-1)-j\nu) \tag{4.21}$$

(if $pq-nr > n+q$),

$$(4.22)\qquad W_{p,r}(s) \simeq \frac{1}{(-1)^p n}(nv)^{1+(qv/2)(n-1)} \sum_{j=0}^{\infty} A_j|_{b=0}$$
$$\times \frac{d}{ds}[(nv)^{-nvs+jv}\Gamma(nvs-(qv/2)(n-1)-jv)]$$
$$(\text{if}\quad pq-nr=n+q),$$

$$(4.23)\qquad W_{p,r}(s) \simeq \frac{1}{(-1)^p(n+q-pq-nr)}(nv)^{1+(qv/2)(n-1)-(n+q-pq+nr)}$$
$$\times (nv)^{-nvs} \sum_{j=0}^{\infty} A_j|_{b=0}(nv)^{jv}$$
$$\times \Gamma(nvs-(qv/2)(n-1) + (n+q-pq+nr)v-jv)$$
$$(\text{if}\quad pq-nr<n+q),$$

as s tends to infinity in any half-strip (4.19).

We shall now attempt to seek explicit solutions of difference equations (4.16), (4.17). At first, from (4.16) and (4.20), we can obtain

$$(4.24)\qquad W_0(s) = (2\pi)^{(1-n)/2} n^{1/2} v^{1+qv(n-1)/2} \cdot v^{-nvs} \prod_{j=0}^{n-1} \Gamma(v(s+j)).$$

In fact, a special solution of the difference equation (4.16) is

$$\Omega_0(s) = v^{-nvs} \prod_{j=0}^{n-1} \Gamma(v(s+j)),\quad v = 1/(n+q).$$

As the general solution of (4.16) is the product of $\Omega_0(s)$ and an arbitrary periodic function $\rho(s)$ of period $n+q$, we can put

$$W_0(s) = \Omega_0(s)\rho(s)$$

and have to determine $\rho(s)$ from the asymptotic representation for $W_0(s)$. Lemma 4.5 provides

$$W_0(s) \simeq (nv)^{1+(qv/2)(n-1)} n^{-nvs} \Gamma(nvs-(qv/2)(n-1))$$

as s tends to infinity in (4.19). This gives

$$\rho(s) \simeq (nv)^{1+(qv/2)(n-1)} n^{-nvs} \Gamma(nvs-(qv/2)(n-1)) \prod_{j=0}^{n-1} \Gamma(v(s+j))^{-1}$$

as s tends to infinity in (4.19).

Applying the multiplication theorem of the Γ-function and the asymptotic property of $\Gamma(z)$:

$$(4.25)\qquad z^{b-a}\frac{\Gamma(z+b)}{\Gamma(z+b)} = 1 + \frac{(a-b)(a+b-1)}{2z} + O(z^{-2}),\quad z \longrightarrow \infty,\ a,\ b \in \mathbf{C},$$

we get

$$\rho(s) \simeq v^{1+(qv/2)(n-1)}(2\pi)^{(1-n)/2} n^{1/2}$$

as s tends to infinity in (4.19). As $\rho(s)$ was supposed to be periodic, we even have equality in this relation and finally obtain (4.24).

Now if we put

$$(4.26) \qquad \Theta_{p,r}(s) = \frac{W_{p,r}(s(n+q))}{W_0(s(n+q))},$$

we can rewrite the difference equation (4.17) in the form

$$(4.27) \qquad \Theta_{p,r}(s+1) = \Theta_{p,r}(s) + \Lambda_{p,r}(s) \qquad (p=2, 3,\dots, n;\ r=0, 1,\dots, q-2),$$

where

$$(4.28) \quad \Lambda_{p,r}(s) = (-1)^p \times \frac{W_0(s(n+q)+p+r)}{[s(n+q)+n-p][s(n+q)+n-p+1]\cdots[s(n+q)+n-1]\, W_0(s(n+q))}.$$

This is an inhomogeneous difference equation of the first order. Applying (4.16), (4.24) and (4.25) to (4.28), we can easily find that

$$(4.29) \qquad \Lambda_{p,r}(s) = O(s^{-(pq-nr)/(n+q)}) = O(s^{-1-(pq-nr(n+q))/(n+q)}) \text{ in } (4.19).$$

Therefore, if

$$(\mathrm{A}_1) \qquad pq - nr > n + q,$$

we can apply the following lemma to the inhomogeneous difference equation (4.27)

LEMMA 4.6. *If $\phi(s)$ is holomorphic in a half-strip* (4.19), *and if*

$$\phi(s) = O(s^{-1-\sigma})$$

with $\sigma>0$, then the difference equation

$$f(s+1) = f(s) + \phi(s)$$

has a solution

(i) $f_0(s) = -\sum_{k=0}^{\infty} \phi(s+k)$,

which is holomorphic in (4.19) *and satisfies*

(ii) $f_0(s) = O(s^{-\sigma})$.

Hence, we can obtain

(4.30)
$$W_{p,r}(s) = - W_0(s) \sum_{j=0}^{\infty} \Lambda_{p,r}(vs+j)$$
$$= -(-1)^p \frac{W_0(s+p+r)}{(s+n-p)\cdots(s+n-1)} - (-1)^p W_0(s) K_{p,r}(s)$$
$$(pq-nr>n+q),$$

where

(4.31)
$$K_{p,r}(s) = (-1)^p \sum_{j=1}^{\infty} \Lambda_{p,r}(vs+j).$$

In fact, by Lemma 4.6, we get a solution $\Theta^*_{p,r}(s)$ of (4.17) in the form

$$\Theta^*_{p,r}(s) = - \sum_{j=0}^{\infty} \Lambda_{p,r}(s+j),$$

which is holomorphic in (4.19) and satisfies $\Theta^*_{p,r}(s)=O(s^{-(pq-nr-n-q)/(n+q)})$ there. Since the general solution of (4.17) is the sum of the special solution $\Theta^*_{p,r}(s)$ and an arbitrary periodic function of period 1, it is clear that $\Theta^*_{p,r}(s)$ is the only solution of (4.17) with this property. On the other hand, since $A_0=1$, we have from Lemma 4.5,

$$\Theta_{p,r}(s) = O(s^{-(pq-nr-n-q)/(n+q)})$$

and this asymptotic condition implies $\Theta_{p,r}(s)=\Theta^*_{p,r}(s)$ $(pq-nr>n+q)$. Rewriting this in terms of $W_0(s)$, we have (4.30).
Furthermore, it follows from (4.28) and (4.31) that

$$K_{p,r}(s) = \sum_{j=1}^{\infty}$$
$$\times \frac{W_0(s+j(n+q)+p+r)}{[s+j(n+q)+n-p][s+(n+q)+n-p+1]\cdots[s+j(n+q)+n-1]}$$
$$\times \frac{1}{W_0(s+j(n+q))}.$$

Therefore, we get from (4.24) that

$$K_{p,r}(s) > 0 \quad \text{for} \quad s > -(n+q).$$

Next we shall consider the case in which

(A_2)
$$pq - nr < n + q.$$

We have the following lemma concerning inhomogeneous difference equations.

LEMMA 4.7 (*Nörlund* [6]). *Let k be a non-negative integer, such that*

(i) $g \in C^k(s_0, \infty)$;

(ii) $g^{(k)}(s) = O(s^{-1-\sigma})$ *for* $s \geqq s_1 > s_0$ *and* $\sigma > 0$.

Then

$$F(s) = \lim_{\varepsilon\to 0}\left[\int_{c'}^{\infty} g(t)e^{-\varepsilon t}dt - \sum_{k=0}^{\infty} g(s+k)e^{-\varepsilon(s+k)}\right] \qquad (c',\ s>s_0)$$

exists and this is a solution of the difference equation

$$f(s+1) = f(s) + g(s).$$

These solutions are called principal solutions.

If we can show that $\Theta_{p,r}(s)$ is a principal solution of the difference equation (4.27), each $W_{p,r}(s)$, which is a solution of (4.17), has the following form:

$$W_{p,r}(s) = W_0(s)[c-(-1)^p K'_{p,r}(s)] - (-1)^p \times \frac{W_0(s+p+r)}{(s+n-p)(s+n-p+1)\cdots(s+n-1)} \qquad (pq-nr<n+q), \tag{4.32}$$

where

$$K'_{p,r}(s) = \lim_{\varepsilon\to 0}\left[\int_0^{\infty} -(-1)^p \Lambda_{p,r}(t)e^{-\varepsilon t}dt + (-1)^p \sum_{j=1}^{\infty} \Lambda_{p,r}(\nu s+j)e^{-\varepsilon(\nu s+j)}\right] \tag{4.33}$$

and

$$c = \int_{c'}^{0} \Lambda_{p,r}(t)e^{-\varepsilon t}dt.$$

Since $\Lambda_{p,r}(s)$ is a real-valued function, c is a constant real number which depends on p, r, n, q and c'.

In order to show that $\Theta_{p,r}(s)$ is a principal solution of (4.27), we use the following lemmas.

LEMMA 4.8 (*Nörlund* [6]). *Let k be a non-negative integer. We assume that $g(s)$ satisfies the following two conditions*:

$$g\in C^k(s_0,\infty) \quad \textit{and} \quad g^{(k)}(s) = O(s^{-1-\sigma}) \qquad \textit{for} \quad s\geqq s_1 > s_0,\ \sigma>0. \tag{4.34}$$

Then each principal solution $f(s)$ of the difference equation

$$f(s+1) = f(s) + g(s) \tag{4.35}$$

satisfies the following two conditions:

(i) $f(s)\in C^k(s_0,\infty)$,

(ii) $\lim_{s\to+\infty} f^{(k)}(s)$ *exists.*

LEMMA 4.9 (*Nörlund* [6]). *Let $f(s)$ be a solution of the difference equation* (4.35) *and let $g(s)$ satisfy* (4.34). *Then, if*

$$\lim_{s\to+\infty} f^{(k)}(s) \quad \textit{exists,}$$

$f(s)$ is a principal solution of the equation (4.35).

LEMMA 4.10 (*Wyrwich* [9]). *We assume that $h(s)$ is holomorphic in* (4.19) *and*

$$h(s) = O(s^{\alpha}) \quad \textit{in (4.19) with } \alpha \in \mathbf{R}.$$

Then it holds that

$$h^{(\mu)}(s) = O(s^{\alpha}) \quad \textit{in (4.19)}.$$

It follows from (4.28) and (4.16) that

$$\Lambda_{p,r}(s) = (-1)^p \frac{s(n+q)[s(n+q)+1]\cdots[s(n+q)+n-q-1]W_0(s(n+q)+p+r)}{W_0((s+1)(n+q))}.$$

Therefore, $\Lambda_{p,r}(s)$ is holomorphic in $\operatorname{Re} s > -(p+r)/(n+q)$.
Furthermore, we get from (4.24), (4.25) and (4.28) that

$$\Lambda_{p,r}(s) = (-1)^p \frac{\nu^{-n\nu(p+r)} s^{(p+r)} \prod_{j=0}^{n-1}\left[1+\dfrac{(p+r)((p+r)\nu+2j\nu-1)}{2s}+O(s^{-2})\right]}{(n+q)^p s^p\left[1+\dfrac{n-p}{s(n+q)}\right]\left[1+\dfrac{n-p+1}{s(n+q)}\right]\cdots\left[1+\dfrac{n-1}{s(n+q)}\right]}$$

$$= (-1)^p \frac{1}{(n+q)^p \nu^{-n\nu(p+r)}} s^{-(pq-nr)/(n+q)} + O(s^{-1-(pq-nr)/(n+q)}).$$

Using Lemma 4.10, we can obtain

$$\Lambda_{p,r}(s) = O(s^{-1-(pq-nr)/(n+q)}).$$

Since $(pq-nr)/(n+q)>0$ under the condition $q(n-2)<2n$, we find that $\Lambda_{p,r}(s)$ satisfies the conditions (4.34). It holds from (4.20), (4.23), (4.26) and (4.28) that

$$\Theta_{p,r}(s) \simeq s^{1-(pq-nr)/(n+q)} \sum_{j=0}^{\infty} \gamma_j s^{-j/(n+q)}.$$

Noting that $(pq-nr)/(n+q)>0$, we can get that

$$\lim_{s\to+\infty}\left[\frac{d}{ds}\Theta_{p,r}(s)\right] \quad \text{exists.}$$

Therefore, we have obtained from Lemma 4.9 that each $\Theta_{p,r}(s)$ is a principal solution of the difference equation (4.27). Thus we have obtained (4.23).

Summarizing these results concerning difference equations (4.16) and (4.17), we have the following

LEMMA 4.11.

$$(4.24)\qquad W_0(s) = (2\pi)^{(1-n)/2}\cdot n^{1/2}\cdot\nu^{1+(q\nu/2)(n-1)}\cdot\nu^{-n\nu s}\prod_{j=0}^{n-1}\Gamma(\nu(s+j)).$$

$$(4.30)\qquad W_{p,r}(s) = -(-1)^p\frac{W_0(s+p+r)}{(s+n-p)(s+n-p+1)\cdots(s+n-1)} - (-1)^p W_0(s)K_{p,r}(s) \quad (\text{if } pq-nr>n+q)$$

and

$$(4.32)\qquad W_{p,r}(s) = W_0(s)[c-(-1)^pK'_{p,r}(s)] - (-1)^p \times \frac{W_0(s+p+r)}{(s+n-p)(s+n-p+1)\cdots(s+n-1)} \quad (\text{if } pq-nr<n+q)$$

for $s>-(p+r)$; $s\neq 0, -1,\dots, -(p+r)+1$, *where* $K_{p,r}(s)$ *and* $K'_{p,r}(s)$ *are given by* (4.31) *and* (4.33), *respectively. Furthermore,*

$$K_{p,r}(s) > 0 \qquad \textit{for} \quad s > -(n+q)$$

and c *is a constant real number which depends on* p, r, n, q *and* c^*.

Now we shall consider $K_{p,r}(s)$ $[K'_{p,r}(s)]$ and $K_{p,r}(s+1)$ $[K'_{p,r}(s+1)]$. From (4.33) and (4.28), we can get

(4.35)

$$\frac{A_{p,r}(\nu s+k)e^{(\nu s+k)}(-1)^p}{A_{p,r}(\nu s'+k)e^{-\varepsilon(\nu s'+k)}(-1)^p}$$
$$= \frac{W_0(m+p+r)e^{-\varepsilon(\nu s+k)}(m+n+p+1)(m+n-p+2)\cdots(m+n)W_0(m+1)}{(m+n-p)(m+n-p+1)\cdots(m+n-1)W_0(m)W_0(m+1+p+r)e^{-\varepsilon(\nu s+\nu+k)}}$$
$$= \frac{(m+n)W_0(m+1)W_0(m+p+r)}{(m+n-p)W_0(m)W_0(m+p+r+1)}e^{\varepsilon\nu},$$

where

$$(4.36)\qquad m = (n+q)(\nu s+k) = s + (n+q)k \quad\text{and}\quad s' = s+1.$$

Using (4.24) and (4.25) for a sufficiently large integer k, we have from

(4.35)

$$\frac{(m+n)W_0(m+1)W_0(m+p+r)}{(m+n-p)W_0(m)W_0(m+p+r+1)}e^{\varepsilon\nu}$$
$$= \frac{(n+s+(n+q)k)\Gamma(k+\nu(n+s))\Gamma(k+\nu(s+p+r)}{(n+s-p+(n+q)k)\Gamma(k+\nu s)\Gamma(k+\nu(s+p+r+n))}e^{\varepsilon\nu}$$

$$= e^{\varepsilon\nu}\left[1+\frac{p}{(n+q)k}+O(k^{-2})\right]\left[1+\frac{\nu n(\nu n+2\nu s-1)}{2k}+O(k^{-2})\right]$$

$$\times\left[1+\frac{-\nu n(2\nu(s+p+r)+\nu n-1}{2k}+O(k^{-2})\right]$$

$$- e^{\varepsilon\nu}\left[1+\frac{1}{k}\left(\frac{p}{n+q}+\frac{\nu n(\nu n+2\nu s-1)}{2}+\frac{-\nu n(2\nu(s+p+r)+\nu n-1}{2}\right)\right.$$

$$\left.+O(k^{-2})\right]$$

$$= e^{\varepsilon\nu}\left[1+\frac{pq-nr}{(n+q)^2}\frac{1}{k}+O(k^{-2})\right].$$

Since

$$(pq-nr)/(n+q)^2 > 0$$

under the condition $q(n-2)<2n$, we can get

$$(-1)^p A_{p,r}(\nu s+k)e^{-\varepsilon(\nu s+k)} > (-1)^p A_{p,r}(\nu s'+k)e^{-\varepsilon(\nu s'+k)} \tag{4.37}$$

for a sufficiently large integer k.

Next we assume that (4.37) holds for $k \geqq k'$ and we consider the $(k'-1)$-th term. Using the fact that $W_0(s)$ satisfies the difference equation (4.16), we can get

$$\frac{(-1)^p A_{p,r}(\nu s+k'-1)e^{-\varepsilon(\nu s+k'-1)}}{(-1)^p A_{p,r}(\nu s'+k'-1)e^{-\varepsilon(\nu s'+k'-1)}}$$

$$= \frac{(m'+n-(n+q))\,W_0(m'-(n+q)+1)\,W_0(m'+p+r-(n+q))}{(m'+n-p-(n+q))\,W_0(m'-(n+q))\,W_0(m'+1+p+r-(n+q))}e^{\varepsilon\nu}$$

$$= \frac{(m'-q)\,W_0(m'+1)(m'-(n+q))\cdots(m'-(n+q)+n-1)}{(m'-p-q)(m'-(n+q)+1)\cdots(m'-(n+q)+1+n-1)\,W_0(m')}\times$$

$$\frac{W_0(m'+p+r)(m'+1+p+r-(n+q))\cdots(m'+1+p+r-(n+q)+n-1)}{(m'+p+r-(n+q))\cdots(m'+p+r-(n+q)+n-1)\,W_0(m'+1+p+r)}e^{\varepsilon\nu}$$

$$= \frac{(m'-n-q)(m'+p+r-q)\,W_0(m'+p+r)\,W_0)(m'+1)}{(m'-p-q)(m'+p+r-(n+q))\,W_0(m'+1+p+r)\,W_0(m')}e^{\varepsilon\nu}$$

$$> \frac{(m'+n-p)(m'-n-q)(m'+p+r-q)}{(m'+n)(m'-p-q)(m'+p+r-p-n)},$$

where

$$m' = s+(n+q)k'.$$

Here we used the assumption of the induction that (4.37) holds for $k=k'$. Since $pq-nr>0$ and

$$[(m'+n-p)(m'-n-q)(m'+p+r-q)]$$
$$-\ [(m'+n)(m'-p-q)(m'+p+r-q-n)]$$
$$= (pq-nr)m' + pq(-2n+p+r-q) + nr(2p-n),$$

we can obtain the following lemma from above results.

LEMMA 4.12. *Suppose that*

$$(4.38)\quad D^*[n, q, p, r; s^*, k^*]$$
$$= (pq-nr)(s^*+(n+q)(k^*+1)) + pq(-2n+p+r-q) + nr(2p-n) \geqq 0.$$

Then (4.37) *holds for* $s \geqq s^*$ *and* $k \geqq k^*$. *Furthermore, if*

$$D^*(n, q, p, r; s^*, 1) \geqq 0,$$

then

$$K_{p,r}(s) > K_{p,r}(s+1)\,[K'_{p,r}(s) > K'_{p,r}(s+1)] \qquad \textit{for} \quad s \geqq s^*.$$

We shall here make preparations for the use in §6. Assuming that

$$pq - nr > n + q \quad \text{and} \quad (p-1)p - n(r+1) < n + q,$$

we shall consider

$$(4.39)\qquad W^*_{p,r}(s) = LW_{p,r}(s) - W_{p-1,r+1}(s),$$

where L is a constant number. From (4.17), $W^*_{p,r}(s)$ satisfies the following difference equation;

$$(4.40)\qquad (-1)^n s(s+1)\cdots(s+n-1)W^*_{p,r}(s) + (-1)^{n+1}W^*_{p,r}(s+n+q)$$
$$+ (-1)^{n-p}s(s+1)\cdots(s+n-p-1)(s+n-p+L)W_0(s+p+r) = 0.$$

Putting

$$\Theta^*(s+1) = W^*_{p,r}(s(n+q))/W_0(s(n+q)),$$

we can easily obtain that (4.40) becomes

$$\Theta^*(s+1) = \Theta^*(s) + \Lambda^*_{p,r}(s),$$

where

$$(4.41)\quad \Lambda^*_{p,r}(s) = (-1)^p \frac{[s(n+q)+n-p-1+L]\,W_0(s(n+q)+p+r)}{[s(n+q)+n-p-1]\cdots[s(n+q)+n-1]\,W_0(s(n+q))}.$$

Therefore, using Lemmas 4.7–4.9, we can similarly obtain

$$(4.42)\qquad W^*_{p,r}(s) = W_0(s)[c^* - (-1)^p K^*_{p,r}(s)]$$
$$- (-1)^p \frac{(s+n-p-1+L)\,W_0(s+p+r)}{(s+n-p-1)\cdots(s+n-1)},$$

where

(4.43)

$$K^*_{p,r}(s) = \lim_{\varepsilon\to 0}\left[\int_0^\infty -(-1)^p \Lambda^*_{p,r}(t)e^{-\varepsilon t}dt + (-1)^p \sum_{j=1}^\infty \Lambda^*_{p,r}(vs+j)e^{-\varepsilon(vs+j)}\right]$$

and if L is a real number, c^* is a constant real number.

Lastly we prove the following lemma, which we often use in §5 and §6.

LEMMA 4.13. *Let a, b $(b>a)$ be non-negative integers. Then, we have*

$$\frac{W_0(a+1)\,W_0(b)}{W_0(a)\,W_0(b+1)} < 1. \tag{4.44}$$

PROOF. From (4.24), it follows that

$$\frac{W_0(a+1)\,W_0(b)}{W_0(a)\,W_0(b+1)} = \frac{\Gamma(v(a+n))\Gamma(vb)}{\Gamma(va)\Gamma(v(b+n))}. \tag{4.45}$$

Since $\log[\Gamma(x)]$ $(x>0)$ is a strictly convex function, we have

$$\begin{aligned}\log\Gamma(vb) &= \log\Gamma\left(\left(\frac{b-a}{b+n-a}\right)v(b+n) + \left(\frac{n}{b+n-a}\right)va\right)\\ &< \left(\frac{b-a}{b+n-a}\right)\log\Gamma(v(b+n)) + \left(\frac{n}{b+n-a}\right)\log\Gamma(va)\end{aligned}$$

and

$$\begin{aligned}\log\Gamma(v(a+n)) &= \log\Gamma\left(\left(\frac{n}{b+n-a}\right)v(b+n) + \left(\frac{b-a}{b+n-a}\right)va\right)\\ &< \left(\frac{n}{b+n-a}\right)\log\Gamma(v(b+n)) + \left(\frac{b-a}{b+n-a}\right)\log\Gamma(va).\end{aligned}$$

Therefore, we can easily obtain

$$\log\Gamma(vb) + \log\Gamma(v(a+n)) < \log\Gamma(v(b+n)) + \log\Gamma(va).$$

Using (4.45), we have obtained Lemma 4.13.

REMARK. If it holds that

$$(pq-nr)(m+n+q) + pq(-2n+p+r-q) + nr(2p-n) \geqq 0, \tag{4.46}$$

then it follows from (4.35) and (4.38) that

$$\frac{(m+n)\,W_0(m+1)\,W_0(m+p+r)}{(m+n-p)\,W_0(m)\,W_0(m+p+r+1)} > 1.$$

Therefore, using Lemma 4.11, we can get

$$(4.47)\qquad 1>\frac{W_0(m+1)\,W_0(m+p+r)}{W_0(m)\,W_0(m+p+r+1)}>\frac{m+n-p}{m+n}$$

under the conditions (4.46) and $m\geqq 0$.
Furthermore, from (4.24), it holds that

$$(4.48)\quad \frac{W_0(m+1)\,W_0(m+p+r)}{W_0(m)\,W_0(m+p+r+1)}=\frac{\Gamma((m+n)/(n+q))\Gamma((m+p+r)/(n+q))}{\Gamma(m/(n+q))\Gamma((m+n+p+r)/(n+q))}\,.$$

Putting $m=(n+q)k$, $p=2$ and $r=q-2$, we get

$$\begin{aligned}\frac{W_0(m+1)\,W_0(m+p+r)}{W_0(m)\,W_0(m+p+r+1)}&=\frac{\Gamma(k+n/(n+q))\Gamma(k+q/(n+q))}{(k-1)!k!}\\&=\frac{(k-1+n/(n+q))(k-2+n/(n+q))\cdots(1+n/(n+q))}{(k-1)!}\\&\qquad\frac{(k-1+q/(n+q))(k-2+q/(n+q))\cdots(1+q/(n+q))}{k!}\times\\&\qquad\times\Gamma(1+n/(n+q))\Gamma(1+q/(n+q))\\&=\frac{(k-1+n/(n+q))\cdots(1+n/(n+q))(k-1+q/(n+q))\cdots(1+q/(n+q))}{(k-1)!k!}\times\\&\qquad\times\frac{nq}{(n+q)^2}\,\frac{\pi}{\sin(n\pi/(n+q))}\,,\end{aligned}$$

Therefore, it follows from (4.27) that

$$\begin{aligned}&\frac{nq(k-1)!k!}{(k-1+n/(n+q))\cdots(1+n/(n+q))(k-1+q/(n+q))\cdots}\\&\qquad\frac{[k(n+q)+n]\pi}{(1+q/(n+q))(n+q)^2[k(n+q)+n-2]}\\&>\sin(n\pi/(n+q))\\&>\frac{nq(k-1)!}{(k-1+n/(n+q))\cdots(1+n/(n+q))(k-1+q/(n+q))\cdots}\\&\qquad\frac{k!\pi}{(1+q/(n+q))(n+q)^2}\,.\end{aligned}$$

REMARK. In the differential equation (1.3), if we put $b=0$, then (1.3) becomes an extended Airy equation. Since $\tilde{y}(x;0)$ is a principally recessive solution on the positive real axis $\arg x=0$, an extended Airy function of the first kind $Ai(x)$ coincides with that. It follows that (See M. Kohno (3))

$$A_i(x)=\sum_{i=1}^{n} d_i\cdot A_i(\omega^{k_i}x)\,,$$

where

$$d_i = \frac{(\omega^{k_1}-1)(\omega^{k_2}-1)\cdots(\omega^{k_{i-1}}-1)(\omega^{k_{i+1}})\cdots(\omega^{k_n}-1)}{(\omega^{k_1}-\omega^{k_i})(\omega^{k_2}-\omega^{k_n})\cdots(\omega^{k_{i-1}}-\omega^{k_i})(\omega^{k_{i+1}}-\omega^{k_i})\cdots(\omega^{k_n}-\omega^{k_i})}$$

and k_i $(i=1, 2,\dots, n)$ are mutually distinct modulo $n+q$. Hence we can easily obtain

$$C_j^k(0) \neq 0 \qquad (k=0, 1,\dots, n+q-1; j=1, 2,\dots, n).$$

§5. Difference equations and Stokes multipliers (II)

In §4, we obtained

$$C_j^k(0) \neq 0 \qquad (k=0, 1,\dots, n+q-1; j=1, 2,\dots, n).$$

In this section, making use of solutions of the difference equations $W_0(s)$ and $W_{p,r}(s)$, we shall prove the following lemma.

LEMMA 5.1. *In the connection formula* (4.1) *it holds that*

$$\left.\frac{\partial C_1^k(b)}{\partial b_{p,r}}\right|_{b=0} \neq 0 \qquad (pq-nr\neq n+q;\ k=0, 1,\dots, n+q-1).$$

To do this, we prove the following

LEMMA 5.2. *Suppose that*

$$\left.\frac{\partial C_1^k(b)}{\partial b_{p,r}}\right|_{b=0} = 0 \qquad (pq-nr\neq n+q). \tag{5.1}$$

Then

$$\frac{\partial}{\partial b_{p,r}}[\tilde{y}^{(p+r-j)}(0; b)]|_{b=0} = 0 \qquad (j=1, 2,\dots, q). \tag{5.2}$$

PROOF. From the Cramer rule and the connection formula (4.1), we see that the Stokes multipliers $C_1^k(b)$ are given by the formula

$$C_1^k(b) = \frac{\text{Wron}\,[y_k(x; b), y_{k+2}(x; b),\dots, y_{k+n}(x; b)]}{\text{Wron}\,[y_{k+1}(x; b), y_{k+2}(x; b),\dots, y_{k+n}(x; b)]}. \tag{5.3}$$

It follows from Lemmas 2.3 and 4.1 that

$$\begin{aligned}&\text{Wron}\,[y_{k+1}(x; b), y_{k+2}(x; b),\dots, y_{k+n}(x; b)]\\ &= \omega^{-\sum_{h=k+1}^{k+n}[h\alpha_{n+q}(G^h(b))-hq(n-1)/2n]} \times \det|(\exp[-2hm\pi i/n])|\\ &\qquad\qquad\qquad\qquad \begin{Bmatrix}h=k+1, k+2,\dots, k+n\\ m=0, 1,\dots, n-1\end{Bmatrix}\end{aligned}$$

does not depend on $b_{p,r}$ $(pq-nr\neq n+q)$. Therefore, we get from (5.1)

$$\frac{\partial}{\partial b_{p,r}}\text{Wron}\,[y_k(0; b), y_{k+2}(0; b),\dots, y_{k+n}(0; b)]|_{b=0} = 0 \tag{5.4}$$

and

$$\frac{\partial}{\partial b_{p,r}} \mathrm{Wron}\,[y_{k+1}(0;b), y_{k+2}(0;b),\dots, y_{k+n}(0;b)]\,|_{b=0} = 0. \tag{5.5}$$

Furthermore, from the definition (2.13), it holds that

$$\tilde{y}_k^{(m)}(0;b)|_{b=0} = \omega^{-km} y^{(m)}(0;b)|_{b=0} \tag{5.6}$$

$$\frac{\partial}{\partial b_{p,r}} y_k^{(m)}(0;b)|_{b=0} = \omega^{(p+r-m)k} \frac{\partial}{\partial b_{p,r}} \tilde{y}^{(m)}(0;b)|_{b=0}$$

$$(m=0, 1,\dots, n-1;\ k=0, 1,\dots, n+q-1).$$

We get from Lemma 4.3

$$\tilde{y}^{(m)}(0;b)|_{b=0} = m!\cdot \mathrm{Res}_{s=-m}\, W_0(s)$$

and

$$\left.\frac{\partial \tilde{y}^{(m)}(0;b)}{\partial b_{p,r}}\right|_{b=0} = m! \times \mathrm{Res}_{s=-m}\, W_{p,r}(s) \qquad (m=0, 1,\dots, n-1).$$

Therefore, we can easily verify from Lemma 4.11 that $\tilde{y}^{(m)}(0;b)|_{b=0}$ and $\partial\tilde{y}^{(m)}(0;b)/\partial b_{p,r}|_{b=0}$ $(pq-nr\neq n+q;\ m=0, 1,\dots, p+r-1)$ have real values. Furthermore, it holds from (4.24) and (4.16) that

$$\mathrm{Res}_{s=-m}\, W_0(s) > 0 \qquad \text{for} \quad m = 0, 2, 4,\dots,$$

and

$$\mathrm{Res}_{s=-m}\, W_0(s) < 0 \qquad \text{for} \quad m = 1, 3, 5,\dots.$$

Noting these facts, we consider the following cases;

Case (I) $q=2$; In this case, using (5.6) and letting k_i be mutually distinct modulo $n+2$, we put

$$\frac{\partial}{\partial b_{p,0}} \mathrm{Wron}\,[y_{k_1}(0;b), y_{k_2}(0;b),\dots, y_{k_n}(0;b)]\,|_{b=0} = \textstyle\sum_{j=0}^{n-1} L_{j+1}^p, \tag{5.7}$$

where

$$L_{j+1}^p = \left[\frac{\partial}{\partial b_{p,0}} \tilde{y}^{(j)}(0;b)|_{b=0} \prod_{\substack{k=0\\k\neq j}}^{n-1} \tilde{y}^{(k)}(0;0) \times\right. \tag{5.8}$$

$$\times \det \begin{vmatrix} 1 & \cdots\cdots & 1 & \cdots\cdots & 1 \\ \lambda_1 & \cdots\cdots & \lambda_i & \cdots\cdots & \lambda_n \\ \vdots & & \vdots & & \vdots \\ \lambda_1^{j-1} & \cdots\cdots & \lambda_i^{j-1} & \cdots\cdots & \lambda_n^{j-1} \\ \lambda_1^{j-p} & \cdots\cdots & \lambda_i^{j-p} & \cdots\cdots & \lambda_n^{j-p} \\ \lambda_1^{j+1} & \cdots\cdots & \lambda_i^{j+1} & \cdots\cdots & \lambda_n^{j+1} \\ \vdots & & \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots\cdots & \lambda_i^{n-1} & \cdots\cdots & \lambda_n^{n-1} \end{vmatrix}.$$

Here we put

$$\lambda_i = \omega^{-k_i}.$$

Now in the determinant of (5.8), for each $j \neq p-1$, $p-2$ (mod $n+2$), there is another row which coincides with the $(j+1)$-th row. So we can get $L^p_{j+1}=0$ for $j \neq p-1$, $p-2$. Hence, in order to seek the value of (5.7), we have only to consider the cases in which $j=p-1$, $p-2$. To do this, the following notation of the determinant is put to use:

$$(5.9) \qquad V_{n+1}(\lambda) = \det \begin{vmatrix} 1 & 1 \cdots 1 \cdots 1 \\ \lambda & \lambda_1 \cdots \lambda_i \cdots \lambda_n \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \lambda^{p-3} & \lambda_1^{p-3} \cdots \lambda_i^{p-3} \cdots \lambda_n^{p-3} \\ \lambda^{p-2} & \lambda_1^{p-2} \cdots \lambda_i^{p-2} \cdots \lambda_n^{p-2} \\ \lambda^{p-1} & \lambda_1^{p-1} \cdots \lambda_i^{p-1} \cdots \lambda_n^{p-1} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \lambda^{n-1} & \lambda_1^{n-1} \cdots \lambda_i^{n-1} \cdots \lambda_n^{n-1} \\ \lambda^n & \lambda_1^n \cdots \lambda_i^n \cdots \lambda_n^n \end{vmatrix} = \sum_{j=0}^{n} d_j(k_1, k_2,\dots, k_n)\lambda^j.$$

Then we can get

$$(5.10) \qquad \begin{aligned} L^p_{p-1} &= \left[\frac{\partial}{\partial b_{p,0}} \tilde{y}^{(p-2)}(0; b)|_{b=0}\right] \prod_{\substack{k=0 \\ k\neq p-2}}^{n-1} \tilde{y}^{(k)}(0; 0)(-1)^{n-1} d_{p-2}(k_1,\dots, k_n), \\ L^p_p &= \frac{\partial}{\partial b_{p,0}} \tilde{y}^{(p-1)}(0; b)|_{b=0} \prod_{\substack{k=0 \\ k\neq p-1}}^{n-1} \tilde{y}^{(k)}(0; 0) \frac{(-1)}{\prod_{i=1}^{n} \lambda_i} d_p(k_1, k_2,\dots, k_n). \end{aligned}$$

In order to consider the conditions (5.4) and (5.5), we must seek the values of $d_p(k, k+2,\dots, k+n)$, $d_p(k+1, k+2,\dots, k+n)$, $d_{p-2}(k, k+2,\dots, k+n)$ and $d_{p-2}(k+1, k+2,\dots, k+n)$. For that purpose, we put

$$(5.11) \qquad \prod_{i=1}^{n} (\lambda - \omega^{-k-i}) = \sum_{j=0}^{n} g_j \lambda^j \quad \text{and}$$
$$\prod_{i=2}^{n} (\lambda - \omega^{-k-i})(\lambda - \omega^{-k}) = \sum_{j=0}^{n} \tilde{g}_j \lambda^j.$$

Using the identities

$$(\lambda - \omega^{-k})(\lambda - \omega^{-k+1}) \sum_{j=0}^{n} g_j \lambda^j = \lambda^{n+2} - 1$$

and

$$(\lambda - \omega^{-k-1})(\lambda - \omega^{-k+1}) \sum_{j=0}^{n} \tilde{g}_j \lambda^j = \lambda^{n+2} - 1,$$

we can easily obtain

$$g_{j-2} - g_{j-1}(\omega^{-k} + \omega^{-k+1}) + g_j \omega^{-k} \omega^{-k+1} = 0 \qquad (j=2, 3,\dots, n),$$

$$g_{n-1} - g_n(\omega^{-k}+\omega^{-k+1}) = 0,$$
$$- g_0(\omega^{-k}+\omega^{-k+1}) + g_1\omega^{-k}\omega^{-k+1} = 0,$$
$$g_n = 1, \quad g_0 = -\omega^{2k-1}$$

and

$$\tilde{g}_{j-2} - \tilde{g}_{j-1}(\omega^{-k-1}+\omega^{-k+1}) + \tilde{g}_j\omega^{-k-1}\omega^{-k+1} = 0 \qquad (j=2, 3,\dots, n),$$
$$\tilde{g}_{n-1} - \tilde{g}_n(\omega^{-k-1}+\omega^{-k+1}) = 0,$$
$$- \tilde{g}_0(\omega^{-k-1}+\omega^{-k+1}) + \tilde{g}_1\omega^{-k-1}\omega^{-k+1} = 0,$$
$$\tilde{g}_n = 1, \quad \tilde{g}_0 = -\omega^{2k}.$$

Therefore, it holds that

$$g_j = \frac{\omega^{2k-1}}{\omega^{-1}-1}\omega^{kj} - \frac{\omega^{2k-2}}{\omega^{-1}-1}\omega^{(k-1)j} \tag{5.12}$$

and

$$\tilde{g}_j = -\frac{\omega^{2k+2}}{\omega^2-1}\omega^{(k+1)j} + \frac{\omega^{2k}}{\omega^2-1}\omega^{(k-1)j}. \tag{5.13}$$

Since $V_{n+1}(\lambda)$ is a Vandermonde determinant, it holds that

$$V_{n+1}(\lambda) = (-1)^{n(n+1)/2} \times \prod_{i=1}^{n}(\lambda-\lambda_i)\prod_{i<j}(\lambda_i-\lambda_j).$$

Therefore, from the definition of $d_p\,(k_1, k_2,\dots, k_n)$, we get

$$d_m(k+1, k+2,\dots, k+n) = (-1)^{n(n+1)/2}g_m \times \prod_{i<j}(\omega^{-k-i}-\omega^{-k-j}) \tag{5.14}$$

and

$$\begin{aligned} d_m(k, k+2,\dots, k+n) = (-1)^{n(n+1)/2}\tilde{g}_m &\times \prod_{i\neq 2}(\omega^{-k}-\omega^{-k-i}) \\ &\times \prod_{2<i<j}(\omega^{-k-i}-\omega^{-k-j}) \end{aligned} \tag{5.15}$$
$$(m=0, 1,\dots, n).$$

Using these results, we can rewrite (5.4) and (5.5) as follows;

$$\begin{aligned} &\frac{\partial}{\partial b_{p,0}}\tilde{y}^{(p-2)}(0;0)\prod_{\substack{k=0\\k\neq p-2}}^{n-1}\tilde{y}^{(k)}(0;0)(-1)^{n-1}d_{p-2}(k, k+2,\dots, k+n) \\ &+ \frac{\partial}{\partial b_{p,0}}\tilde{y}^{(p-1)}(0;0)\prod_{\substack{k=0\\k\neq p-1}}^{n-1}\tilde{y}^{(k)}(0;0)\frac{(-1)d_p(k, k+2,\dots, k+n)}{\omega^{-k}\prod_{i=2}^{n}\omega^{-k-i}} = 0, \end{aligned} \tag{5.16}$$

$$\begin{aligned} &\frac{\partial}{\partial b_{p,0}}\tilde{y}^{(p-2)}(0;0)\prod_{\substack{k=0\\k\neq p-2}}^{n-1}\tilde{y}^{(k)}(0;0)(-1)^{n-1}d_{p-2}(k+1, k+2,\dots, k+n) \\ &+ \frac{\partial}{\partial b_{p,0}}\tilde{y}^{(p-1)}(0;0)\prod_{\substack{k=0\\k\neq p-1}}^{n-1}\tilde{y}^{(k)}(0;0)\frac{(-1)d_p(k+1, k+2,\dots, k+n)}{\prod_{i=1}^{n}\omega^{-k-i}} \\ &= 0. \end{aligned} \tag{5.17}$$

Since

$$\begin{aligned}\tilde{y}^{(k)}(0;0) &= k!\,\mathrm{Res}_{s=-k}\, W_0(s)\\ &= k!(-1)^k(2\pi)^{(1-n)/2}n^{1/2}\nu^{\nu(n-1)}\nu^{n\nu k}\prod_{\substack{j=0\\ j\neq k}}^{n-1}\Gamma(\nu(-k+j)) \neq 0,\end{aligned}$$

in order to prove (5.2) from (5.16) and (5.17), we have only to show that the following determinant of the coefficient matrix does not vanish. In fact, we get

$$\begin{aligned}&\begin{vmatrix} (-1)^{n-1}d_{p-2}(k, k+2,\dots, k+n) & \dfrac{(-1)d_p(k, k+2,\dots, k+n)}{\omega^{-k}\prod_{i=2}^n \omega^{-k-i}} \\ (-1)^{n-1}d_{p-2}(k+1, k+2,\dots, k+n) & \dfrac{(-1)d_p(k, k+2,\dots, k+n)}{\prod_{i=1}^n \omega^{-k-i}} \end{vmatrix}\\ &= \prod_{i=2}^n \omega^{k+i}\prod_{i<j}(\omega^{-k-i}-\omega^{-k-j})\prod_{i=2}^n(\omega^{-k}-\omega^{-k-i})\\ &\qquad\times\prod_{2<i<j}(\omega^{-k-i}-\omega^{-k-j})\omega^k\begin{vmatrix}\tilde{g}_{p-2} & \tilde{g}_p\\ g_{p-2} & \omega g_p\end{vmatrix}.\end{aligned}$$

Furthermore, using (5.12) and (5.13), we get

$$\begin{aligned}\begin{vmatrix}\tilde{g}_{p-2} & \tilde{g}_p\\ g_{p-2} & \omega g_p\end{vmatrix} &= \frac{\omega^{2k-2+2pk}}{(\omega^{-1}-1)(\omega^2-1)}\begin{vmatrix}-\omega^p+\omega^{2-p} & -\omega^{p+2}+\omega^{-p}\\ \omega^{-1}-\omega^{-p} & 1-\omega^{-1-p}\end{vmatrix}\\ &= \frac{1}{(\omega^{-1}-1)(\omega^2-1)}\,\omega^{2k-2+2pk-p}(\omega-1)(\omega^p-1)(\omega^p-\omega)(\omega^p-\omega^{-1}).\end{aligned}$$

Therefore, it holds that

$$\begin{vmatrix}\tilde{g}_{p-2} & \tilde{g}_p\\ g_{p-2} & \omega g_p\end{vmatrix} \neq 0 \qquad \text{for} \quad p=2, 3,\dots, n.$$

Hence, we have obtained (5.2) for case (I)

Case (II) $n=3$ and $q=3$; In this case, $pq-nr\neq n+q$ means $(p, r)=(3, 0)$, $(2,1)$. Since it holds from (2.13) and (4.1) that

$$C_1^k(b) = C_1^0(G^k(b)) \qquad (k=0, 1,\dots, n+q-1),$$

we put $k=0$ in (5.4) and (5.5). Then (5.4) and (5.5) become

$$\begin{aligned}&\frac{\partial}{\partial b_{p,r}}\mathrm{Wron}\,[y_0(0; b), y_2(0; b), y_3(0; b)]|_{b=0}\\ &= \begin{vmatrix}1 & \omega^{2(p+r)} & \omega^{3(p+r)}\\ 1 & \omega^{-2} & \omega^{-3}\\ 1 & \omega^{-4} & \omega^{-6}\end{vmatrix}\frac{\partial}{\partial b_{p,r}}\tilde{y}(0; b)|_{b=0}\,\tilde{y}'(0;0)\tilde{y}''(0; 0)\end{aligned}$$

$$+\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^{2(p+r-1)} & \omega^{3(p+r-1)} \\ 1 & \omega^{-4} & \omega^{-6} \end{vmatrix} \tilde{y}(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}'(0;b)|_{b=0}\,\tilde{y}''(0;0)$$

$$+\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{2(p+r-2)} & \omega^{3(p+r-2)} \end{vmatrix} \tilde{y}(0;0)\tilde{y}'(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}''(0;b)|_{b=0}$$

$$= (2\omega^{-2}-2)\frac{\partial}{\partial b_{p,r}}\tilde{y}(0;b)|_{b=0}\,\tilde{y}'(0;0)\tilde{y}''(0;0)$$

$$+ (2\omega^{2}-2\omega^{-2})\tilde{y}(0;0)\,\tilde{y}'(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}''(0;b)|_{a=0} = 0,$$

(5.18)

$$\frac{\partial}{\partial b_{p,r}}\text{Wron}\,[y_1(0;b),\,y_2(0;b),\,y_3(0;b)]|_{b=0}$$

$$= \begin{vmatrix} \omega^{(p+r)} & \omega^{2(p+r)} & \omega^{3(p+r)} \\ \omega^{-1} & \omega^{-2} & \omega^{-3} \\ \omega^{-2} & \omega^{-4} & \omega^{-6} \end{vmatrix} \frac{\partial}{\partial b_{p,r}}\tilde{y}(0;b)|_{b=0}\,\tilde{y}'(0;0)\tilde{y}''(0;0)$$

$$+\begin{vmatrix} 1 & 1 & 1 \\ \omega^{(p+r-1)} & \omega^{2(p+r-1)} & \omega^{3(p+r-1)} \\ \omega^{-2} & \omega^{-4} & \omega^{-6} \end{vmatrix} \tilde{y}(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}'(0;b)|_{b=0}\,\tilde{y}''(0;0)$$

$$+\begin{vmatrix} 1 & 1 & 1 \\ \omega^{-1} & \omega^{-2} & \omega^{-3} \\ \omega^{(p+r-2)} & \omega^{2(p+r-2)} & \omega^{3(p+r-2)} \end{vmatrix} \tilde{y}(0;0)\tilde{y}'(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}''(0;b)|_{a=0}$$

$$= (-\omega^{-2}-\omega^{-1})\frac{\partial}{\partial b_{p,r}}\tilde{y}(0;b)|_{b=0}\,\tilde{y}'(0;0)\tilde{y}''(0;0)$$

$$+ (1-\omega)\tilde{y}(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}'(0;b)|_{b=0}\,\tilde{y}''(0;0)$$

$$+ (\omega^{2}-\omega^{-2})\tilde{y}(0;0)\tilde{y}'(0;0)\frac{\partial}{\partial b_{p,r}}\tilde{y}''(0;b)|_{b=0} = 0,$$

for $(p, r)=(3,0), (2,1)$.

Noting that $\tilde{y}^{(m)}(0;0)$ and $\frac{\partial}{\partial b_{p,r}}\tilde{y}^{(m)}(0;b)|_{b=0}$ $(m=0, 1, 2)$ have real values, we can easily obtain from (5.18)

$$\frac{\partial}{\partial b_{p,r}}\tilde{y}(0;b)|_{b=0} = \frac{\partial}{\partial b_{p,r}}\tilde{y}'(0;b)|_{b=0} = \frac{\partial}{\partial b_{p,r}}\tilde{y}''(0;b)|_{b=0} = 0.$$

Case (III) $n=3$ and $q=4$; In this case, $pq-nr\neq n+q$ means that $(p, r)=(2, 0), (3, 0), (2, 1), (3, 1), (2, 2), (3, 2)$. Then, by a similar calculation to case (II), we get

$$(5.19)\quad \begin{cases} (\omega^6-\omega^5-\omega^4+\omega^2+\omega-1)\tilde{y}_b\tilde{y}'\tilde{y}'' + (-\omega^6+3\omega^3-\omega^2-\omega)\tilde{y}\tilde{y}'_b\tilde{y}''=0, \\ (\omega^6-\omega^4-2\omega^2+2\omega)\tilde{y}_b\tilde{y}'\tilde{y}''+(-\omega^6+\omega^4+\omega^3-\omega^2+\omega-1)\tilde{y}\tilde{y}'_b\tilde{y}''=0 \end{cases}$$

$$\text{for}\quad (p, r)=(2, 0),$$

$$(5.20)\quad \begin{cases} (\omega^6+\omega^5+\omega^3-3)\tilde{y}_b\tilde{y}'\tilde{y}'' + (\omega^6+\omega^5-\omega^4+\omega^3-\omega^2-\omega)\tilde{y}\tilde{y}'_b\tilde{y}'' + \\ \qquad + (\omega^6-\omega^5+\omega^4-\omega^3+\omega^2+\omega)\tilde{y}\tilde{y}'\tilde{y}''_b = 0, \\ -(\omega^6+\omega^5-\omega^4+\omega^3-\omega^2-\omega)\tilde{y}_b\tilde{y}'\tilde{y}''+(2\omega^5+\omega^4-\omega^3-2\omega^2)\tilde{y}\tilde{y}'_b\tilde{y}''+ \\ \qquad + (-2\omega^6+\omega^5-\omega^2+2\omega)\tilde{y}\tilde{y}'\tilde{y}''_b = 0 \end{cases}$$

$$\text{for}\quad (p, r)=(3, 0) \text{ and } (2, 1),$$

$$(5.21)\quad \begin{cases} (\omega^5-\omega^3-\omega^2+1)\tilde{y}_b\tilde{y}'\tilde{y}'' + (-\omega^6-\omega^5+\omega^3+\omega^2-\omega+1)\tilde{y}\tilde{y}'_b\tilde{y}'' + \\ \qquad + (-\omega^6+3\omega^4-\omega^5-\omega)\tilde{y}\tilde{y}'\tilde{y}''_b = 0, \\ (-\omega^4+2\omega^3-2\omega+1)\tilde{y}_b\tilde{y}'\tilde{y}'' + (-\omega^6-\omega^5+2)\tilde{y}\tilde{y}'_b\tilde{y}'' + \\ \qquad + (\omega^6-\omega^5+\omega^4+\omega^3-\omega-1)\tilde{y}\tilde{y}'\tilde{y}''_b = 0 \end{cases}$$

$$\text{for}\quad (p, r)=(3, 1) \text{ and } (2, 2),$$

$$(5.22)\quad \begin{cases} (\omega^5+\omega^3+\omega^2-3\omega)\tilde{y}\tilde{y}'_b\tilde{y}'' + (\omega^6-\omega^5+\omega^4-\omega^3-\omega^2+1)\tilde{y}\tilde{y}'\tilde{y}''_b = 0, \\ (-\omega^6-\omega^5+\omega^3+\omega^2-\omega+1)\tilde{y}\tilde{y}'_b\tilde{y}'' + (\omega^5-\omega^3-2\omega+2)\tilde{y}\tilde{y}'\tilde{y}''_b = 0 \end{cases}$$

$$\text{for}\quad (p, r)=(3, 2).$$

Similarly, noting that $\tilde{y}^{(m)}=\tilde{y}^{(m)}(0; 0)$ and $\tilde{y}_b^{(m)}=\dfrac{\partial}{\partial b_{p,r}}\tilde{y}^{(m)}(0; b)\,|_{b=0}$ $(m=0, 1, 2)$ have real values, we can obtain

$$\begin{cases} \tilde{y}_b(0; 0) = \tilde{y}'_b(0; 0) = 0 & \text{for}\quad (p, r) = (2,0), \\ \tilde{y}_b(0; 0) = \tilde{y}'_b(0; 0) = \tilde{y}''_b(0; 0) = 0 & \text{for}\quad (p, r) = (3, 0), (2, 1), (3, 1), (2, 2), \\ \tilde{y}'_b(0; 0) = \tilde{y}''_b(0; 0) = 0 & \text{for}\quad (p, r) = (3, 2). \end{cases}$$

Hence we have proved (5.2) for case (III).

For other cases, by a quite similar manner, we can prove (5.2). Thus we have finished the proof of Lemma 5.2.

PROOF of LEMMA 5.1. *Case (I) and $2p>n+2$;* If we assume (5.1), then, using Lemmas 4.3 and 5.2, we can get

$$\text{(5.23)} \qquad \mathrm{Res}_{s=1-p} W_{p,0}(s) = \mathrm{Res}_{s=2-p} W_{p,0}(s) = 0.$$

Furthermore, in this case, (4.30) becomes

$$\text{(5.24)} \qquad W_{p,0}(s) = (-1)^{p+1}\left[\frac{W_0(s+p)}{(s+n-p)(s+n-p+1)\cdots(s+n-1)} + W_0(s)K_{p,0}(s)\right].$$

Since

$$\begin{cases} \left.\dfrac{s+p-1}{(s+n-p)(s+n-p+1)\cdots(s+n-1)}\right|_{s=1-p} = \dfrac{(-1)^{-n-1}}{(n-p)!(2p-n-1)!}, \\ \left.\dfrac{s+p-2}{(s+n-p)(s+n-p+1)\cdots(s+n-1)}\right|_{s=2-p} = \dfrac{(-1)^{-n-2}}{(n-p+1)!(2p-n-2)!}, \end{cases}$$

it holds from (5.23) that

$$\text{(5.25)} \qquad \begin{cases} (-1)^{p+1}\left[\dfrac{(-1)^{-n-1}W_0(1)}{(n-p)!(2p-n-1)!} + \mathrm{Res}_{s=1-p} W_0(s)K_{p,0}(1-p)\right] = 0, \\ (-1)^{p+1}\left[\dfrac{(-1)^{-n-2}W_0(2)}{(n-p+1)!(2p-n-2)!} + \mathrm{Res}_{s=2-p} W_0(s)K_{p,0}(2-p)\right] = 0. \end{cases}$$

Therefore, (5.25) means that

$$\text{(5.26)} \qquad \frac{\mathrm{Res}_{s=1-p} W_0(s)K_p(2-p)}{\mathrm{Res}_{s=1-p} W_0(s)K_p(1-p)} = \frac{(n-p)!(2p-n-1)!(-1)^{-n-2}W_0(2)}{(-1)^{-n-1}W_0(1)(n-p+1)!(2p-n-2)!}.$$

Since, using the difference equation (4.16), we have

$$\begin{aligned} \mathrm{Res}_{s=-k} W_0(s) &= \lim_{s=-k}\left[\frac{(s+k)\,W_0(s+n+2)}{s(s+1)\cdots(s+k)\cdots(s+n-1)}\right] \\ &= \frac{W_0(-k+n+2)}{(-1)^k k!(n-1-k)!} \qquad (k=0,\ 1,\dots,\ n-1), \end{aligned}$$

we can see from (5.26) that

$$\text{(5.27)} \qquad \frac{(p-1)\,W_0(2-p+n+2)K_{p,0}(2-p)}{(1-p+n)\,W_0(1-p+n+2)K_{p,0}(1-p)} = \frac{(2p-n-1)\,W_0(2)}{(n-p+1)\,W_0(1)}$$

In this case, we have

$$D^*(n,\ 2,\ p,\ 0;\ s^*,\ 1) = 2p(s^*+p+2)$$

and then, using Lemma 4.10,

$$K_{p,0}(1-p) > K_{p,0}(2-p).$$

Therefore, it must hold from (5.27) that

$$\frac{(p-1)\,W_0(2-p+n+2)\,W_0(1)}{(2p-n-1)\,W_0(1-p+n+2)\,W_0(2)} < 1,$$

that is,

$$\frac{(p-1)(n+1)\,W_0(2-p+n+2)\,W_0(n+3)}{(2p-n-1)\,W_0(1-p+n+2)\,W_0(n+4)} < 1. \tag{5.28}$$

Applying (4.47) to (5.28) [put $m=1-p+n+2$], we can get

$$\frac{(p-1)(n+1)(2n+3-2p)}{(2p-n-1)(2n+3-p)} < 1. \tag{5.29}$$

Noting the condition $2p>n+2$, we then have from (5.29)

$$-\,np^2+(n^2+n-1)p<0,$$

that is,

$$p>n+1-1/n.$$

This is a contradiction. Thus we have proved Lemma 5.1 for the case in which $2p>n+2$.

Case (I) and $2p<n+2$; If we assume (5.1), then we can obtain from Lemma 4.3 and Lemma 5.2

$$\mathrm{Res}_{s=1-p}\,W_0(s)=0 \quad\text{and}\quad \mathrm{Res}_{s=2-p}\,W_0(s)=0. \tag{5.23}$$

Furthermore, noting $2p<n+2$, we can easily derive

$$s(s+1)\cdots(s+n-p-1)|_{s=1-p}=0 \quad\text{and}\quad s(s+1)\cdots(s+n-p-1)|_{s=2-p}=0.$$

Using (5.23) and (5.24), if we put $s=1-p$ and $s=2-p$ in the difference equation (4.17), we get

$$W_{p,0}(n+3-p)=0 \quad\text{and}\quad W_{p,0}(n+4-p)=0. \tag{5.25}$$

Therefore, it follows from (4.32) that

$$\frac{W_0(n+3)}{(2n+3-2p)\cdots(2n+2-p)\,W_0(n+3-p)}=(-1)^p c-K'_{p,0}(n+3-p), \tag{5.26}$$

$$\frac{W_0(n+4)}{(2n+4-2p)\cdots(2n+3-p)\,W_0(n+4-p)}=(-1)^p c-K'_{p,0}(n+4-p). \tag{5.27}$$

Since

$$D^*(n,\,2,\,p,\,0;\,n+3-p,\,1)=6p>0,$$

it holds from Lemma 4.10 that

$$K'_{p,0}(n+3-p) > K'_{p,0}(n+4-p),$$

that is,

$$(-1)^p c - K'_{p,0}(n+3-p) < (-1)^p c - K'_{p,0}(n+4-p).$$

Therefore, we can easily obtain from (5.26) and (5.27)

$$\text{(5.28)} \qquad \frac{W_0(n+3)\,W_0(n+4-p)}{W_0(n+4)\,W_0(n+3-p)} > \frac{2n+3-2p}{2n+3-p}.$$

On the other hand, applying (4.47) to the left hand member of (5.28), we get [put $m=n+3-p$.]

$$\frac{W_0(n+3)\,W_0(n+4-p)}{W_0(n+4)\,W_0(n+3-p)} > \frac{2n+3-2p}{2n+3-p}.$$

This is a contradiction. Hence we have proved Lemma 5.1 for case (I).

Case (*II*) *and* $pq-nr>n+q$ *i.e.*, $(p, r)=(3, 0)$; Assuming (5.1), we get from Lemma 4.3 and Lemma 5.2

$$\text{(5.29)} \qquad \mathrm{Res}_{s=0}\, W_{3,0}(s) = \mathrm{Res}_{s=-1}\, W_{3,0}(s) = \mathrm{Res}_{s=-2}\, W_{3,0}(s) = 0.$$

Then, using (4.30) [put $n=3$, $q=3$, $p=3$, $r=0$.], we have from (5.29)

$$\left.\frac{s\,W_0(s+3)}{s(s+1)(s+2)}\right|_{s=0} + \mathrm{Res}_{s=0}\, W_0(s) \times K_{3,0}(0) = 0,$$

$$\left.\frac{(s+1)\,W_0(s+3)}{s(s+1)(s+2)}\right|_{s=-1} + \mathrm{Res}_{s=-1}\, W_0(s) \times K_{3,0}(-1) = 0,$$

and

$$\left.\frac{(s+2)\,W_0(s+3)}{s(s+1)(s+2)}\right|_{s=-2} + \mathrm{Res}_{s=-2}\, W_0(s) \times K_{3,0}(-2) = 0.$$

Since it holds from the difference equation (4.16) that

$$W_0(-m+n+q) = (-1)^m m!(n-m-1)!\mathrm{Res}_{s=-m}\, W_0(s) \qquad (m=0, 1,\dots, n-1),$$

it follows from these conditions that

$$K_{3,0}(0),\ K_{3,0}(-1),\ K_{3,0}(-2) < 0.$$

This is a contradiction. (Lemma 4.11.)

Case (*II*) *and* $pq-nr<n+q$ *i.e.*, $(p, r)=(2, 1)$; In this case, assuming (5.1), we can similarly obtain

(5.30) $\mathrm{Res}_{s=0} W_{2,1}(s) = \mathrm{Res}_{s=-1} W_{2,1}(s) = \mathrm{Res}_{s=-2} W_{2,1}(s) = 0.$

Using (4.32), we get

$$\mathrm{Res}_{s=0} W_0(s)[c - K'_{2,1}(0)] - \frac{s \cdot W_0(s+3)}{(s+1)(s+2)}\Big|_{s=0} = 0$$

and

$$\mathrm{Res}_{s=-1} W_0(s)[c - K'_{2,1}(-1)] - \frac{(s+1)\, W_0(s+3)}{(s+1)(s+2)}\Big|_{s=-1} = 0.$$

Therefore, using the difference equation (4.16), we have

(5.31) $c = K'_{2,1}(0)$ and $c - K'_{2,1}(-1) = \dfrac{W_0(2)}{\mathrm{Res}_{s=-1} W_0(s)} = -\dfrac{W_0(2)}{W_0(5)}.$

On the other hand, it follows from the condition

$$D^*(3, 3, 2, 1;\ -1, 1) = 0$$

that

(5.32) $$K'_{2,1}(0) > K'_{2,1}(5).$$

Noting that

(5.33) $K'_{2,1}(-1) = \dfrac{W_0(8)}{6\cdot 7\, W_0(5)} + K'_{2,1}(-1+6) = \dfrac{4\, W_0(2)}{7\, W_0(5)} + K'_{2,1}(5),$

we can obtain from (5.31), (5.32) and (5.33)

$$\frac{4\, W_0(2)}{7\, W_0(5)} > \frac{4\, W_0(2)}{7\, W_0(5)} + K'_{2,1}(5) - K'_{2,1}(0) = K'_{2,1}(-1) - K'_{2,1}(0) = \frac{W_0(2)}{W_0(5)}.$$

Since it holds from (4.24) that

$$W_0(s) > 0 \qquad \text{for} \quad s > 0,$$

this is a contradiction. Hence we have obtained Lemma 5.1 for case (II).

Case (III) and $pq - nr > n + q$; (i) $(p, r) = (2, 0)$; Assuming (5.1), we get

(5.34) $$\mathrm{Res}_{s=0} W_{2,0}(s) = 0, \quad \mathrm{Res}_{s=-1} W_{2,0}(s) = 0.$$

It follows from (4.30) that

(5.35) $$\frac{s \cdot W_0(s+2)}{(s+1)(s+2)}\Big|_{s=0} + \mathrm{Res}\, W_0(s) \times K_{2,0}(0) = 0.$$

The condition (5.35) contradicts the fact that

$$\mathrm{Res}_{s=0}\, W_0(s) > 0 \quad \text{and} \quad K_{2,0}(0) > 0.$$

(*ii*) $(p, r)=(3, 0)$; In this case, we can similarly obtain from (5.1)

$$\mathrm{Res}_{s=0}\, W_{3,0}(s) = \mathrm{Res}_{s=-1}\, W_{3,0}(s) = \mathrm{Res}_{s=-2}\, W_{3,0}(s) = 0.$$

It follows from (4.30) that

$$\begin{cases} \left.\dfrac{s\cdot W_0(s+3)}{s(s+1)(s+2)}\right|_{s=0} + \mathrm{Res}_{s=0} W_0(s) \times K_{3,0}(0) = 0, \\ \left.\dfrac{(s+1)\, W_0(s+3)}{s(s+1)(s+2)}\right|_{s=-1} + \mathrm{Res}_{s=1} W_0(s) \times K_{3,0}(-1) = 0, \\ \left.\dfrac{(s+2)\, W_0(s+3)}{s(s+1)(s+2)}\right|_{s=-2} + \mathrm{Res}_{s=-2} W_0(s) \times K_{3,0}(-2) = 0. \end{cases}$$

From (4.16), these conditions mean that

$$\begin{cases} W_0(3) + W_0(7)K_{3,0}(0) = 0, \\ W_0(2) + W_0(6)K_{3,0}(-1) = 0, \\ W_0(1) + W_0(5)K_{3,0}(-2) = 0. \end{cases}$$

Using the condition $D^*(3, 4, 3, 0; -2, 1)=60>0$, we can get

$$-W_0(2)/W_0(6) = K_{3,0}(-1) > K_{3,0}(0) = -W_0(3)/W_0(7).$$

Therefore, it must holds that

$$\frac{W_0(2)W_0(7)}{W_0(3)W_0(6)} < 1,$$

which contradicts Lemma 4.13.

(*iii*) $(p, r)=(3, 1)$; Assuming (5.1), we get

$$\mathrm{Res}_{s=0}\, W_{3,1}(s) = \mathrm{Res}_{s=-1}\, W_{3,1}(s) = \mathrm{Res}_{s=-2}\, W_{3,1}(s) = 0.$$

Therefore, using (4.30) and (4.16), we get

$$W_0(4) + W_0(7)K_{3,1}(0) = 0, \quad W_0(3) + W_0(6)K_{3,1}(-1) = 0.$$

It follows from the condition $D^*(3, 4, 3, 1; -1, 1)=6>0$ that

$$-W_0(4)/W_0(7) = K_{3,1}(0) < K_{3,1}(-1) = -W_0(3)/W_0(6),$$

that is,

$$\frac{W_0(4)W_0(6)}{W_0(7)W_0(3)} > 1.$$

This a contradiction. Thus we have proved Lemma 5.1 for $pq-nr>n+q$.

Case (III) and $pq-nr<n+q$; (*i*) $(p, r)=(2, 1)$; If we assume (5.1) in this case, we can similarly obtain

$$\mathrm{Res}_{s=0} W_{2,1}(s) = \mathrm{Res}_{s=-1} W_{2,1}(s) = \mathrm{Res}_{s=-2} W_{2,1}(s) = 0.$$

Putting $s=0, -1, -2$, in the difference equation (4.17), we get

$$W_{2,1}(7) = W_{2,1}(6) + W_0(2) = W_{2,1}(5) + 2W_0(1) = 0. \tag{5.36}$$

Furthermore, using (4.32), we get

$$\begin{aligned} &\mathrm{Res}_{s=0} W_0(s)[c-K'_{2,1}(0)] - \frac{s\cdot W_0(s+3)}{(s+1)(s+2)}\Big|_{s=0} = 0, \\ &\mathrm{Res}_{s=-1} W_0(s)[c-K'_{2,1}(-1)] - \frac{(s+1)\,W_0(s+3)}{(s+1)(s+2)}\Big|_{s=-1} = 0, \\ &\mathrm{Res}_{s=-2} W_0(s)[c-K'_{2,1}(-2)] - \frac{(s+2)\,W_0(s+3)}{(s+1)(s+2)}\Big|_{s=-2} = 0. \end{aligned} \tag{5.37}$$

Since

$$W_{2,1}(6) = W_0(6)[c-K'_{2,1}(6)] - W_0(9)/7\cdot 8,$$

we can obtain from (5.36) and (5.37)

$$\begin{aligned} W_0(6)[K'_{2,1}(0)-K'_{2,1}(6)] &= W_0(6)[c-K'_{2,1}(6)] \\ &= W_0(9)/7\cdot 8 - W_{2,1}(6) = W_0(9)/7\cdot 8 - W_0(2) = -4W_0(2)/7 < 0. \end{aligned}$$

Here we used the difference equation (4.16). It follows that

$$K'_{2,1}(0) < K'_{2,1}(6). \tag{5.38}$$

On the other hand, the condition $D^*(3, 4, 2, 1; -2, 1)=7>0$ means that

$$K'_{2,1}(0) > K'_{2,1}(6).$$

This fact contradicts (5.38).

(*ii*) $(p, r)=(2, 2)$; Assuming (5.1), we get

$$\mathrm{Res}_{s=0} W_{2,2}(s) = \mathrm{Res}_{s=-1} W_{2,2}(s) = \mathrm{Res}_{s=-2} W_{2,2}(s) = 0.$$

Putting $s=0, -1, -2$ in (4.17), we get

$$W_{2,2}(7) = W_{2,2}(6) + W_0(3) = W_{2,2}(5) + 2W_0(2) = 0.$$

Since

$$\begin{cases} W_{2,2}(7) = W_0(7)[c-K'_{2,2}(7)] - W_0(11)/8\times 9, \\ W_{2,2}(6) = W_0(6)[c-K'_{2,2}(6)] - W_0(10)/7\times 8, \end{cases}$$

we get

$$c - K'_{2,2}(7) = W_0(11)/72W_0(7), \quad c - K'_{2,2}(6) = [-W_0(3)+W_0(10)/56]/W_0(6).$$

On the other hand, the condition $D^*(3, 4, 2, 2; 0, \underline{2})=0$ means that

$$c - \left[k'_{2,2}(7) - \frac{W_0(18)}{15\times 16\times W_0(14)}\right] > c - \left[k'_{2,2}(6) - \frac{W_0(17)}{14\times 15\times W_0(13)}\right].$$

Therefore, we get

$$\frac{W_0(11)}{72\,W_0(7)} + \frac{W_0(18)}{240\,W_0(14)} > \frac{W_0(3)}{W_0(6)} + \frac{W_0(10)}{56\,W_0(6)} + \frac{W_0(17)}{210\,W_0(13)},$$

thatis,

$$\text{(5.39)} \qquad \frac{W_0(18)W_0(13)}{W_0(14)W_0(17)} > \frac{3443\cdot 41184}{76230\cdot 1132} = 1.643\dots.$$

Here we used the difference equation (4.16). It holds from (4.47) that

$$\frac{W_0(18)W_0(13)}{W_0(17)W_0(14)} < \frac{16}{14} = 1.42\dots.$$

Therefore, (5.39) contradicts this fact.

(*iii*) $(p, r)=(3, 2)$; Assuming (5.1), we can similarly obtain

$$\mathrm{Res}_{s=-1} W_{3,2}(s) = 0 \quad \text{and} \quad \mathrm{Res}_{s=-2} W_{3,2}(s) = 0.$$

It follows from (4.32) that

$$\begin{cases} \mathrm{Res}_{s=-1} W_0(s)[c-(-1)^3K'_{3,2}(-1)] - (-1)^3(s+1)W_0(s+5)|_{s=-1} = 0, \\ \mathrm{Res}_{s=-2} W_0(s)[c-(-1)^3K'_{3,2}(-2)] - (-1)^3(s+2)W_0(s+5)|_{s=-2} = 0, \end{cases}$$

that is,

$$\text{(5.40)} \qquad K'_{3,2}(-1) = K'_{3,2}(-2).$$

On the other hand, the condition $D^*(3, 4, 3, 2; -2, 1)=30>0$ means

$$K'_{3,2}(-1) < K'_{3,2}(-2),$$

which contradicts (5.40). Thus we have proved Lemma 5.1 for case (III).

For other cases [cases (IV), (V), (VI)], by a quite similar manner to the proof of cases (II) and (III), we can prove Lemma 5.1.

§ 6. Difference equations and Stokes multipliers (III)

In the case $q \neq 2$, in order to prove the uniform simplification theorem, we need the following

LEMMA 6.1.

$$(6.1) \qquad \det \begin{vmatrix} \dfrac{\partial C_1^0(b)}{\partial b_{p,r}} & \dfrac{\partial C_1^0(b)}{\partial b_{p-1,r+1}} \\ \dfrac{\partial C_2^0(b)}{\partial b_{p,r}} & \dfrac{\partial C_2^0(b)}{\partial b_{p-1,r+1}} \end{vmatrix}_{b=0} \neq 0$$

for $p=3, 4, \ldots, n$; $r=0, 1, \ldots, n-1$.

PROOF. *Case* (*II*) *i.e.*, $(p, r)=(3, 0)$ *and* $(p-1, r+1)=(2, 1)$; We assume that (6.1) does not hold. Then, from Lemma 5.1, there exists a constant L ($\neq 0$) such that

$$(6.2) \qquad L\frac{\partial C_1^0(b)}{\partial b_{3,0}}\Big|_{b=0} + \frac{\partial C_1^0(b)}{\partial b_{2,1}}\Big|_{b=0} = 0, \quad L\frac{\partial C_2^0(b)}{\partial b_{3,0}}\Big|_{b=0} + \frac{\partial C_2^0(b)}{\partial b_{2,1}}\Big|_{b=0} = 0.$$

Noting (5.3)–(5.6) and

$$C_2^0(b) = \frac{\mathrm{Wron}\,[y_1(x; b), y_0(x; b), y_3(x; b)]}{\mathrm{Wron}\,[y_1(x; b), y_2(x; b), y_3(x; b)]},$$

we can get from (6.2)

$$\begin{cases} (2\omega^{-2}-2)\,[Ly_{b_{3,0}}(0; 0) - y_{b_{2,1}}(0; 0)]y'(0; 0)y''(0; 0) \\ \quad + (2\omega^2-2\omega^{-2})y(0; 0)\,[Ly''_{b_{3,0}}(0; 0) - y''_{b_{2,1}}(0; 0)]y'(0; 0) = 0, \\ (-\omega^{-2}-\omega^{-1})\,[Ly_{b_{3,0}}(0; 0) - y_{b_{2,1}}(0; 0)]y'(0; 0)y''(0; 0) \\ \quad + (1-\omega)y(0; 0)\,[Ly'_{b_{3,0}}(0; 0) - y'_{b_{2,1}}(0; 0)]y''(0; 0) \\ \quad + (\omega^2-\omega^{-2})y(0; 0)y'(0; 0)\,[Ly''_{b_{3,0}}(0; 0) - y''_{b_{2,1}}(0; 0)] = 0, \\ (-2)-2\omega^{-1})\,[Ly_{b_{3,0}}(0; 0) - y_{b_{2,1}}(0; 0)]y'(0; 0)y''(0; 0) \\ \quad + (2\omega^{-1}-2\omega)y(0; 0)y'(0; 0)\,[Ly''_{b_{3,0}}(0; 0) - y''_{b_{2,1}}(0; 0)] = 0, \end{cases}$$

where

$$\omega = \exp\,[2\pi i/(n+q)] = \exp\,[\pi i/3].$$

Since

$$\begin{vmatrix} 2\omega^{-2}-2 & 0 & 2\omega^2-2\omega^{-2} \\ -\omega^{-2}-\omega^{-1} & 1-\omega & \omega^2-\omega^{-2} \\ -2-2\omega^{-1} & 0 & 2\omega^{-1}-2\omega \end{vmatrix} \neq 0,$$

we can obtain

$$\text{(6.3)} \qquad L\tilde{y}_{b_{3,0}}(0;0) - \tilde{y}_{b_{2,1}}(0;0) = L\tilde{y}'_{b_{3,0}}(0;0) - \tilde{y}'_{b_{2,1}}(0;0) = L\tilde{y}''_{b_{3,0}}(0;0) - \tilde{y}''_{b_{2,1}}(0;0) = 0,$$

that is,

$$\text{(6.4)} \qquad \mathrm{Res}_{s=0}\, W^*_{3,0}(s) = \mathrm{Res}_{s=-1}\, W^*_{3,0}(s) = \mathrm{Res}_{s=-2}\, W^*_{3,0}(s) = 0.$$

Furtheremore, noting that $\tilde{y}_b^{(m)}(0;0)$ $(m=0, 1, 2)$ have real values, we see from (6.3) that the constant L must be a real number.

Next we shall show that the constant L is a positive real number. In fact, it holds from (6.3) that

$$L \times \mathrm{Res}_{s=0}\, W_{3,0}(s) = \mathrm{Res}_{s=0}\, W_{2,1}(s), \quad L \times \mathrm{Res}_{s=-1}\, W_{3,0}(s) = \mathrm{Res}_{s=-1}\, W_{2,1}(s),$$

that is,

(6.5)

$$\begin{cases} L[W_0(3)/2 + \mathrm{Res}_{s=0}\, W_0(s)K_{3,0}(0)] = \mathrm{Res}_{s=0}\, W_0(s)\,[c - K'_{2,1}(0)], \\ L[\mathrm{Res}_{s=-1}\, W_0(s)K_{3,0}(-1) - W_0(2)] = \mathrm{Res}_{s=-1}\, W_0(s)\,[c - K'_{2,1}(-1)] - W_0(2), \end{cases}$$

where we used (4.30) and (4.32). Using the difference equation (4.16), we have from (6.5)

$$\text{(6.6)} \qquad \begin{aligned} &L[K_{3,0}(-1) - K_{3,0}(0) + W_0(2)/W_0(5) - W_0(3)/W_0(6)] \\ &\quad = K'_{2,1}(0) - K'_{2,1}(-1) + W_0(2)/W_0(5) \\ &\quad = K'_{2,1}(0) - K'_{2,1}(5) + 3W_0(2)/7W_0(5). \end{aligned}$$

Since $D^*(3, 3, 3, 0; -1, 1) = 45 > 0$ and $D^*(3, 3, 2, 1; 0; 1) = 3 > 0$, we get from Lemma 4.13

$$K_{3,0}(-1) - K_{3,0}(0) + W_0(2)/W_0(5) - W_0(3)/W_0(6) > 0$$

and

$$K'_{2,1}(0) - K'_{2,1}(5) + 3W_0(2)/7W_0(5) > 0.$$

Therefore, we see from (6.6) that the constant L is a positive real number in this case.

Since the constant L is positive, using the difference equation

$$\Theta^*(s) = \Theta^*(s+1) - \Lambda^*_{p,r}(s),$$

we get

$$\text{(6.7)} \qquad \begin{aligned} \frac{W^*_{3,0}(5)}{W_0(5)} - \frac{W^*_{3,0}(6)}{W_0(6)} &= \frac{W^*_{3,0}(11)}{W_0(11)} - \frac{W^*_{3,0}(12)}{W_0(12)} + \Lambda^*_{3,0}(5) - \Lambda^*_{3,0}(6) \\ &> \Lambda^*_{3,0}(5) - \Lambda^*_{3,0}(6) + \Lambda^*_{3,0}(11) - \Lambda^*_{3,0}(12), \end{aligned}$$

$$(6.8)\qquad \frac{W^*_{3,0}(4)}{W_0(4)} - \frac{W^*_{3,0}(5)}{W_0(5)} = \frac{W^*_{3,0}(10)}{W_0(10)} - \frac{W^*_{3,0}(11)}{W_0(11)} + \Lambda^*_{3,0}(4) - \Lambda^*_{3,0}(5)$$
$$< \Lambda^*_{3,0}(4) - \Lambda^*_{3,0}(5) + \Lambda^*_{3,0}(10).$$

Furthermore, it follows from (4.40) and (6.4) that

$$(6.9)\qquad W^*_{3,0}(6) + LW_0(3) = 0, \quad W^*_{3,0}(5) + (L-1)W_0(2) = 0,$$
$$W^*_{3,0}(4) + (L-2)W_0(1) = 0.$$

Using (6.9), (4.16) and (4.41), we get from (6.7)

$$(1-L)\frac{W_0(2)}{W_0(5)} + L\frac{W_0(3)}{W_0(6)} > \frac{2\times3\times4}{4\times5\times6\times7}(4+L)\frac{W_0(2)}{W_0(5)}$$
$$+ \frac{8\times9\times10\times2\times3\times4}{5\times6\times7\times10\times11\times12\times13}(10+L)\frac{W_0(2)}{W_0(5)} - \frac{3\times4\times5}{5\times6\times7\times8}$$
$$\times (5+L)\frac{W_0(3)}{W_0(6)} - \frac{9\times10\times11\times3\times4\times5}{11\times12\times13\times14\times6\times7\times8}(11+L)\frac{W_0(3)}{W_0(6)},$$

that is,

$$(6.10)\qquad 1 - \frac{4}{35} - \frac{48}{1001} + \frac{5}{28}W + \frac{825}{20384}W$$
$$> L\left[1 - W + \frac{1}{35} + \frac{24}{5005} - \frac{1}{28}W - \frac{75}{20384}W\right],$$

where we put

$$W = W_0(3)W_0(5)/W_0(6)W_0(2).$$

Since

$$W = 16W_0(9)W_0(11)/25W_0(12)W_0(8) = 1792W_0(15)W_0(17)/3025W_0(18)W_0(14),$$

it follows from (4.47) that

$$(6.11)\qquad 1792 \times 15/3025 \times 17 = 0.5227\ldots < W < 1792/3025 = 0.59239\ldots.$$

Therefore, it must hold from (6.10) that

$$(6.12)\qquad L < 2.33\ldots.$$

Similarly, we get from (6.8)

$$(2-L)\frac{W_0(1)}{W_0(4)} - (1-L)\frac{W_0(2)}{W_0(5)} < \frac{2(3+L)\,W_0(1)}{4\times5\times6\times W_0(4)}$$
$$- \frac{2\times3\times4\times(4+L)\,W_0(2)}{4\times5\times6\times7\times W_0(5)} - \frac{7\times8\times9\times1\times2\times3\times(9+L)\,W_0(1)}{9\times10\times11\times12\times4\times5\times6\times W_0(4)}.$$

Putting

$$W' = W_0(2)W_0(4)/W_0(5)W_0(1),$$

we get

(6.13) $2 - W' - 1/20 - 21/1100 + 4/35 < L[1 - W' + 1/60 + 7/3300 - 1/35W']$.

Since

$$W' = \frac{W_0(2)W_0(4)}{W_0(5)W_0(1)} = \frac{7W_0(8)W_0(10)}{16W_0(11)W_0(7)},$$

it follows from (4.47) that

(6.14) $$0.35 = 7/20 < W' < 7/16 = 0.4375.$$

Therefore, it must hold from (6.12) that

$$L > 2.44\ldots.$$

This contradicts (6.12). Thus we have proved Lemma 6.1 for case (II).

Case (III) and $(p, r) = (3, 0)$, $(p-1, r+1) = (2, 1)$; If we assume that (6.1) does not hold, then there exists a constant $L(\neq 0)$ such that

(6.15) $$L\frac{\partial C_1^0(b)}{\partial b_{3,0}}\Big|_{b=0} + \frac{\partial C_1^0(b)}{\partial b_{2,1}}\Big|_{b=0} = 0, \quad L\frac{\partial C_2^0(b)}{\partial b_{3,0}}\Big|_{b=0} + \frac{\partial C_2^0(b)}{\partial b_{2,1}}\Big|_{b=0} = 0.$$

Similarly, we get

$$\begin{cases} (\omega^6+\omega^3+\omega^5-3)[Ly_{b_{3,0}}(0;0)-y_{b_{2,1}}(0;0)]y'(0;0)y''(0;0) \\ \quad +(\omega^5+\omega^6+\omega^3-\omega^4-\omega^2-\omega)y(0;0)[Ly'_{b_{3,0}}(0;0)-y'_{b_{2,1}}(0;0)]y''(0;0) \\ \quad +(\omega+\omega^4+\omega^2-\omega^5-\omega^6-\omega^3)y(0;0)y'(0;0)[Ly''_{b_{3,0}}(0;0)-y''_{b_{2,1}}(0;0)]=0, \\ (\omega^4+\omega^2+\omega-3)[Ly_{b_{3,0}}(0;0)-y_{b_{2,1}}(0;0)]y'(0;0)y''(0;0) \\ \quad +(\omega+\omega^4+\omega^2-\omega^5-\omega^6-\omega^3)y(0;0)[Ly'_{b_{3,0}}(0;0)-y'_{b_{2,1}}(0;0)]y''(0;0) \\ \quad +(\omega^3+\omega^5+\omega^6-\omega-\omega^4-\omega^2)y(0;0)y'(0;0)[Ly''_{b,3}(0;0)-y''_{b_{2,1}}(0;0)]=0, \\ (\omega^2+\omega+1-\omega^5-\omega^3-\omega^2)[Ly_{b_{3,0}}(0;0)-y_{b_{2,1}}(0;0)]y'(0;0)y''(0;0) \\ \quad +(\omega^5+\omega^4+\omega^5-\omega^2-\omega^2-\omega^3)y(0;0)[Ly'_{b_{3,0}}(0;0)-y'_{b_{2,1}}(0;0)]y''(0;0) \\ \quad +(\omega+\omega^5+\omega-\omega^6-\omega^6-\omega^2)y(0;0)y'(0;0)[Ly''_{b_{3,0}}(0;0)-y''_{b,21}(0;0)]=0. \end{cases}$$

Then we can easily obtain

(6.16) $$Ly_{b_{3,0}}(0;0) = y_{b_{2,1}}(0;0), \quad Ly'_{b_{3,0}}(0;0) = y'_{b_{2,1}}(0;0),$$
$$Ly''_{b_{3,0}}(0;0) = y''_{b_{2,1}}(0;0),$$

that is,

$$\text{(6.17)}\qquad \operatorname{Res}_{s=0} W^*_{3,0}(s) = \operatorname{Res}_{s=-1} W^*_{3,0}(s) = \operatorname{Res}_{s=-2} W^*_{3,0}(s) = 0.$$

It follows from (6.16) that

$$L\left[\frac{s\,W_0(s+3)}{s(s+1)(s+2)}\Big|_{s=0} + \operatorname{Res}_{s=0} W_0(s)K_{3,0}(0)\right] = \operatorname{Res}_{s=0} W_0(s)(c-K'_{2,1}(0))$$

and

$$L\left[\frac{(s+1)\,W_0(s+3)}{s(s+1)(s+2)}\Big|_{s=-1} + \operatorname{Res}_{s=-1} W_0(s)K_{3,0}(-1)\right]$$
$$= \operatorname{Res}_{s=-1} W_0(s)(c-K'_{2,1}(-1)) - W_0(2).$$

Then, using the difference equation (4.16), we get

$$\text{(6.18)}\qquad L[W_0(2)/W_0(6)-W_0(2)/W_0(6)+K_{3,0}(0)-K_{3,0}(-1)]$$
$$= K'_{2,1}(-1) - K'_{2,1}(0) - W_0(2)/W_0(6)$$
$$= -\,4W_0(2)/7W_0(6) + K'_{2,1}(6) - K'_{2,1}(0).$$

Since $D^*(3, 4, 3, 0; -1, 1)>0$ and $D^*(3, 4, 2, 1; 0, 1)>0$, it holds that

$$[W_0(3)/W_0(7)] - [W_0(2)/W_0(6)] + K_{3,0}(0) - K_{3,0}(-1) < 0$$

and

$$[-4W_0(2)/7W_0(6)] + K'_{2,1}(6) - K'_{2,1}(0) < 0.$$

Therefore, the constant L must be positive.

From (4.40) and (6.17), it follows that

$$\text{(6.19)}\qquad W^*_{3,0}(7) + LW_0(3) = 0,\quad W^*_{3,0}(6) + (L-1)W_0(2) = 0,$$
$$W^*_{3,0}(5) + (L-2)W_0(1) = 0.$$

Since the constant L is positive, we can similarly obtain

$$\text{(6.20)}\quad -\frac{W^*_{3,0}(7)}{W_0(7)} + \frac{W^*_{3,0}(6)}{W_0(6)} > \frac{(5+L)\,W_0(9)}{5\times6\times7\times8\times W_0(6)} - \frac{(6+L)\,W_0(10)}{6\times7\times8\times9\times W_0(7)}$$
$$+\frac{(12+L)\,W_0(16)}{12\times13\times14\times15\times W_0(13)} - \frac{(13+L)\,W_0(17)}{13\times14\times15\times16\times W_0(14)}$$

and

$$\text{(6.21)}\qquad \frac{W^*_{3,0}(5)}{W_0(5)} - \frac{W^*_{3,0}(6)}{W_0(6)} < \frac{(4+L)\,W_0(8)}{4\times5\times6\times7\times W_0(5)}.$$

From (6.19) and (4.16), (6.20) and (6.21) become

$$(6.22)\quad L\frac{W_0(3)W_0(6)}{W_0(7)W_0(2)}+(1-L)>\frac{(5+L)2\times 3\times 4}{5\times 6\times 7\times 8}$$
$$-\frac{(4+L)1\times 2\times 3}{4\times 5\times 6\times 7}\frac{W_0(3)W_0(6)}{W_0(7)W_0(2)}+\frac{(12+L)9\times 10\times 11\times 2\times 3\times 4}{12\times 13\times 14\times 15\times 6\times 7\times 8}$$
$$-\frac{(13+L)10\times 11\times 12\times 3\times 4\times 5}{13\times 14\times 15\times 16\times 7\times 8\times 9}\frac{W_0(3)W_0(6)}{W_0(7)W_0(2)},$$

$$(6.23)\qquad (2-L)-(1-L)\frac{W_0(2)W_0(5)}{W_0(6)W_0(1)}<\frac{1\times 2\times 3\times(4+L)}{4\times 5\times 6\times 7}.$$

Since

$$\frac{W_0(3)W_0(6)}{W_0(7)W_0(2)}=\frac{3W_0(10)W_0(13)}{5W_0(14)W_0(9)}=\frac{36W_0(17)W_0(20)}{65W_0(21)W_0(16)}<\frac{36}{65},$$

it follows from (6.22) that

$$(6.24)\qquad L<2.08\ldots.$$

On the other hand, it holds that

$$\frac{W_0(2)W_0(5)}{W_0(6)W_0(1)}=\frac{2W_0(9)W_0(12)}{5W_0(13)W_0(8)}=\frac{4W_0(16)W_0(19)}{11W_0(20)W_0(15)}.$$

Therefore, we get from (4.47)

$$\frac{4}{11}>\frac{W_0(2)W_0(5)}{W_0(6)W_0(1)}>\frac{32}{99},$$

and then we get from (6.23)

$$(6.25)\qquad L>2.34\ldots.$$

This fact contradicts (6.24). Thus we proved (6.1) for this case.

Case (III) and $(p, r)=(3, 1)$, $(p-1, r+1)=(2, 2)$; If we assume that (6.1) does not hold in this case, there exists a constant L ($\neq 0$) such that

$$(6.26)\quad L\frac{\partial C_1^0(b)}{\partial b_{3,1}}\Big|_{b=0}+\frac{\partial C_1^0(b)}{\partial b_{2,2}}\Big|_{b=0}=0,\quad L\frac{\partial C_2^0(b)}{\partial b_{3,1}}\Big|_{b=0}+\frac{\partial C_2^0(b)}{\partial b_{2,2}}\Big|_{b=0}=0.$$

Then we get

$$\begin{cases}(\omega^6+\omega^5+\omega-\omega^3-1-\omega^2)[Ly_{b_{3,1}}(0;0)-y_{b_{2,2}}(0;0)]y'(0;0)y''(0;0)\\ \quad+(1+\omega^2+\omega^3-\omega^6-\omega^5-\omega)y(0;0)[Ly'_{b_{3,1}}(0;0)-y'_{b_{2,2}}(0;0)]y''(0;0)\\ \quad+(3\omega^4-\omega^5-\omega-\omega^6)y(0;0)y'(0;0)[Ly''_{b_{3,1}}(0;0)-y''_{b_{2,2}}(0;0)]=0,\\ (\omega^5+\omega^2+\omega^4-\omega^3-\omega-1)[Ly_{b_{3,1}}(0;0)-y_{b_{2,2}}(0;0)]y'(0;0)y''(0;0)\end{cases}$$

$$\begin{cases} \quad + (\omega+1+\omega^3-\omega^5-\omega^2-\omega^4)y(0;0)[Ly'_{b_{3,1}}(0;0)-y'_{b_{2,2}}(0;0)]y''(0;0) \\ \quad + (3\omega^6-\omega^2-\omega^4-\omega^5)y(0;0)y'(0;0)[Ly''_{b_{3,1}}(0;0)-y''_{b_{2,2}}(0;0)] = 0, \\ (3\omega^3-\omega-2\omega^4)[Ly_{b_{3,1}}(0;0)-y_{b_{2,2}}(0;0)]y'(0;0)y''(0;0) \\ \quad + (2+\omega^6-\omega^4-\omega^5-\omega^4)y(0;0)[Ly'_{b_{3,1}}(0;0)-y'_{b_{2,2}}(0;0)]y''(0;0) \\ \quad + (\omega^4+\omega^3+\omega^6-1-\omega-\omega^5)y(0;0)y'(0;0)[Ly''_{b_{3,1}}(0;0)-y''_{b_{2,2}}(0;0)] = 0. \end{cases}$$

It follows that

$$(6.27) \qquad Ly_{b_{3,1}}(0;0) = y_{b_{2,2}}(0;0), \quad Ly'_{b_{3,1}}(0;0) = y'_{b_{2,2}}(0;0)$$
$$Ly''_{b_{3,1}}(0;0) = y''_{b_{2,2}}(0;0),$$

that is,

$$(6.28) \quad \mathrm{Res}_{s=0}\, W^*_{3,1}(s) = 0, \quad \mathrm{Res}_{s=-1}\, W^*_{3,1}(s) = 0 \quad \text{and} \quad \mathrm{Res}_{s=-2}\, W^*_{3,1}(s) = 0.$$

Furthermore, (6.27) means that the constant L is a real number.

Next we shall show that the constant L is positive. In fact, it follows from (6.27) that

$$L[W_0(4)/W_0(7)+K_{3,1}(0)] = c - K'_{2,2}(0),$$

and

$$L[W_0(3)/W_0(6)+K_{3,1}(-1)] = c - K'_{2,2}(-1) + W_0(3)/W_0(6).$$

Here we used (4.16), (4.30) and (4.32).
Then we get

$$\begin{aligned} &L[K_{3,1}(-1)-K_{3,1}(0)+W_0(3)/W_0(6)-W_0(4)/W_0(7) \\ &\quad = -K'_{2,2}(-1) + K'_{2,2}(0) + W_0(3)/W_0(6) \\ &\quad = -K'_{2,2}(6) - 3\times4\times5\times W_0(3)/6\times7\times8\times W_0(6) + K'_{2,2}(2) \\ &\qquad + W_0(3)/W_0(6) \\ &\quad > W_0(11)/7\times8\times9\times W_0(7) - W_0(17)/13\times14\times15\times W_0(13) \\ &\qquad + W_0(3)/W_0(6) - 3\times4\times5\times W_0(3)/6\times7\times8\times W_0(6) \\ &\quad = 4\times5\times6\times W_0(4)/7\times8\times9\times W_0(7) \\ &\qquad + (1-1/60-5/28)\, W_0(3)/W_0(6) > 0. \end{aligned}$$

Here we used the condition $D^*(3, 4, 2, 2; 0, \underline{2})=0$ and (4.16). Since

$$K_{3,1}(-1) > K_{3,1}(0) \quad \text{and} \quad W_0(3)/W_0(6) > W_0(4)/W_0(7),$$

we could find that the constant L is a positive real number.

From (6.28) and (4.40), it follows that

$$\text{(6.29)}\qquad W^*_{3,1}(7) + LW_0(4) = 0, \quad W^*_{3,1}(6) + (L-1)W_0(3) = 0,$$
$$W^*_{3,1}(5) + (L-2)W_0(2) = 0.$$

Since the constant L is positive, we can obtain

$$\text{(6.30)}\qquad \frac{W^*_{3,1}(6)}{W_0(6)} - \frac{W^*_{3,1}(7)}{W_0(7)} > \frac{(5+L)\,W_0(10)}{5\times6\times7\times8\times W_0(6)} - \frac{(6+L)\,W_0(11)}{6\times7\times8\times9\times W_0(7)}$$
$$+ \frac{(12+L)\,W_0(17)}{12\times13\times14\times15\times W_0(13)} - \frac{(13+L)\,W_0(18)}{13\times14\times15\times16\times W_0(14)}$$

and

$$\text{(6.31)}\qquad \frac{W^*_{3,1}(5)}{W_0(5)} - \frac{W^*_{3,1}(6)}{W_0(6)} < \frac{(4+L)\,W_0(9)}{4\times5\times6\times7\times W_0(5)} - \frac{(5+L)\,W_0(10)}{5\times6\times7\times8\times W_0(6)}$$
$$+ \frac{(11+L)\,W_0(16)}{11\times12\times13\times14\times W_0(12)}.$$

From (6.28) and (4.16), (6.30) and (6.31) become

$$\text{(6.32)}\qquad (1-L) + L\frac{W_0(4)\,W_0(6)}{W_0(7)\,W_0(3)} > \frac{(5+L)3\times4\times5}{5\times6\times7\times8} - \frac{(6+L)4\times5\times6}{6\times7\times8\times9}$$
$$\times \frac{W_0(4)\,W_0(6)}{W_0(7)\,W_0(3)} + \frac{(12+L)10\times11\times12\times3\times4\times5}{12\times13\times14\times15\times6\times7\times8}$$
$$- \frac{(13+L)11\times12\times13\times4\times5\times6\,W_0(4)\,W_0(6)}{13\times14\times15\times16\times7\times8\times9\,W_0(7)\,W_0(3)},$$

$$\text{(6.33)}\qquad (2-L) - (1-L)\frac{W_0(3)\,W_0(5)}{W_0(2)\,W_0(6)} < \frac{(4+L)2\times3\times4}{4\times5\times6\times7}$$
$$- \frac{(5+L)3\times4\times5}{5\times6\times7\times8}\,\frac{W_0(3)\,W_0(5)}{W_0(2)\,W_0(6)} + \frac{(11+L)9\times10\times11\times2\times3\times4}{11\times12\times13\times14\times5\times6\times7}.$$

Furthermore, since

$$\frac{W_0(4)W_0(6)}{W_0(7)W_0(3)} = \frac{3W_0(11)W_0(13)}{4W_0(14)W_0(10)} = \frac{120W_0(18)W_0(20)}{169W_0(21)W_0(17)}$$

and

$$\frac{W_0(3)W_0(5)}{W_0(2)W_0(6)} = \frac{16W_0(10)W_0(12)}{25W_0(9)W_0(13)} = \frac{3W_0(17)W_0(19)}{5W_0(16)W_0(20)},$$

we get from (4.47)

$$\frac{108}{169} < \frac{W_0(4)W_0(6)}{W_0(7)W_0(3)} < \frac{120}{169}, \quad \frac{51}{95} < \frac{W_0(3)W_0(5)}{W_0(2)W_0(6)} < \frac{3}{5}.$$

Therefore, we get from (6.32) and (6.33)

$$L < 3{,}32\ldots \quad \text{and} \quad L > 3.334\ldots.$$

This is a contradiction. Thus we have proved (6.1) for this case.

For other cases;

$$\begin{cases} \text{case (IV)} & \text{(i)} \quad (p, r) = (3, 0), \quad (p-1, r+1) = (2, 1) \\ & \text{(ii)} \quad (p, r) = (3, 1), \quad (p-1, r+1) = (2, 2) \\ & \text{(iii)} \quad (p, r) = (3, 2), \quad (p-1, r+1) = (2, 3) \\ \text{case (V)} & \text{(i)} \quad (p, r) = (3, 0), \quad (p-1, r+1) = (2, 1) \\ & \text{(ii)} \quad (p, r) = (4, 0), \quad (p-1, r+1) = (3, 1) \\ \text{case (VI)} & \text{(i)} \quad (p, r) = (3, 0), \quad (p-1, r+1) = (2, 1) \\ & \text{(ii)} \quad (p, r) = (4, 0), \quad (p-1, r+1) = (3, 1) \\ & \text{(iii)} \quad (p, r) = (5, 0), \quad (p-1, r+1) = (4, 1), \end{cases}$$

we can prove Lemma 6.1 by a similar manner to the proof of cases (II) and (III). So we omit them.

REMARK. In (6.11), for example, it holds that

$$W = \frac{W_0(3)W_0(5)}{W_0(6)W_0(2)} = \frac{\Gamma(5/6)\Gamma(5/6)}{\Gamma(8/6)\Gamma(2/6)} = 0.5354\ldots.$$

(See [1] pp. 267–270.)

§7. Relations between Stokes multipliers

Case (I) $q=2$; Let $x_k(t)$ and $\tilde{x}_k(t)$ be solutions of the differential equation (1.3) and

(7.1) $$x_k(t) = x_h(t), \quad \tilde{x}_k(t) = \tilde{x}_h(t) \qquad (k=h \pmod{n+2}).$$

Furthermore, we write their connection formulas as follows;

(7.2) $$x_k(t) = \sum_{i=1}^{n} a_i^k x_{k+i}(t) \quad \text{and} \quad \tilde{x}_k(t) = \sum_{i=1}^{n} b_i^k \tilde{x}_{k+i}(t).$$

In this case, we shall derive the following

LEMMA 7.1. *Suppose that*

(7.3) $$a_n^j = b_n^j \neq 0 \qquad (j=0, 1, 2,\ldots, n+1)$$

and

(7.4) $$a_1^s = b_1^s \qquad (s=0, 1, 2,\ldots, n-2).$$

Then it holds that

(7.5) $$a_m^j = b_m^j \qquad (j=0, 1,\dots, n+1;\ m=1, 2,\dots, n).$$

Utilizing the Cramer rule, we get from (7.2)

(7.6) $$\begin{aligned}&\mathrm{Wron}\,[x_k(t), x_{k+1}(t),\dots, x_{k+n-1}(t)]\\ &\quad = (-1)^{n-1}a_n^k\,\mathrm{Wron}\,[x_{k+1}(t), x_{k+2}(t),\dots, x_{k+n}(t)]\end{aligned}$$

and

(7.7) $$a_m^k = \frac{\mathrm{Wron}\,[x_{k+1}(t),\dots, x_{k+m-1}(t), x_k(t), x_{k+m+1}(t),\dots, x_{k+n}(t)]}{\mathrm{Wron}\,[x_{k+1}(t),\dots, x_{k+m-1}(t), x_{k+m}(t), x_{k+m+1}(t),\dots, x_{k+n}(t)]}.$$

Noting (7.1), we consider the following connection formula

$$\begin{aligned}x_{k+m+1}(t) &= a_1^{k+m+1}x_{k+m+2}(t) + \cdots + a_{n-m}^{k+m+1}x_{k+n+1}(t) +\\ &\quad + a_{n-m+1}^{k+m+1}x_k(t) + \cdots + a_n^{k+m+1}x_{k+m+2+n-1}(t).\end{aligned}$$

Then, from the Cramer rule, it holds that

(7.8) $$a_{n-m}^{k+m+1} = \frac{\mathrm{Wron}\,[x_{k+m+2},\dots, x_{k+n}, x_{+m+1}, x_k,\dots, x_{k+m+2+n-1}]}{\mathrm{Wron}\,[x_{k+m+2}(t),\dots, x_{k+m+2+n-1}(t)]}.$$

Here, using (7.6), we can get

(7.9) $$\begin{aligned}&\mathrm{Wron}\,[x_{k+1}(t),\dots, x_{k+n}(t)]\\ &\quad = (-1)^{n-1}a_n^{k+1}\,\mathrm{Wron}\,[x_{k+2}(t),\dots, x_{k+n+1}(t)]\\ &\quad = \cdots\cdots\cdots\cdots\\ &\quad = (-1)^{(n-1)(m+1)}\textstyle\prod_{s=1}^{m+1} a_n^{k+s}\,\mathrm{Wron}\,[x_{k+m+2}(t),\dots, x_{k+m+2+n-1}(t)].\end{aligned}$$

Since

$$\begin{aligned}&\mathrm{Wron}\,[x_{k+m+2},\dots, x_{k+n}, x_{k+m+1}, x_k,\dots, x_{k+m+2+n-1}]\\ &\quad = (-1)^{n-m-1}\,\mathrm{Wron}\,[x_{k+m+1},\dots, x_{k+n}, x_k, x_{k+1},\dots, x_{k+m-1}]\\ &\quad = (-1)^{-nm+m+n}\,\mathrm{Wron}\,[x_{k+1},\dots, x_{k+m-1}, x_k, x_{k+m+1},\dots, x_{k+n}],\end{aligned}$$

(7.7), (7.8) and (7.9) mean that

(7.10) $$a_m^k = -\textstyle\prod_{s=1}^{m+1}(a_n^{k+s})^{-1}a_{n-m}^{k+m+1} \qquad (k=0, 1,\dots, n+1;\ m=1, 2,\dots, n-1).$$

This relation is important in the proof of Lemma 7.1.

Next we shall seek other relations of the Stokes multipliers. Let us put $X_k(t) = (x_k(t), x_{k+1}(t),\dots, x_{k+n-1}(t))$ and

(7.11) $$\tilde{A}_k = A_k + J,$$

where

$$(7.12)\qquad A_k=\begin{pmatrix} a_1^k & 0\cdots\cdots 0 \\ a_2^k & 0\cdots\cdots 0 \\ \vdots & \vdots\ \ddots\ \vdots \\ a_n^k & 0\cdots\cdots 0 \end{pmatrix},\quad J=\begin{pmatrix} 0 & 1 & 0\cdots 0 \\ 0 & 0 & 1\cdots 0 \\ \vdots & \vdots & \ddots\ \ddots\ \vdots \\ \vdots & \vdots & \ddots\ 1 \\ 0 & 0\cdots\cdots\cdots & 0 \end{pmatrix}$$

and suppose that

$$(7.13)\qquad a_n^k\neq 0 \qquad (k=0, 1,\dots, n+1).$$

Then n by n matrices $X_k(t)$ are fundamental sets of solutions of (3.1) and from the connection formula (7.2) and (7.1) it holds that

$$(7.14)\quad \prod_{h=0}^{n+1}\tilde{A}_{k+h}=\prod_{h=0}^{n+1}(A_{k+h}+J)=(A_{k+n+1}+J)\cdots(A_{k+1}+J)(A_k+J)=I.$$

We shall introduce the following notation. In this section, an n by n matrix A is called *j-th column matrix* ($j=1, 2,\dots, n$), if only j-th column elements may have non-zero elements. Then, in (7.12) each matrix A_k is a first column matrix and $A_kJ^m(m=0, 1,\dots, n-1)$ are $(m+1)$-th column matrices. Furthermore, let B be any n by n matrix and A a first column matrix, then BA is a first column matrix. Noting these results and (7.14), we put

$$(7.15)\qquad \prod_{j=0}^{n-1}(A_{k+j}+J)=\sum_{j=0}^{n}U_j(k+n-1)J^{n-j},$$

where

$$(7.16)\qquad U_0(k+n-1)=I,\quad U_1(k+n-1)=A_{k+n-1},$$
$$U_j(k+n-1)=\sum_{i=0}^{j-1}(A_{k+n-i}+J)A_{k+n-j}\qquad (j=2, 3,\dots, n).$$

Then we can easily obtain

$$\prod_{i=0}^{n-1}(A_{k+i}+J)=\begin{pmatrix} q_1^n(k+n-1) & q_1^{n-1}(k+n-1)\cdots q_1^2(k+n-1) & q_1^1(k+n-1) \\ q_2^n(k+n-1) & q_2^{n-1}(k+n-1)\cdots q_2^2(k+n-1) & q_2^1(k+n-1) \\ \vdots & \vdots\qquad\qquad \vdots & \vdots \\ q_n^n(k+n-1) & q_n^{n-1}(k+n-1)\cdots q_n^2(k+n-1) & q_n^1(k+n-1) \end{pmatrix},$$

where we put

$$(7.17)\qquad U_j(k+n-1)=\begin{pmatrix} q_1^j(k+n-1) & 0\cdots 0 \\ q_2^j(k+n-1) & 0\cdots 0 \\ \vdots & \vdots\ \ \vdots \\ q_n^j(k+n-1) & 0\cdots 0 \end{pmatrix}\quad (j=1, 2,\dots, n).$$

Furthermore, the condition (7.14) can be written as

$$\begin{pmatrix} a_2^{k+n}+a_1^{k+n+1}a_1^{k+n} & a_1^{k+n+1} & 1 \cdots\cdots 0 \\ a_3^{k+n}+a_2^{k+n+1}a_1^{k+n} & a_2^{k+n+1} & 0 \ \ddots \ \vdots \\ \vdots & \vdots & \vdots \ \ddots \ 1 \\ a_n^{k+n}+a_{n-1}^{k+n+1}a_1^{k+n} & a_{n-1}^{k+n+1} & 0 \cdots\cdots 0 \\ a_n^{k+n+1}a_1^{k+n} & a_n^{k+n+1} & 0 \cdots\cdots 0 \end{pmatrix} \begin{pmatrix} q_1^n & q^{n-1} \cdots q_1^1 \\ q_2^n & q_2^{n-1} \cdots q_2^1 \\ \vdots & \vdots \quad\ \vdots \\ q_n^n & \vdots \quad\ \vdots \\ q_1^n & q_n^{n-1} \cdots q_n^1 \end{pmatrix} = I.$$

From this relation we get

$$(7.18) \quad a_n^{k+n+1}a_1^{k+n}q_1^j(k+n-1) + a_n^{k+n+1}q_2^j(k+n-1) = \begin{cases} 0 & (j \neq 1) \\ 1 & (j=1), \end{cases}$$

$$(7.19) \quad [a_n^{k+n}+a_{n-1}^{k+n+1}a_1^{k+n}]q_1^j(k+n-1) + a_{n-1}^{k+n+1}q_2^j(k+n-1) = \begin{cases} 0 & (j \neq 2) \\ 1 & (j=2), \end{cases}$$

$$(7.20) \quad [a_{n-m+1}^{k+n}+a_{n-m}^{k+n+1}a_1^{k+n}]q_1^j(k+n-1) + a_{n-m}^{k+n+1}q_2^j(k+n-1)$$
$$+ q_{n-m+2}^j(k+n-1) = \begin{cases} 0 & (j \neq m+1) \\ 1 & (j=m+1) \qquad (m=2, 3, \dots, n-1). \end{cases}$$

From (7.13) and (7.18), the conditions (7.19) and (7.20) become

$$(7.21) \qquad q_1^j(k+n-1) = 0 \qquad \text{for} \quad j \neq 1, 2;$$

$$(7.22) \qquad a_n^{k+n+1}a_n^{k+n}q_1^1(k+n-1) + a_{n-1}^{k+n+1} = 0;$$

$$(7.23) \qquad a_n^{k+n}q_1^2(k+n-1) = 1;$$

and

$$(7.24) \qquad a_{n-m+1}^{k+n}q_1^j(k+n-1) + q_{n-m+2}^j(k+n-1) = 0 \qquad (j \neq m+1, 1);$$

$$(7.25) \qquad a_{n-m+1}^{k+n}q_1^1(k+n-1) + a_{n-m}^{k+n+1}/a_n^{k+n+1} + q_{n-m+2}^1(k+n-1) = 0;$$

$$(7.26) \qquad a_{n-m+1}^{k+n}q_1^{m+1}(k+n-1) + q_{n-m+2}^{m+1}(k+n-1) = 1 \quad (m=2, 3, \dots, n-1).$$

Therefore, we can obtain from (7.18), (7.21), (7.24) and (7.26)

$$(7.27) \qquad \prod_{i=0}^{n-1} (A_{k+i}+J) = \begin{pmatrix} 0 & 0 \cdots 0 & q_1^2(k+n-1) & q_1^1(k+n-1) \\ 0 & 0 \cdots 0 & q_2^2(k+n-1) & q_2^1(k+n-1) \\ 1 & 0 \cdots 0 & q_3^2(k+n-1) & q_3^1(k+n-1) \\ \vdots & 1 \quad \vdots & \vdots & \vdots \\ \vdots & \vdots \ \ddots \ \vdots & & \\ 0 & 0 \cdots 1 & q_n^2(k+n-1) & q_n^1(k+n-1) \end{pmatrix}.$$

It holds from (7.16) that

(7.28) $$q_j^1(k+n-1) = a_j^{k+n-1} \qquad (j=1, 2,\dots, n)$$

and

(7.29) $$\begin{cases} q_j^2(k+n-1) = a_j^{k+n-1}a_1^{k+n-2} + a_{j+1}^{k+n-2} & (j=1, 2,\dots, n-1), \\ q_n^2(k+n-1) = a_n^{k+n-1}a_1^{k+n-2}. \end{cases}$$

Putting (7.28) into (7.18) and (7.25), we get

(7.30) $$a_2^{k+n-1} = 1/a_n^{k+n+1} - a_1^{k+n}a_1^{k+n-1}$$

and

(7.31) $$a_{n-m+2}^{k+n-1} = -a_{n-m}^{k+n+1}/a_n^{k+n+1} - a_1^{k+n-1}a_{n-m+1}^{k+n} \qquad (m=2, 3,\dots, n-1).$$

Similarly, we can get (7.10), (7.30) and (7.31) for b_j^k.

We are now in a position to prove Lemma 7.1.

PROOF OF LEMMA 7.1. From (7.4) and (7.30), we can easily obtain

(7.32) $$a_2^s = b_2^s \qquad (s=0, 1,\dots, n-3).$$

Using this relation and putting $m=n-1$ in (7.31), we can get

(7.33) $$a_3^s = b_3^s \qquad (s=0, 1,\dots, n-4).$$

Similarly, putting $m=n-2, n-3,\dots, 2$, we get from (7.31)

(7.34) $$a_m^s = b_m^s \qquad (s=0, 1,\dots, n-m-1;\ m=1, 2,\dots, n-1).$$

Next, using the relation (7.10), we get from (7.34)

(7.35) $$a_{n-m}^s = b_{n-m}^s \qquad (s=m+1, m+2,\dots, n;\ m=1, 2,\dots, n-1).$$

Therefore, we have obtained

(7.36) $$a_m^s = b_m^s \qquad (m=1, 2,\dots, n-1;\ s=0, 1,\dots, n-m-1, n-m+1,\dots, n).$$

Now we have obtained from (7.36)

(7.37) $$A_0 = B_0 \quad \text{and} \quad A_n = B_n.$$

Then, putting $k=1$ in (7.14), we have

(7.38) $$\tilde{A}_{n+1} \prod_{i=0}^{n-1} \tilde{A}_{i+1} = \tilde{B}_{n+1} \prod_{i=0}^{n-1} \tilde{B}_{i+1},$$

that is,

$$(7.39)\quad \begin{pmatrix} a_1^{n+1} & 1 & 0 \cdots\cdots 0 \\ a_2^{n+1} & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \vdots \ddots \\ \vdots & \vdots & \vdots \quad \ddots\ 1 \\ a_n^{n+1} & 0 & 0 \cdots\cdots 0 \end{pmatrix} \begin{pmatrix} 0 \cdots\cdots 0 & q_1^2(n) & q_1^1(n) \\ 0 \cdots\cdots 0 & q_2^2(n) & q_2^1(n) \\ 1 \quad \vdots & \vdots & \vdots \\ \ddots \quad \vdots & \vdots & \vdots \\ 0 \quad \ddots\ 1 & q_n^2(n) & q_n^1(n) \end{pmatrix}$$

$$= \begin{pmatrix} b_1^{n+1} & 1 & 0 \cdots\cdots 0 \\ b_2^{n+1} & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \vdots \ddots \\ \vdots & \vdots & \vdots \quad \ddots\ 1 \\ b_n^{n+1} & 0 & 0 \cdots\cdots 0 \end{pmatrix} \begin{pmatrix} 0 \cdots\cdots 0 & \tilde{q}_1^2(n) & \tilde{q}_1^1(n) \\ 0 \cdots\cdots 0 & \tilde{q}_2^2(n) & \tilde{q}_2^1(n) \\ 1 \quad \vdots & \vdots & \vdots \\ \ddots \quad \vdots & \vdots & \vdots \\ 0 \quad \ddots\ 1 & \tilde{q}_n^2(n) & \tilde{q}_n^1(n) \end{pmatrix},$$

where we put

$$\prod_{i=0}^{n-1} (B_{k+i}+J) = \begin{pmatrix} 0 & 0 \cdots\cdots 0 & \tilde{q}_1^2(k+n-1) & \tilde{q}_1^1(k+n-1) \\ 0 & 0 \cdots\cdots 0 & \tilde{q}_2^2(k+n-1) & \tilde{q}_2^1(k+n-1) \\ 1 & \quad \vdots & \vdots & \vdots \\ & \ddots \quad \vdots & \vdots & \vdots \\ 0 & 0 \cdots\cdots 1 & \tilde{q}_n^2(k+n-1) & \tilde{q}_n^1(k+u-1) \end{pmatrix}.$$

Since it holds from (7.28) that

$$q_j^1(n) = a_j^n = b_j^n = \tilde{q}_j^1(n) \qquad (j=1, 2,\dots, n),$$

the (j, n)-elements of (7.39) give us

$$(7.40)\qquad a_j^{n+1} = b_j^{n+1} \qquad (j=1, 2,\dots, n) \ \text{ i.e., } \ A_{n+1} = B_{n+1}.$$

Then, from (7.39), it holds that

$$(7.41)\quad q_j^2(n) = \tilde{q}_j^2(n) \qquad (j=1, 2,\dots, n) \ \text{ i.e., } \ (A_n+J)A_{n-1} = (B_n+J)B_{n-1}.$$

The conditions (7.41) and (7.37) mean $A_{n-1}=B_{n-1}$. Therefore, from (7.4), (7.37) and (7.40), we have obtained

$$(7.42)\qquad a_1^s = b_1^s \qquad (s=0, 1,\dots, n+1).$$

Hence, we can easily obtain (7.5) by the use of (7.30) and (7.31).

Now we shall consider the case in which $q \neq 2$. For case (II)–(VI), we shall derive the following lemmas.

LEMMA 7.2 (Case (II) $n=3$ *and* $q=3$). *Let* a_i, b_i, c_i, a_i', b_i' $(\neq 0)$ *be complex numbers and*

$$M_i = \begin{pmatrix} a_i & 1 & 0 \\ b_i & 0 & 1 \\ c_i & 0 & 0 \end{pmatrix}, \quad M'_i = \begin{pmatrix} a'_i & 1 & 0 \\ b'_i & 0 & 1 \\ c_i & 0 & 0 \end{pmatrix} \qquad (i=0, 1,\dots, 5).$$

Then, if

$$M_0M_1M_2M_3M_4M_5 = M'_0M'_1M'_2M'_3M'_4M'_5 = I$$

and

$$a_1 = a'_1, \quad a_3 = a'_3, \quad a_0 = a'_0, \quad b_0 = b'_0,$$

we have

$$a_i = a'_i \quad \text{and} \quad b_i = b'_i \qquad (i=0, 1,\dots, 5).$$

LEMMA 7.3 (*Case (III)$n=3$ and $q=4$*). *Let a_i, b_i, c_i, a'_i, b'_i ($\neq 0$) be complex numbers and*

$$M_i = \begin{pmatrix} a_i & 1 & 0 \\ b_i & 0 & 1 \\ c_i & 0 & 0 \end{pmatrix}, \quad M'_i = \begin{pmatrix} a'_i & 1 & 0 \\ b'_i & 0 & 1 \\ c_i & 0 & 0 \end{pmatrix} \qquad (i=0, 1,\dots, 5, 6).$$

Then, if

$$M_0M_1M_2M_3M_4M_5M_6 = M'_0M'_1M'_2M'_3M'_4M'_5M'_6 = I$$

and

$$a_i = a'_i \qquad (i=1, 2, 3, 4), \quad b_j = b'_j \qquad (j=3, 4),$$

we have

$$M_i = M'_i \qquad (i=0, 1,\dots, 5, 6).$$

LEMMA 7.4 (*Case (IV) $n=3$ and $q=5$*). *Let a_i, b_i, c_i, a'_i, b'_i ($\neq 0$) be complex numbers and*

$$M_i = \begin{pmatrix} a_i & 1 & 0 \\ b_i & 0 & 0 \\ c_i & 0 & 0 \end{pmatrix}, \quad M'_i = \begin{pmatrix} a'_i & 1 & 0 \\ b'_i & 0 & 1 \\ c_i & 0 & 0 \end{pmatrix} \qquad (i=0, 1,\dots, 5, 6, 7).$$

Then, if

$$M_0M_1M_2M_3M_4M_5M_6M_7 = M'_0M'_1M'_2M'_3M'_4M'_5M'_6M'_7 = I$$

and

$$a_i = a'_i \qquad (i=1, 2, 3, 4, 5), \quad b_j = b'_j \qquad (j=3, 4, 5),$$

we have

$$M_i = M'_i \qquad (i=0, 1,\ldots, 5, 6, 7).$$

LEMMA 7.5. *(Case (V) $n=4$ and $q=3$). Let a_i, b_i, c_i, d_i, a'_i, b'_i, c'_i, $(\neq 0)$ be complex numbers and*

$$M_i = \begin{pmatrix} a_i & 1 & 0 & 0 \\ b_i & 0 & 1 & 0 \\ c_i & 0 & 0 & 1 \\ d^i & 0 & 0 & 0 \end{pmatrix}, \quad M'_i = \begin{pmatrix} a'_i & 1 & 0 & 0 \\ b'_i & 0 & 1 & 0 \\ c'_i & 0 & 0 & 1 \\ d_i & 0 & 0 & 0 \end{pmatrix} \qquad (i=0, 1,\ldots, 6).$$

Then, if

$$M_0M_1M_2M_3M_4M_5M_6 = M'_0M'_1M'_2M'_3M'_4M'_5M'_6 = I$$

and

$$a_j = a'_j \qquad (j=0, 1, 2, 3), \quad b_k = b'_k \qquad (k=2, 3),$$

we have

$$M_i = M'_i \qquad (i=0, 1,\ldots, 5, 6).$$

LEMMA 7.6 *(Case (VI) $n=5$ and $q=3$). Let a_i, b_i, c_i, d_i, e_i; a'_i, b'_i, c'_i, d'_i $(\neq 0)$ be complex numbers and*

$$M_i = \begin{pmatrix} a_i & 1 & 0 & 0 & 0 \\ b_i & 0 & 1 & 0 & 0 \\ c_i & 0 & 0 & 1 & 0 \\ d_i & 0 & 0 & 0 & 1 \\ e_i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M'_i = \begin{pmatrix} a'_i & 1 & 0 & 0 & 0 \\ b'_i & 0 & 1 & 0 & 0 \\ c'_i & 0 & 0 & 1 & 0 \\ d'_i & 0 & 0 & 0 & 1 \\ e_i & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (i=0, 1,\ldots, 7).$$

Then, if

$$M_0M_1M_2M_3M_4M_5M_6M_7 = M'_0M'_1M'_2M'_3M'_4M'_5M'_6M'_7 = I,$$

and

$$a_j = a'_j \qquad (j=0, 1, 2, 3, 4), \quad b_k = b'_k \qquad (k=2, 3, 4),$$

we have

$$M_i = M'_i \qquad (i=0, 1,\ldots, 6, 7).$$

PROOF OF LEMMA 7.2. Since $M_0 = M'_0$ and

$$M_1M_2M_3M_4M_5 = \begin{pmatrix} a_1(a_2a_3+b_3)+b_2a_3+c_3 & a_1a_2+b_2 & a_1 \\ b_1(a_2a_3+b_3)+c_2a_3 & b_1a_2+c_2 & b_1 \\ c_1(a_2a_3+b_3) & c_1a_2 & c_1 \end{pmatrix}\begin{pmatrix} a_4a_5+b_5 & a_4 & 1 \\ b_4a_5+c_5 & b_4 & 0 \\ c_4a_5 & c_4 & 0 \end{pmatrix},$$

it follows from the condition $M_1M_2M_3M_4M_5=M_1'M_2'M_3'M_4'M_5'$ that

$$\begin{cases} a_1(a_2a_3+b_3) + b_2a_3 + c_3 = a_1(a_2'a_3+b_3') + b_2'a_3 + c_3, \\ b_1(a_2a_3+b_3) + c_2a_3 = b_1'(a_2'a_3+b_3') + c_2a_3, \\ c_1(a_2a_3+b_3) = c_1(a_2'a_3+b_3'). \end{cases}$$

Therefore, it holds that $b_1=b_1'$ or $a_2a_3+b_3=a_2'a_3+b_3'=0$. We assume that $a_2a_3+b_3=0$. Then, from the condition $M_0M_1M_2M_3M_4M_5=I$, we can get

$$a_0(b_2a_3+c_3) + c_2a_3 = 0, \quad b_0(b_2a_3+c_3) = 0, \quad c_0(b_2a_3+c_3) = 1.$$

This is a contradiction. Thus we have obtained $b_1=b_1'$. (i.e., $M_1=M_1'$)

Since

$$M_2M_3M_4M_5 = \begin{pmatrix} a_2(a_3a_4+b_4)+b_3a_4+c_4 & a_2a_3+b_3 & a_2 \\ b_2(a_3a_4+b_4)+c_3a_4 & b_2a_3+c_3 & b_2 \\ c_2(a_3a_4+b_4) & c_2a_3 & c_2 \end{pmatrix}\begin{pmatrix} a_5 & 1 & 0 \\ b_5 & 0 & 1 \\ c_5 & 0 & 0 \end{pmatrix},$$

we can get from the condition $M_2M_3M_4M_5=M_2'M_3'M_4'M_5'$

$$\begin{cases} a_2a_3 + b_3 = a_2'a_3 + b_3', \quad a_2(a_3a_4+b_4) + b_3a_4 + c_4 \\ \qquad\qquad\qquad\qquad = a_2'(a_3a_4'+b_4') + b_3'a_4' + c_4, \\ b_2a_3 + c_3 = b_2'a_3 + c_3, \quad b_2(a_3a_4+b_4) + c_3a_4 = b_2'(a_3a_4'+b_4') + c_3a_4', \\ \qquad\qquad\qquad\qquad c_2(a_3a_4+b_4) = c_2(a_3a_4'+b_4'). \end{cases}$$

From the conditions, we can easily obtain

$$a_2 = a_2', \quad a_4 = a_4', \quad b_2 = b_2', \quad b_3 = b_3' \quad \text{and} \quad b_4 = b_4'.$$

Thus we have obtained Lemma 7.2.

PROOF OF LEMMA 7.3. It holds from the assumption that

$$I = (M_0M_1)(M_2M_3M_4)(M_5M_6)$$
$$= \begin{pmatrix} a_0a_1+b_1 & a_0 & 1 \\ b_0a_1+c_1 & b_0 & 0 \\ c_0a_1 & c_0 & 0 \end{pmatrix}\begin{pmatrix} g_1 & * & * \\ g_2 & * & * \\ g_3 & * & * \end{pmatrix}\begin{pmatrix} * & * & 1 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}$$

and

$$I = (M_0'M_1')(M_2'M_3'M_4')(M_5'M_6')$$

$$= \begin{pmatrix} a_0'a_1+b_1' & a_0' & 1 \\ b_0'a_1+c_1 & b_0' & 0 \\ c_0a_1 & c_0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & * & * \\ g_2' & * & * \\ g_3 & * & * \end{pmatrix} \begin{pmatrix} * & * & 1 \\ * & * & 0 \\ * & * & 0 \end{pmatrix},$$

where

$$g_1 = a_2(a_3a_4+b_4) + b_3a_4 + c_4, \quad g_2 = b_2(a_3a_4+b_4) + c_3a_4,$$
$$g_3 = c_2(a_3a_4+b_4), \quad g_2' = b_2'(a_3a_4+b_4) + c_3a_4.$$

Then we can easily obtain from this relation

$$\begin{cases} c_0a_1g_1 + c_0g_2 = 1 = c_0a_1g_1 + c_0g_2', \\ (b_0a_1+c_1)g_1 + b_0g_2 = 0 = (b_0'a_1+c_1)g_1 + b_0'g_2', \\ (a_0a_1+b_1)g_1 + a_0g_2 + g_3 = 0 = (a_0'a_1+b_1')g_1 + a_0'g_2' + g_3. \end{cases}$$

These conditions means that

$$g_2 = g_2' \quad \text{(i.e., } b_2 = b_2'\text{)} \quad \text{and} \quad b_0 = b_0'.$$

Therefore, it holds that $M_5M_6M_0M_1 = M_5'M_6'M_0'M_1'$, that is,

$$\begin{pmatrix} a_5a_6+b_6 & a_5 & 1 \\ b_5a_6+c_6 & b_5 & 0 \\ c_5a_6 & c_5 & 0 \end{pmatrix} \begin{pmatrix} a_0a_1+b_1 & a_0 & 1 \\ b_0a_1+c_1 & b_0 & 0 \\ c_0a_1 & c_0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a_5'a_6'+b_6' & a_6' & 1 \\ b_5'a_6'+c_6 & a_5' & 0 \\ c_5b_5' & c_5 & 0 \end{pmatrix} \begin{pmatrix} a_0'a_1+b_1' & a_0' & 1 \\ b_0a_1+c_1 & b_0 & 0 \\ c_0a_1 & c_0 & 0 \end{pmatrix}.$$

It follows that

$$c_5a_6 = c_5a_6', \quad c_5a_6a_0 + c_5b_0 = c_5a_6'a_0' + c_5b_0,$$

and

$$c_5a_6(a_0a_1+b_1) + c_5(b_0a_1+c_1) = c_5a_6'(a_0'a_1+b_1') + c_5(b_0a_1+c_1).$$

Hence we get $a_6 = a_6'$, $a_0 = a_0'$ and $b_1 = b_1'$. Thus we have obtained Lemma 7.3.

By a similar manner to the proof of Lemma 7.3, we can prove Lemmas7.6–7.6. Therefore we here omit them.

§8. Uniform simplification ($q=2$)

In this section we shall prove the main Theorem 1.3 for the case in which $q=2$. (i.e., case (I)) To do this, we shall consider the systems of differential equations (3.7), (3.12), (3.41). Let us put their connection formulas as follows;

(8.1) $$z_k(t, \varepsilon) = \sum_{j=1}^{n} d_j^k(\varepsilon) z_{k+j}(t, \varepsilon),$$

(8.2) $$\tilde{z}_k(t, \varepsilon) = \sum_{j=1}^{n} c_j^k(\varepsilon) \tilde{z}_{k+j}(t, \varepsilon),$$

(8.3) $$\hat{z}_k(t, \varepsilon) = \sum_{j=1}^{n} e_j^k(\varepsilon) \hat{z}_{k+j}(t, \varepsilon) \qquad (k=0, 1,\dots, n+q-1).$$

Each solution $y_k(x; b)$ of the differential equation (1.3) has the connection formula (4.1). Noting that

(8.4) $$C_j^k(b) = C_j^0(G^k(b)) \qquad (k=0, 1,\dots, n+q-1; j=1, 2,\dots, n)$$

in (4.1), we can show the following

LEMMA 8.1. *Let δ and M be the same as in Theorem 3.1. Then, in the connection formulas* (8.1), (8.2), (8.3), *the Stokes multipliers $d_j^k(\varepsilon)$ $c_j^k(\varepsilon)$ and $e_j^k(\varepsilon)$ are holomorphic in* (3.29) *and satisfy the following conditions*:

(8.5) $$d_j^k(\varepsilon) - C_j^k(b(\varepsilon)) = d_j^k(\varepsilon) - C_j^0(G^k(b(\varepsilon))) \simeq 0,$$

(8.6) $$c_j^k(\varepsilon) - C_j^k(b(\varepsilon)) = c_j^k(\varepsilon) - C_j^0(G^k(b(\varepsilon))) = 0,$$

(8.7) $$e_j^k(\varepsilon) = C_j^k(b(\varepsilon)+\psi(\varepsilon)) = C_j^0(G^k(b(\varepsilon)+\psi(\varepsilon)) \simeq C_j^0(G^k(b(\varepsilon)))$$
$$(j=1, 2,\dots, n;\ k=0, 1,\dots, n+q-1)$$

as μ tends to infinity in (3.29).

PROOF. From (3.13), we can easily obtain (8.6). Similarly, (3.47) means that

$$e_j^k(\varepsilon) = C_j^k(b(\varepsilon)+\psi(\varepsilon)).$$

From the Cramer rule, $d_j^k(\varepsilon)$, $c_j^k(\varepsilon)$ and $e_j^k(\varepsilon)$ are given by

(8.8) $$\begin{cases} d_j^k(\varepsilon) = \dfrac{\text{Wron}\,[z_{k+1},\dots, z_{k+j-1}, z_k, z_{k+j+1},\dots, z_{k+n}]}{\text{Wron}\,[z_{k+1},\dots, z_{k+n}]}, \\[2ex] c_j^k(\varepsilon) = \dfrac{\text{Wron}\,[\tilde{z}_{k+1},\dots, \tilde{z}_{k+j-1}, \tilde{z}_k, \tilde{z}_{k+j+1},\dots, \tilde{z}_{k+n}]}{\text{Wron}\,[\tilde{z}_{k+1},\dots, \tilde{z}_{k+n}]}, \\[2ex] e_j^k(\varepsilon) = \dfrac{\text{Wron}\,[\hat{z}_{k+1},\dots, \hat{z}_{k+j-1}, \hat{z}_k, \hat{z}_{k+j+1},\dots, \tilde{z}_{k+n}]}{\text{Wron}\,[\hat{z}_{k+1},\dots, \tilde{z}_{k+n}]}. \end{cases}$$

Since

$$\operatorname{trace}[B(t, \varepsilon)+F(t, \varepsilon)] = \operatorname{trace} B(t, \varepsilon) = \operatorname{trace} \hat{B}(t, \varepsilon) = 0,$$

each right hand member of (8.8) is independent of t. Theorefore, from (3.27) and (3.48), we can prove

$$d_j^k(\varepsilon) \simeq c_j^k(\varepsilon) = C_j^k(b(\varepsilon)) \simeq e_j^k(\varepsilon) \quad \text{as} \quad \mu \to \infty \quad \text{in} \quad (3.29).$$

In §3, we defined the n by n matrices $\Phi_k(t, \varepsilon)$, $\Psi_k(t, \varepsilon)$ and $\hat{\Psi}_k(t, \varepsilon)$. ((3.15), (3.50)) Furthermore, we put

$$(8.9) \qquad \tilde{\Gamma}_k(\varepsilon) = \begin{pmatrix} c_1^k(\varepsilon) & 1 & 0 \cdots\cdots 0 \\ c_2^k(\varepsilon) & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \quad 1 \\ c_n^k(\varepsilon) & 0 & 0 \cdots\cdots 0 \end{pmatrix},$$

$$(8.10) \qquad \Gamma_k(\varepsilon) = \begin{pmatrix} d_n^k(\varepsilon) & 1 & 0 \cdots\cdots 0 \\ d_2^k(\varepsilon) & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \quad 1 \\ d_n^k(\varepsilon) & 0 & 0 \cdots\cdots 0 \end{pmatrix}$$

and

$$(8.11) \qquad \hat{\Gamma}_k(\varepsilon) = \begin{pmatrix} e_1^k(\varepsilon) & 1 & 0 \cdots\cdots 0 \\ e_2^k(\varepsilon) & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \quad 1 \\ e_n^k(\varepsilon) & 0 & 0 \cdots\cdots 0 \end{pmatrix} \qquad (k=0, 1,\dots, n+q-1).$$

Then it holds from connection formulas (8.1), (8.2) and (8.3) that

$$(8.12) \qquad \Phi_k(t, \varepsilon) = \Phi_{k+1}(t, \varepsilon)\tilde{\Gamma}_k(\varepsilon), \quad \Psi_k(t, \varepsilon) = \Psi_{k+1}(t, \varepsilon)\Gamma_k(\varepsilon),$$
$$\hat{\Psi}_k(t, \varepsilon) = \hat{\Psi}_{k+1}(t, \varepsilon)\hat{\Gamma}_k(\varepsilon).$$

Since

$$\Phi_k(t, \varepsilon) = \Phi_h(t, \varepsilon), \quad \Psi_k(t, \varepsilon) = \Psi_h(t, \varepsilon), \quad \hat{\Psi}_k(t, \varepsilon) = \hat{\Psi}_h(t, \varepsilon)$$
$$\text{for} \quad k = h \pmod{n+q},$$

we can obtain from (8.12)

$$(8.13) \qquad \prod_{j=0}^{n+q-1} \tilde{\Gamma}_k(t, \varepsilon) = I, \quad \prod_{j=0}^{n+q-1} \Gamma_k(t, \varepsilon) = I, \quad \prod_{j=0}^{n+q-1} \hat{\Gamma}_k(t, \varepsilon) = I.$$

By computing the determinants on both sides of (8.13), we get

(8.14) $\prod_{j=0}^{n+q-1} c_n^j(\varepsilon) = 1, \quad \prod_{j=0}^{n+q-1} (\varepsilon) = 1, \quad \prod_{j=0}^{n+q-1} e_n^j(\varepsilon) = 1.$

From Lemma 8.1 and (8.9)–(8.11), it holds that

$$\tilde{\Gamma}_k(\varepsilon) \simeq \Gamma_k(\varepsilon) \simeq \hat{\Gamma}_k(\varepsilon) \tag{8.15}$$

as μ tends to infinity in (3.29). Then, if we obtain the relation

$$\Gamma_k(\varepsilon) = \hat{\Gamma}_k(\varepsilon) \qquad (k=0, 1,\dots, n+q-1), \tag{8.16}$$

it follows from the definition (3.52) and (8.12) that

$$\begin{aligned} T_k(t, \varepsilon) &= \Psi_k(t, \varepsilon)\hat{\Psi}_k(t, \varepsilon)^{-1} = \Psi_{k+1}(t, \varepsilon)\Gamma_k(\varepsilon)\hat{\Gamma}_k(\varepsilon)^{-1}\hat{\Psi}_{k+1}(t, \varepsilon)^{-1} \\ &= \Psi_{k+1}(t, \varepsilon)\hat{\Psi}_{k+1}(t, \varepsilon)^{-1} = T_{k+1}(t, \varepsilon). \end{aligned} \tag{8.17}$$

Putting

$$T(t, \varepsilon) = T_k(t, \varepsilon) = T_{k+1}(t, \varepsilon) = \cdots = T_{k+n+q-1}(t, \varepsilon),$$

we define

$$Q(t, \varepsilon) = \tilde{P}(t, \varepsilon) \exp\left[\frac{1}{n}\varepsilon^{-1}\int_0^t \operatorname{trace} E(s, \varepsilon)ds\right]T(t, \varepsilon).$$

Since, from the above definition,

$$Q(t, \varepsilon) = Q_k(t, \varepsilon) = Q_{k+1}(t, \varepsilon) = \cdots = Q_{k+n+q-1}(t, \varepsilon),$$

we finish the proof of the uniform simplification in a full neighborhood of the turning point $t=0$.

Therefore, we have only to prove (8.16), by choosing $\psi_{p,r}(\varepsilon)$ ($p=2, 3,\dots, n$; $r=0, 1,\dots, q-2$) and the modification $\gamma(\mu)z_k(t, \varepsilon)$. Here $\gamma(\mu)$ is a scalar function of μ such that

(i) $\gamma(\mu)$ is holomorphic in (3.29)

and

(ii) $\gamma(\mu) \simeq 1$ as μ tends to infinity in (3.29).

Then $\gamma(\mu)z_k(t, \varepsilon)$ is a holomorphic solution of the system (3.7) and furthermore, satisfies the condition (ii) in Theorem 3.1. In fact, from (3.27) it follows that

$$\begin{aligned} &\exp[-E_k(x; b)][\gamma(\mu)z_k(t, \varepsilon) - \tilde{z}_k(t, \varepsilon)] \\ &\quad= \exp[-E_k(x; b)][\gamma(\mu)z_k(t, \varepsilon) - \gamma(\mu)\tilde{z}_k(t, \varepsilon)] \\ &\qquad+ \exp[-E_k(x; b)][(\gamma(\mu) - 1)z_k(t, \varepsilon)] \\ &\quad\simeq 0 \end{aligned}$$

uniformly for (3.28) as μ tends to infinity in (3.29).

Case (I) and n is odd. In the connection formula (4.1), from (2.17) and the

Cramer rule, the Stokes multipliers $C_n^k(b)$ are given by

$$(8.18)\qquad C_n^k(b) = (-1)^{n-1}\frac{\text{Wron}\,[y_k(x;b),\, y_{k+1}(x;b),\ldots,\, y_{k+n-1}(x;b)]}{\text{Wron}\,[y_{k+1}(x;b),\, y_{k+2}(x;b),\ldots,\, y_{k+n}(x;b)]}$$
$$= \omega^{n\alpha_{n+2}(b)\exp[-2k\pi i/n]-(n-1)}.$$

Since n is odd and

$$\prod_{j=0}^{n+1} C_n^j(b) = 1,$$

we get from (8.18)

$$(8.19)\qquad C_n^k(b) = \omega^{-(n-1)} \qquad (k=0, 1,\ldots, n+1).$$

It follows from Lemma 8.1 that for $k=0, 1,\ldots, n+1$,

$$(8.20)\qquad c_n^k(\varepsilon) = \omega^{-(n-1)},\quad d_n^k(\varepsilon) \simeq \omega^{-(n-1)},\quad e_n^k(\varepsilon) \simeq \omega^{-(n-1)}$$

as $\mu=\varepsilon^{-1/(n+2)}$ tends to infinity in (3.29).

If we put

$$(8.21)\qquad \gamma_k(\varepsilon) = \frac{d_n^k(\varepsilon)}{\omega^{-(n-1)}} \qquad (k=0, 1,\ldots, n+1),$$

the quantities $\gamma_k(\varepsilon)$ are holomorphic in (3.29) and

$$(8.22)\qquad \gamma_k(\varepsilon) \simeq 1$$

as μ tends to infinity in (3.29). From (8.14) it holds that

$$(8.23)\qquad \gamma_{n+1}(\varepsilon)\gamma_n(\varepsilon)\cdots\gamma_1(\varepsilon)\gamma_0(\varepsilon) = 1.$$

Let us put

$$(8.24)\quad \zeta_0(t,\varepsilon) = z_0(t,\varepsilon),\ \zeta_{mn}(t,\varepsilon) = [\textstyle\prod_{i=0}^{m-1}\gamma_{in}(\varepsilon)]z_{mn}(t,\varepsilon)\quad (m=1, 2,\ldots, n+1),$$

where

$$\zeta_{mn}(t,\varepsilon) = \zeta_{m'n}(t,\varepsilon) \qquad \text{for}\quad mn = m'n \,(\text{mod}\, n+2).$$

Then $\zeta_k(t,\varepsilon)$ $(k=0, 1,\ldots, n+1)$ are solutions of the system (3.7) which satisfy the same conditions as $z_k(t,\varepsilon)$ $(k=0, 1,\ldots, n+1)$. If we substitute $\zeta_k(t,\varepsilon)$ for $z_k(t,\varepsilon)$, the connection formula (8.1) becomes

$$(8.25)\quad \zeta_k(t,\varepsilon) = \textstyle\sum_{j=1}^{n-1} \hat{d}_j^k(\varepsilon)\zeta_{k+j}(t,\varepsilon) + \omega^{-(n-1)}\zeta_{k+n}(t,\varepsilon)\quad (k=0, 1,\ldots, n+1),$$

where $\hat{d}_j^k(\varepsilon)$ are holomorphic in (3.29) and

$$(8.26)\quad \hat{d}_j^k(\varepsilon) - c_j^k(\varepsilon) = \hat{d}_j^k(\varepsilon) - C_j^k(b(\varepsilon)) \simeq 0 \quad \text{as}\quad \mu \longrightarrow \infty \text{ in } (3.29).$$

Here we used (8.22) and the fact that n is odd. Furthermore, (8.3) is written as

(8.27) $\hat{z}_k(t, \varepsilon) = \sum_{j=1}^{n-1} e_j^k(\varepsilon)\hat{z}_{k+j}(t, \varepsilon) + \omega^{-(n-1)}\hat{z}_{k+n}(t, \varepsilon) \quad (k=0, 1,\dots, n+1)$.

We shall now construct $n-1$ functions $\psi_{p,0}(\varepsilon)$ $(p=2, 3,\dots, n)$ so that

(a) each $\psi_{p,0}(\varepsilon)$ is holomorphic for $|\arg\mu| \leqq \delta/(n+2)$, $|\mu| \geqq M'$, where M' is a sufficiently large positive number;

(b) $\psi_{p,0}(\varepsilon) \simeq 0$ as μ tends to infinity in (3.29);

(c) $C_1^k(b(\varepsilon)+\psi(\varepsilon)) = \hat{d}_1^k(\varepsilon) \quad (k=0, 1,\dots, n-2)$.

To do this, we need the following lemma.

LEMMA 8.2. *Let us put*

$$b = (0, b_{2,0}, b_{3,0},\dots, b_{n,0}), \quad \tilde{b} = (\tilde{b}_0, \tilde{b}_1,\dots, \tilde{b}_{n-2}),$$
$$\psi = (0, \psi_{2,0}, \psi_{3,0},\dots, \psi_{n,0}),$$

and

$$f_k(b, \psi) = C_1^0(G^k(b+\psi)) - C_1^0(G^k(b)) \quad k = 0, 1,\dots, n-2.$$

Assume that ε_1 and ε_2 are fufficiently small positive numbers. Then, if

$$\sum_{p=2}^{n} |b_{p,0}| \leqq \varepsilon_1, \quad \sum_{j=0}^{n-2} |\tilde{b}_j| \leqq \varepsilon_2, \tag{8.28}$$

there exists a unique solution

$$\psi = g(b, \tilde{b}) = (0, g_2(b, \tilde{b}), g_3(b, \tilde{b}),\dots, g_n(b, \tilde{b})) \tag{8.29}$$

of the system of equations

$$f_k(b, \psi) = \tilde{b}_k \qquad (k=0, 1,\dots, n-2) \tag{8.30}$$

so that $g_2(b, \tilde{b}),\dots, g_n(b, \tilde{b})$ are holomorphic in the domain (8.28) *and*

$$g_j(b, 0) = 0 \qquad (j=2, 3,\dots, n). \tag{8.31}$$

PROOF. Since

$$G^k(b) = (0, \omega^{2k}b_{2,0}, \omega^{3k}b_{3,0},\dots, \omega^{nk}b_{n,0}),$$

we can easily get

$$\left.\frac{\partial f_k(b, \psi)}{\partial \psi_{p,0}}\right|_{\substack{b=0\\ \psi=0}} = \omega^{pk}\left.\frac{\partial C_1^0(b)}{\partial b_{p,0}}\right|_{b=0}.$$

This means that the Jacobian determinant of the system (8.30) with respect to $\psi_{p,0}$ $(p=2, 3,\dots, n)$ at $b=0$, $\psi=0$ is given by

$$\begin{vmatrix} 1 & 1 & \cdots\cdots 1 \\ \omega^2 & \omega^3 & \cdots\cdots \omega^n \\ \omega^4 & \omega^6 & \cdots\cdots \omega^{2n} \\ \vdots & \vdots & \vdots \\ \omega^{2(n-2)} & \omega^{3(n-2)} & \cdots\cdots \omega^{n(n-2)} \end{vmatrix} \prod_{j=2}^{n} \frac{\partial C_1^0(b)}{\partial b_{j,0}}\bigg|_{b=0}.$$

By virtue of Lemma 5.1, this Jacobian determinant is different from zero. This proves Lemma 8.2.

Let us now put

$$\tilde{b}_k(\varepsilon) = \hat{a}_1^k(\varepsilon) - C_1^0(G^k(b)) = \hat{a}_1^k(\varepsilon) - C_1^k(b) \qquad (k=0, 1,\dots, n-2), \tag{8.32}$$

and

$$\psi_{p,0}(\varepsilon) = g_p(b(\varepsilon), \tilde{b}(\varepsilon)) \qquad (p=2, 3,\dots, n). \tag{8.33}$$

Since

$$f_k(b(\varepsilon), \psi(\varepsilon)) = \tilde{b}_k(\varepsilon) \qquad (k=0, 1,\dots, n-2),$$

we get

$$\hat{a}_1^k(\varepsilon) = C_1^k(b(\varepsilon)+\psi(\varepsilon)) = e_1^k(\varepsilon) \quad (k=0, 1,\dots, n-2). \tag{8.34}$$

Hence, in the connection formulas (8.25) and (8.27), utilizing Lemma 7.1 and (8.34), we can easily obtain

$$\hat{a}_m^j(\varepsilon) = e_m^j(\varepsilon) \qquad (j=0, 1,\dots, n+1;\ m=1, 2,\dots, n-1). \tag{8.35}$$

If we put again

$$\Psi_k^*(t, \varepsilon) = (\zeta_k(t, \varepsilon), \zeta_{k+1}(t, \varepsilon),\dots, \zeta_{k+n-1}(t, \varepsilon)),$$

$$\Gamma_k^*(\varepsilon) = \begin{pmatrix} \hat{a}_1^k(\varepsilon) & 1 & & \mathbf{0} & \\ \hat{a}_2^k(\varepsilon) & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \hat{a}_{n-1}^k(\varepsilon) & 0 & & \ddots & 1 \\ \omega^{-(n-1)} & 0 & \cdots\cdots & & 0 \end{pmatrix}$$

and

$$T_k^*(t, \varepsilon) = \Psi_k^*(t, \varepsilon)\hat{\Phi}_k(t, \varepsilon)^{-1} \qquad (k=0, 1,\dots, n+1),$$

we get

$$\Gamma_k^*(\varepsilon) = \hat{\Gamma}_k(\varepsilon) \ \text{ and } \ T_k^*(t, \varepsilon) = T_{k+1}^*(t, \varepsilon) \qquad (k=0, 1,\dots, n+1).$$

Since

$$T_k^*(t, \varepsilon) \simeq I$$

uniformly for (3.49) as μ tends to infinity in (3.29), and sectors (3.49) for $k = 0, 1, \dots, n+1$ cover a full neighborhood of the turning point $t=0$,

$$T^*(t, \varepsilon) = T_k^*(t, \varepsilon) = T_{k+1}^*(t, \varepsilon) = \cdots = T_{k+n+1}^*(t, \varepsilon)$$

is a desirable transformation.

Case (I) and n is even; From (8.14) and (8.18), it holds that

$$\omega^{n\alpha_{n+2}(b)\ \Sigma_{j=k}^{k+n+1} \exp[-2j\pi i/n] - (n-1)(n+2)} = 1,$$

that is,

(8.36) $$\omega^{n\alpha_{n+2}(b)}(1+e^{-2\pi i/n}) = 1.$$

Using (8.5) and (8.18), we get

(8.37) $$d_n^k(\varepsilon) \simeq \omega^{n\alpha_{n+2}(b)\exp[-2k\pi i/n]-(n-1)} \qquad (k=0, 1, \dots, n+1)$$

as μ tends to infinity in (3.29). Therefore,

(8.38) $$\prod_{h=0}^{(n+2)/2-1} d_n^{nh}(\varepsilon) \simeq \prod_{h=0}^{(n+2)/2-1} [\omega^{n\alpha_{n+2}(b)\exp[-2h\pi i]-(n-1)}]$$
$$\simeq (-1)^{n-1}\omega^{(n+2)n\alpha_{n+2}(b)/2}$$

as μ tends to infinity in (3.29). Here we put

$$d_n^{nh}(\varepsilon) = d_n^{h'}(\varepsilon) \qquad \text{if} \quad nh = h' \pmod{n+2}.$$

We now define a function $\tilde{d}(\varepsilon)$ which is holomorphic in (3.29) and satisfies the following two conditions:

(8.39) $$(\tilde{d}(\varepsilon))^{(n+2)/2} = (-1)^{n-1}[\prod_{h=0}^{(n+2)/2-1} d_n^{nh}(\varepsilon)]$$

and

(8.40) $$\tilde{d}(\varepsilon) \simeq \omega^{n\alpha_{n+2}(b)} \quad \text{as} \quad \mu \text{ tends to infinity in (3.29)}.$$

Similarly, it holds form (8.37) that

$$\prod_{h=0}^{(n+2)/2-1} d_n^{nh+1}(\varepsilon) \simeq (-1)^{n-1}\omega^{(1/2)n(n+2)\alpha_{n+2}(b)\exp[-2\pi i/n]}$$

as μ tends to infinity in (3.29). Therefore, there exists a function $\tilde{d}^*(\varepsilon)$ which is holomorphic in (3.29) and satisfies the following two conditions:

(8.42) $$(\tilde{d}^*(\varepsilon))^{-(n+2)/2} = (-1)^{n-1}[\prod_{h=0}^{(n+2)/2-1} d_n^{nh+1}(\varepsilon)]$$

and

(8.43) $$\tilde{d}^*(\varepsilon) \simeq \omega^{-n\alpha_{n+2}(b)\exp[-2\pi i/n]}$$

as μ tends to infinity in (3.29). Since

$$\prod_{h=0}^{n+1} d_n^h(\varepsilon) = \prod_{h=0}^{(n+2)/2-1} d_n^{nh}(\varepsilon) \times \prod_{h=0}^{(n+2)/2-1} d_n^{nh+1}(\varepsilon),$$

it follows from (8.14), (8.39) and (8.32) that

$$\left(\frac{\tilde{d}(\varepsilon)}{\tilde{d}^*(\varepsilon)}\right)^{(n+2)/2} = 1.$$

Therefore, there exists an integer p^* ($p^*=0, 1,\ldots, (n+2)/2-1$) such that

$$\frac{\tilde{d}(\varepsilon)}{\tilde{d}^*(\varepsilon)} = \exp\left[\frac{2p^*}{(n+2)/2}\pi i\right] = \omega^{2p^*}.$$

Then, using asymptotic conditions (8.40), (8.43) and (8.36), we get

(8.44) $$\omega^{2p^*} = 1, \text{ i.e., } \tilde{d}(\varepsilon) = \tilde{d}^*(\varepsilon).$$

Let us put

$$\gamma_{nh}(\varepsilon) = \frac{d_n^{nh}(\varepsilon)}{\omega^{-(n-1)}\tilde{d}(\varepsilon)}, \quad \gamma_{nh+1}(\varepsilon) = \omega^{-(n-1)} d_n^{nh+1}(\varepsilon)\tilde{d}^*(\varepsilon)$$
$$(h=0, 1,\ldots, (n+2)/2-1).$$

Then the quantities $\gamma_j(\varepsilon)$ are holomorphic in (3.29) and

(8.46) $$\gamma_j(\varepsilon) \simeq 1 \quad \text{as} \quad \mu \text{ tends to infinity in (3.29)}.$$

Furthermore, from (8.39) and (8.42), it holds that

(8.47) $$\prod_{h=0}^{(n+2)/2-1} \gamma_{nh}(\varepsilon) = 1, \quad \prod_{h=0}^{(n+2)/2-1} \gamma_{nh+1}(\varepsilon) = 1.$$

Now, let us put

(8.48) $$\begin{cases} \eta_0(t, \varepsilon) = z_0(t, \varepsilon), & \eta_{nm}(t, \varepsilon) = [\prod_{h=0}^{m-1} \gamma_{hn}(\varepsilon)] z_{nm}(t, \varepsilon), \\ \eta_1(t, \varepsilon) = z_1(t, \varepsilon), & \eta_{nm+1}(t, \varepsilon) = [\prod_{h=0}^{m-1} \gamma_{nh+1}(\varepsilon)] z_{nm+1}(t, \varepsilon), \end{cases}$$
$$(m=1, 2,\ldots, (n+2)/2-1).$$

Then $\eta_k(t, \varepsilon)$ ($k=0, 1,\ldots, n+1$) are solutions of the system (3.7) which satisfy the same conditions as $z_k(t, \varepsilon)$ ($k=0, 1,\ldots, n+1$). Furthermore, $\eta_k(t, \varepsilon)$ ($k=0, 1,\ldots, n+1$) admit connection formulas:

(8.49) $$\begin{cases} \eta_{nm}(t, \varepsilon) = \sum_{j=1}^{n-1} \hat{d}_j^{nm}(\varepsilon)\eta_{nm+j}(t, \varepsilon) + \omega^{-(n-1)}\tilde{d}(\varepsilon)\eta_{n(m+1)}(t, \varepsilon), \\ \eta_{nm+1}(t, \varepsilon) = \sum_{j=1}^{n-1} \hat{d}_j^{nm+1}(\varepsilon)\eta_{nm+1+j}(t, \varepsilon) + \dfrac{\omega^{-(n-1)}}{\tilde{d}^*(\varepsilon)}\eta_{n(m+1)+1}(t, \varepsilon) \end{cases}$$
$$(m=0, 1,\ldots, (n+2)/2-1),$$

where

$$\hat{d}_j^{nm}(\varepsilon) \simeq d_j^{nm}(\varepsilon) \simeq C_j^0(G^{nm}(b)), \tag{8.50}$$
$$\hat{d}_j^{nm+1}(\varepsilon) \simeq d_j^{nm+1}(\varepsilon) \simeq C_j^0(G^{nm+1}(b)),$$

as μ tends to infinity in the sector (3.29). On the other hand, from the asymtpotic property (8.40) of $\tilde{d}(\varepsilon)$, there exists a function $\phi(\varepsilon)$ which is holomorphic in (3.29) and satisfies the following two conditions:

$$\tilde{d}(\varepsilon)\omega^{-n\alpha_{n+2}(b)} = \omega^{\phi(\varepsilon)} \tag{8.51}$$

and

$$\phi(\varepsilon) \simeq 0 \quad \text{as } \mu \text{ tends to infinity in (3.29)}. \tag{8.52}$$

We shall now construct $n-1$ functions $\psi_{p,0}(\varepsilon)$ $(p=2, 3,\dots, n)$ so that

(a) each $\psi_{p,0}(\varepsilon)$ is holomorphic for $|\arg \mu| \leqq \delta/(n+2)$, $|\mu| \geqq M''$, where M'' is a sufficiently large positive number;

(b) $\psi_{p,0}(\varepsilon) \simeq 0$ as μ tends to infinity in (3.29).

(c)
$$\begin{cases} \dfrac{\hat{d}_1^{2h}(\varepsilon)}{\hat{d}_1^0(\varepsilon)} = \dfrac{C_1^0(G^{2h}(b(\varepsilon)+\psi(\varepsilon)))}{C_1^0(b(\varepsilon)+\psi(\varepsilon))} & (h=1, 2,\dots, n/2-1), \\ \hat{d}_1^0(\varepsilon)\hat{d}_1^{2h+1}(\varepsilon) = C_1^0(b(\varepsilon)+\psi(\varepsilon))C_1^0(G^{2h+1}(b(\varepsilon)+\psi(\varepsilon))) & \\ \qquad\qquad (h=0, 1,\dots, n/2-2), & \\ n\alpha_{n+2}(b(\varepsilon)+\psi(\varepsilon)) = n\alpha_{n+2}(b(\varepsilon)) + \phi(\varepsilon), & \end{cases} \tag{8.53}$$

where

$$\psi(\varepsilon) = (0, \psi_{2,0}(\varepsilon), \psi_{3,0}(\varepsilon),\dots, \psi_{n,0}(\varepsilon)).$$

To do this, we shall prove the following

LEMMA 8.3. *Let us put*

$$b = (0, b_{2,0}, b_{3,0},\dots, b_{n,0}), \quad \tilde{b} = (\tilde{b}_1, \tilde{b}_2,\dots, \tilde{b}_n),$$
$$\psi = (0, \psi_{2,0}, \psi_{3,0},\dots, \psi_{n,0}),$$

and

$$\begin{cases} f_{2h}(b, \psi) = \dfrac{C_1^0(G^{2h}(b+\psi))}{C_1^0(b+\psi)} - \dfrac{C_1^0(G^{2h}(b))}{C_1^0(b)} & (h=1, 2,\dots, n/2-1), \\ f_{2h+1}(b, \psi) = C_1^0(b+\psi)C_1^0(G^{2h+1}(b+\psi)) - C_1^0(b)C_1^0(G^{2h+1}(b)) & \\ \qquad\qquad (h=0, 1,\dots, n/2-2). & \end{cases}$$

Assume that ε_1 and ε_2 are sufficiently small positive numbers. Then, if

(8.54) $$\sum_{p=2}^{n} |b_{p,0}| \leqq \varepsilon_1, \quad \sum_{j=1}^{n-1} |\tilde{b}_j| \leqq \varepsilon_2,$$

there exists a unique solution

(8.55) $$\psi = g(b, \tilde{b}) = (0, g_2(b, \tilde{b}), g_3(b, \tilde{b}),\dots, g_n(b, \tilde{b}))$$

of the system of equations

(8.56) $$f_k(b, \psi) = \tilde{b}_k \quad (k=1, 2,\dots, n-2), \qquad \alpha_{n+2}(b+\psi) - \alpha_{n+2}(b) = b_{n-1},$$

such that $g_2(b, \tilde{b}), g_3(b, \tilde{b}),\dots, g_n(b, \tilde{b})$ *are holomorphic in the domain* (8.54) *and*

(8.57) $$g_j(b, 0) = 0 \qquad (j=2, 3,\dots, n).$$

PROOF. Since

$$G^k(b) = (0, \omega^{2k} b_{2,0}, \omega^{3k} b_{3,0},\dots, \omega^{nk} b_{n,0}),$$

we can easily get

$$\frac{\partial f_{2h}(b, \psi)}{\partial \psi_{j,0}}\Big|_{\substack{b=0\\ \psi=0}} = (\omega^{2hj}-1)\frac{1}{C_1^0(0)}\frac{\partial C_1^0(b)}{\partial b_{j,0}}\Big|_{b=0},$$

$$\frac{\partial f_{2h+1}(b, \psi)}{\partial \psi_{j,0}}\Big|_{\substack{b=0\\ \psi=0}} = (\omega^{(2h+1)j}+1)C_1^0(0)\frac{\partial C_1^0(b)}{\partial b_{j,0}}\Big|_{b=0}.$$

We also derive from Lemma 4.1

$$\frac{\alpha_{n+2}(b+\psi)}{\partial \psi_{j,0}}\Big|_{\substack{b=0\\ \psi=0}} = \begin{cases} 0 & (j \neq n/2+1) \\ (-1)^j \dfrac{1}{n} & (j=n/2+1). \end{cases}$$

Putting

$$\omega_{j,k} = \omega^{jk} - (-1)^k \qquad \begin{pmatrix} j=2, 3,\dots, n/2, n/2+2,\dots, n \\ k=1, 2,\dots, n-2 \end{pmatrix},$$

denote by D the determinant of the $(n-2)$ by $(n-2)$ matrix whose components are the $\omega_{j,k}$. Then

$$D = V\Big[\prod_{\substack{j=2\\ j\neq n/2+1}}^{n} (\omega^j+1)\Big],$$

where

$$V = \begin{vmatrix} 1 & \cdots\cdots 1 & 1 & \cdots\cdots 1 \\ \omega^2 & \cdots\cdots \omega^{n/2} & \omega^{n/2+2} & \cdots\cdots \omega^n \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{2(n-2)} & \cdots\cdots \omega^{n(n-2)/2} & \omega^{(n/2+2)(n-2)} & \cdots\cdots \omega^{n(n-2)} \end{vmatrix}.$$

Since

$$\omega^j + 1 \neq 0 \qquad (j=2, 3,\dots, n/2, n/2+2,\dots, n),$$

we get

$$D \neq 0.$$

Now it is easily shown that the Jacobian determinant of system (8.56) with respect to $\psi_{p,0}$ $(p=2, 3,\dots, n)$ at $b=0$, $\psi=0$ is given by

$$\frac{1}{n} \times D \times \left[\prod_{\substack{j=2 \\ j\neq n/2+1}}^{n} \frac{\partial C_1^0(b)}{\partial b_{j,0}} \bigg|_{b=0} \right].$$

Therefore, by virtue of Lemma 5.1, this Jacobian determinant is deifferent from zero. This proves Lemma 8.3.

Let us put

$$(8.58) \quad \begin{cases} \tilde{b}_{2h}(\varepsilon) = \dfrac{\hat{a}_1^{2h}(\varepsilon)}{\hat{a}_1^0(\varepsilon)} - \dfrac{C_1^0(G^{2h}(b))}{C_1^0(b)} & (h=1, 2,\dots, n/2-1), \\ \tilde{b}_{2h+1}(\varepsilon) = \hat{a}_1^0(\varepsilon)\hat{a}_1^{2h+1}(\varepsilon) - C_1^0(b)C_1^0(G^{2h+1}(b)) & \\ & (h=0, 1,\dots, n/2-2), \\ \tilde{b}_{n-1}(\varepsilon) = \dfrac{1}{n}\phi(\varepsilon), & \end{cases}$$

and

$$(8.59) \qquad \psi_j(\varepsilon) = g_j(b(\varepsilon), \tilde{b}(\varepsilon)) \qquad (j=2, 3,\dots, n).$$

Then, using Lemma 8.3, we can obtain (8.53). If we put

$$(8.60) \qquad \beta(\varepsilon) = \frac{C_1^0(b(\varepsilon)+\psi(\varepsilon))}{\hat{a}_1^0(\varepsilon)},$$

then, using (8.50), we can get that the function $\beta(\varepsilon)$ is holomorphic for

$$(8.61) \qquad |\arg \mu| \leqq \delta/(n+2), \quad |\mu| \geqq M'',$$

and

$$(8.62) \qquad \beta(\varepsilon) \simeq 1 \quad \text{as } \mu \text{ tends to infinity in (7.62)}.$$

If M'' is sufficiently large, from this asymptotic property (8.62), we may assume that $\beta(\varepsilon)\neq 0$ in (8.61). Therefore, from (8.53), it holds that

$$(8.63) \quad \begin{cases} \beta(\varepsilon)\hat{a}_1^{2h}(\varepsilon) = C_1^0(G^{2h}(b(\varepsilon)+\psi(\varepsilon))) & (h=1, 2,\dots, n/2-1), \\ \dfrac{1}{\beta(\varepsilon)}\hat{a}_1^{2h+1}(\varepsilon) = C_1^0(G^{2h+1}(b(\varepsilon)+\psi(\varepsilon))) & (h=0, 1,\dots, n/2-2). \end{cases}$$

Now, if we put

(8.64)
$$\zeta_{nm}(t,\varepsilon)=\beta(\varepsilon)\eta_{nm}(t,\varepsilon),\ \zeta_{nm+1}(t,\varepsilon)=\eta_{nm+1}(t,\varepsilon)\qquad (m=0,1,\dots,(n+2)/2-1),$$

then connection formulas (8.49) for $\eta_k(t, \varepsilon)$ become

(8.65)
$$\begin{cases}\zeta_{nm}(t,\varepsilon)=\sum_{j=1}^{n-1}\beta(\varepsilon)\hat{d}_j^{nm}(\varepsilon)\eta_{nm+j}(t,\varepsilon)+\\ \qquad +\ \omega^{-(n-1)+n\alpha_{n+2}(b+\psi)}\zeta_{n(m+1)}(t,\varepsilon),\\ \zeta_{nm+1}(t,\varepsilon)=\sum_{j=1}^{n-1}\hat{d}_j^{nm+1}(\varepsilon)\eta_{nm+1+j}(t,\varepsilon)+\\ \qquad +\ \omega^{-(n-1)-n\alpha_{n+2}(b+\psi)}\zeta_{nn(m+1)+1}(t,\varepsilon).\end{cases}$$

On the other hand, in the connection formulas (8.3) for $\hat{z}_k(t, \varepsilon)$:

(8.66)
$$\begin{cases}\hat{z}_{nm}(t,\varepsilon)=\sum_{j=1}^{n}e_j^{nm}(\varepsilon)\hat{z}_{nm+j}(t,\varepsilon),\\ \hat{z}_{nm+1}(t,\varepsilon)=\sum_{j=1}^{n}e_{jm}^{nm+1}(\varepsilon)\hat{z}_{nm+j+1}(t,\varepsilon)\end{cases}\qquad (m=0,1,\dots,n/2),$$

we can derive from (8.36) that

(8.67)
$$\begin{cases}e_n^{nm}(\varepsilon)=C_n^{nm}(b+\psi)=\omega^{n\alpha_{n+2}(b+\psi)\exp[-2nm\pi i/n]-(n-1)}\\ \qquad =\omega^{-(n-1)+n\alpha_{n+2}(b+\psi)},\\ e_n^{nm+1}(\varepsilon)=C_n^{nm+1}(b+\psi)=\omega^{n\alpha_{n+2}(b+\psi)\exp[-2(nm+1)\pi i/n]-(n-1)}\\ \qquad =\omega^{-n\alpha_{n+2}(b+\psi)-(n-1)}\end{cases}\qquad (m=0,1,\dots,n/2).$$

Furthermore, in the connection formulas (8.65), it holds that

(8.68)
$$\begin{cases}\beta(\varepsilon)\hat{d}_1^{nm'}(\varepsilon)\eta_{nm'+1}(t,\varepsilon)=\beta(\varepsilon)\hat{d}_1^{nm'}(\varepsilon)\zeta_{nm'+1}(t,\varepsilon)\\ \quad =C_1^0(G^{nm'}(b+\psi))\zeta_{nm'+1}(t,\varepsilon)\\ \quad \text{for}\quad nm'=2,4,\dots,n-2\pmod{n+2},\\ \hat{d}_1^{nm''+1}(\varepsilon)\eta_{(nm''+1)+1}(t,\varepsilon)=\dfrac{\hat{d}_1^{nm''+1}(\varepsilon)}{\beta(\varepsilon)}\zeta_{(nm''+1)+1}(t,\varepsilon)\\ \quad =C_1^0(G^{nm''+1}(b+\psi))\zeta_{(nm''+1)+1}(t,\varepsilon)\\ \quad \text{for}\quad nm''=0,2,\dots,n-4\pmod{n+2}.\end{cases}$$

Hence, if we rewrite two connection formulas (8.65), (8.66) as

(8.69)
$$\begin{cases}\zeta_k(t,\varepsilon)=\sum_{j=1}^{n}d_j^{*k}(\varepsilon)\zeta_{k+j}(t,\varepsilon),\\ \hat{z}_k(t,\varepsilon)=\sum_{j=1}^{n}e_j^k(\varepsilon)\hat{z}_{k+j}(t,\varepsilon),\end{cases}\qquad (k=0,1,\dots,n+1)$$

then it holds from (8.67) and (8.68) that

$$d_n^{*k}(\varepsilon) = C_n^k(b+\psi) = e_n^k(\varepsilon) \qquad (k=0, 1,\dots, n+1)$$

and

$$d_1^{*j}(\varepsilon) = C_1^k(b+\psi) = e_1^j(\varepsilon) \qquad (j=0, 1,\dots, n-2).$$

Then, using Lemma 7.1, we can derive

$$d_j^{*k}(\varepsilon) = e_j^k(\varepsilon) \qquad (j=1, 2,\dots, n;\ k=0, 1,\dots, n+1).$$

Similarly, if we put again

$$\Psi_k^*(t, \varepsilon) = (\zeta_k(t, \varepsilon), \zeta_{k+1}(t, \varepsilon),\dots, \zeta_{k+n-1}(t, \varepsilon)),$$

$$\Gamma_k^*(\varepsilon) = \begin{pmatrix} d_1^{*k}(\varepsilon) & 1 & & & 0 \\ d_2^{*k}(\varepsilon) & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ d_{n-1}^{*k}(\varepsilon) & 0 & & \ddots & 1 \\ d_n^{*k}(\varepsilon) & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

and

$$T_k^*(t, \varepsilon) = \Psi_k^*(t, \varepsilon)\hat{\Psi}_k(t, \varepsilon)^{-1} \qquad (k=0, 1,\dots, n+1),$$

we get

$$\Gamma_k^*(\varepsilon) = \Gamma_k^*(\varepsilon) \quad \text{and} \quad T_k^*(t, \varepsilon) = T_{k+1}^*(t, \varepsilon) \qquad (k=0, 1,\dots, n+1).$$

Therefore,

$$T^*(t, \varepsilon) = T_k^*(t, \varepsilon) = T_{k+1}^*(t, \varepsilon) = \cdots = T_{k+n+1}^*(t, \varepsilon)$$

is a desirable transformation. Thus we have obtained Theorem 1.3 for the case in which $q=2$.

§9. Uniform simplification ($q \neq 2$)

Case (II) $n=3$ and $q=3$; In this case, connection formulas (8.1) and (8.3) become

$$(9.1) \qquad \begin{cases} z_0(t, \varepsilon) = d_1^0(\varepsilon)z_1(t, \varepsilon) + d_2^0(\varepsilon)z_2(t, \varepsilon) + d_3^0(\varepsilon)z_3(t, \varepsilon), \\ z_1(t, \varepsilon) = d_1^1(\varepsilon)z_2(t, \varepsilon) + d_2^1(\varepsilon)z_3(t, \varepsilon) + d_3^1(\varepsilon)z_4(t, \varepsilon), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_5(t, \varepsilon) = d_1^5(\varepsilon)z_0(t, \varepsilon) + d_2^5(\varepsilon)z_1(t, \varepsilon) + d_3^5(\varepsilon)z_2(t, \varepsilon), \end{cases}$$

and

$$\begin{cases} \tilde{z}_0(t, \varepsilon) = e_1^0(\varepsilon)\tilde{z}_1(t, \varepsilon) + e_2^0(\varepsilon)\tilde{z}_2(t, \varepsilon) + e_3^0(\varepsilon)\tilde{z}_3(t, \varepsilon), \\ \tilde{z}_1(t, \varepsilon) = e_1^1(\varepsilon)\tilde{z}_2(t, \varepsilon) + e_2^1(\varepsilon)\tilde{z}_3(t, \varepsilon) + e_3^1(\varepsilon)\tilde{z}_4(t, \varepsilon), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \tilde{z}_5(t, \varepsilon) = e_1^5(\varepsilon)\tilde{z}_0(t, \varepsilon) + e_2^5(\varepsilon)\tilde{z}_1(t, \varepsilon) + e_3^5(\varepsilon)\tilde{z}_2(t, \varepsilon). \end{cases}$$

Furthermore, using (2.17) and the Cramer rule, we get

$$C_3^k(b) = \omega^{3\alpha_6(b)\exp[-2k\pi i/3]-3} \qquad (k=0, 1,\dots, 5). \tag{9.3}$$

Therefore, using Lemma 8.1, we can get

$$\begin{cases} d_3^k(\varepsilon), \quad d_3^{k+3}(\varepsilon) \simeq \omega^{3\alpha_6(b)\exp[-2k\pi i/3]-3}, \\ d_3^k(\varepsilon) \times d_3^{k+3}(\varepsilon) \simeq [\omega^{3\alpha_6(b)\exp[-2k\pi i/3]-3}]^2 \end{cases}$$

as μ tends to infinity in (3.29). From these results, we now define functions $\tilde{d}(\varepsilon)$, $\tilde{d}^*(\varepsilon)$, $\tilde{d}^{**}(\varepsilon)$ which are holomorphic in (3.29) and satisfy the following conditions:

$$\tilde{d}(\varepsilon) \simeq \omega^{3\alpha_6(b)-3}, \quad [\tilde{d}(\varepsilon)]^2 = d_3^0(\varepsilon)d_3^3(\varepsilon) \tag{9.4}$$

$$\tilde{d}^*(\varepsilon) \simeq \omega^{3\alpha_6(b)\exp[-2\pi i/3]-3}, \quad [\tilde{d}^*(\varepsilon)]^2 = d_3^1(\varepsilon)d_3^4(\varepsilon) \tag{9.5}$$

$$\tilde{d}^{**}(\varepsilon) \simeq \omega^{3\alpha_6(b)\exp[-4\pi i/3]-3}, \quad [\tilde{d}^{**}(\varepsilon)]^2 = d_3^2(\varepsilon)d_3^5(\varepsilon) \tag{9.6}$$

as μ tends to infinity in (3.29). Then it hods from the asymptotic property of $\tilde{d}(\varepsilon)$, $\tilde{d}^*(\varepsilon)$, $\tilde{d}^{**}(\varepsilon)$ that

$$\tilde{d}(\varepsilon)\tilde{d}^*(\varepsilon)\tilde{d}^{**}(\varepsilon) = -1. \tag{9.7}$$

Let us put

$$\begin{cases} \gamma_0(\varepsilon) = d_3^0(\varepsilon)/\tilde{d}(\varepsilon), & \gamma_3(\varepsilon) = d_3^3(\varepsilon)/\tilde{d}(\varepsilon), \\ \gamma_1(\varepsilon) = d_3^1(\varepsilon)/\tilde{d}^*(\varepsilon), & \gamma_4(\varepsilon) = d_3^4(\varepsilon)/\tilde{d}^*(\varepsilon), \\ \gamma_2(\varepsilon) = d_3^2(\varepsilon)/\tilde{d}^{**}(\varepsilon), & \gamma_5(\varepsilon) = d_3^5(\varepsilon)/\tilde{d}^{**}(\varepsilon). \end{cases} \tag{9.8}$$

Then the quantities $\gamma_j(\varepsilon)$ $(j=0, 1,\dots, 5)$ are holomorphic in (3.29) and

$$\begin{gathered} \gamma_0(\varepsilon)\gamma_3(\varepsilon) = 1, \quad \gamma_1(\varepsilon)\gamma_4(\varepsilon) = 1, \quad \gamma_2(\varepsilon)\gamma_5(\varepsilon) = 1, \\ \gamma_j(\varepsilon) \simeq 1 \quad \text{as} \quad \mu \text{ tends to infinity in (3.29)}. \end{gathered} \tag{9.9}$$

Now, let us put

$$\begin{cases} \eta_0(t, \varepsilon) = z_0(t, \varepsilon), \quad \eta_1(t, \varepsilon) = z_1(t, \varepsilon), \quad \eta_2(t, \varepsilon) = z_2(t, \varepsilon), \\ \eta_3(t, \varepsilon) = z_3(t, \varepsilon)\gamma_0(\varepsilon), \quad \eta_4(t, \varepsilon) = z_4(t, \varepsilon)\gamma_1(\varepsilon), \\ \eta_5(t, \varepsilon) = z_5(t, \varepsilon)\gamma_2(\varepsilon). \end{cases}$$

Then $\eta_k(t, \varepsilon)$ $(k=0, 1,\dots, 5)$ are solutions of the system (3.7) which satisfy the same condition as $z_k(t, \varepsilon)$. Furthermore, from the condition (9.9), the connection formulas (9.1) become

$$
(9.10)\quad \begin{cases}
\eta_0(t, \varepsilon) = d_1^0(\varepsilon)\eta_1(t, \varepsilon) + d_2^0(\varepsilon)\eta_2(t, \varepsilon) + \tilde{d}(\varepsilon)\eta_3(t, \varepsilon), \\
\eta_1(t, \varepsilon) = d_1^1(\varepsilon)\eta_2(t, \varepsilon) + \dfrac{d_2^1(\varepsilon)}{\gamma_0(\varepsilon)}\eta_3(t, \varepsilon) + \tilde{d}^*(\varepsilon)\eta_4(t, \varepsilon), \\
\eta_2(t, \varepsilon) = \dfrac{d_1^2(\varepsilon)}{\gamma_0(\varepsilon)}\eta_3(t, \varepsilon) + \dfrac{d_2^2(\varepsilon)}{\gamma_1(\varepsilon)}\eta_4(t, \varepsilon) + \tilde{d}^{**}(\varepsilon)\eta_5(t, \varepsilon), \\
\eta_3(t, \varepsilon) = \dfrac{\gamma_0(\varepsilon)}{\gamma_1(\varepsilon)}d_1^3(\varepsilon)\eta_4(t, \varepsilon) + \dfrac{\gamma_0(\varepsilon)}{\gamma_2(\varepsilon)}d_2^3(\varepsilon)\eta_5(t, \varepsilon) + \tilde{d}(\varepsilon)\eta_0(t, \varepsilon), \\
\eta_4(t, \varepsilon) = \dfrac{\gamma_1(\varepsilon)}{\gamma_2(\varepsilon)}d_1^4(\varepsilon)\eta_5(t, \varepsilon) + \gamma_1(\varepsilon)d_2^4(\varepsilon)\eta_0(t, \varepsilon) + \tilde{d}^*(\varepsilon)\eta_1(t, \varepsilon), \\
\eta_5(t, \varepsilon) = \gamma_2(\varepsilon)d_1^5(\varepsilon)\eta_0(t, \varepsilon) + \gamma_2(\varepsilon)d_2^5(\varepsilon)\eta_1(t, \varepsilon) + \tilde{d}^{**}(\varepsilon)\eta_2(t, \varepsilon).
\end{cases}
$$

On the other hand, from the definition of $\tilde{d}(\varepsilon)$ and $\tilde{d}^*(\varepsilon)$, there exist functions $\phi_1(\varepsilon)$ and $\phi_2(\varepsilon)$ which are holomorphic in (3.29) and satisfy the following conditions;

$$\tilde{d}(\varepsilon) = \omega^{3\alpha_6(b)-3+\phi_1(\varepsilon)}, \quad \tilde{d}^*(\varepsilon) = \omega^{3\alpha_6(b)\exp[-2\pi i/3]-3+\phi_2(\varepsilon)}$$

and

$$\phi_1(\varepsilon),\ \phi_2(\varepsilon) \simeq 0 \quad \text{as} \quad \mu \text{ tends to infinity in (3.29)}.$$

Then, from (9.7), it follows that

$$(9.11)\qquad \tilde{d}^{**}(\varepsilon) = \omega^{3\alpha_6(b)\exp[-4\pi i/3]-3-\phi_1(\varepsilon)-\phi_2(\varepsilon)}.$$

We shall construct functions $\psi_{2,0}(\varepsilon)$, $\psi_{2,1}(\varepsilon)$, $\psi_{3,0}(\varepsilon)$, $\psi_{3,1}(\varepsilon)$ so that

(a) each $\psi_{p,r}(\varepsilon)$ is holomorphic for (3.29);

(b) $\psi_{p,r}(\varepsilon) \simeq 0$ as μ tends to infinity in (3.29);

(c)
$$
(9.12)\quad \begin{cases}
\dfrac{d_1^3(\varepsilon)\gamma_0(\varepsilon)}{d_1^0(\varepsilon)\gamma_1(\varepsilon)} = \dfrac{C_1^0(G^3(b(\varepsilon)+\psi(\varepsilon)))}{C_1^0(b(\varepsilon)+\psi(\varepsilon))}, \\
\dfrac{d_2^0(\varepsilon)d_1^2(\varepsilon)}{\gamma_0(\varepsilon)} = C_2^0(b(\varepsilon) + \psi(\varepsilon))C_1^0(G^2(b(\varepsilon)+\psi(\varepsilon))), \\
3\alpha_6(b(\varepsilon) + \psi(\varepsilon)) = 3\alpha_6(b(\varepsilon)) + \phi_1(\varepsilon), \\
3\alpha_6(b(\varepsilon) + \psi(\varepsilon))\exp[-2\pi i/3] = 3\alpha_6(b(\varepsilon))\exp[-2\pi i/3] + \phi_2(\varepsilon).
\end{cases}
$$

To do this, we shall prove the following lemma.

LEMMA 9.1. *Let*

$$b = (b_{2,0}, b_{2,1}, b_{3,0}, b_{3,1}), \quad \tilde{b} = (\tilde{b}_{2,0}, \tilde{b}_{2,1}, \tilde{b}_{3,0}, \tilde{b}_{3,1}),$$
$$\psi = (\psi_{2,0}, \psi_{2,1}, \psi_{3,0}, \psi_{3,1})$$

and

$$F(b, \psi) = \frac{C_1^0(G^3(b+\psi))}{C_1^0(b+\psi)} - \frac{C_1^0(G^3(b))}{C_1^0(b)},$$

$$F^*(b, \psi) = C_2^0(b+\psi)C_1^0(G^2(b+\psi)) - C_2^0(b)C_1^0(G^2(b)).$$

Assume that ε_1 and ε_2 are sufficiently small positive numbers. Then, if

$$(9.13) \qquad \begin{aligned} &|b_{2,0}| + |b_{2,1}| + |b_{3,0}| + |b_{3,1}| \leqq \varepsilon_1, \\ &|\tilde{b}_{2,0}| + |\tilde{b}_{2,1}| + |\tilde{b}_{3,0}| + |\tilde{b}_{3,1}| \leqq \varepsilon_2, \end{aligned}$$

there exists a unique solution

$$\psi = g(b, \tilde{b}) = (g_{2,0}(b, \tilde{b}), g_{2,1}(b, \tilde{b}), g_{3,0}(b, \tilde{b}), g_{3,1}(b, \tilde{b}))$$

of the systems of equations

$$(9.14) \qquad \begin{aligned} &F(b, \psi) = \tilde{b}_{2,0}, \quad F^*(b, \psi) = \tilde{b}_{2,1}, \\ &\alpha_6(b+\psi) - \alpha_6(b) = \tilde{b}_{3,0}, \ \alpha_6(G(b+\psi)) - \alpha_6(G(b)) = \tilde{b}_{3,1}, \end{aligned}$$

such that $g_{p,r}(b, \tilde{b})$ are holomorphic in the domain (3.29), *and*

$$g_{p,r}(b, 0) = 0 \qquad (p=2, 3;\ r=0, 1).$$

PROOF. We defined $G^k(b)$ $(k=0, 1, \dots, 5)$ by

$$G^k(b) = (\omega^{2k}b_{2,0}, \omega^{3k}b_{2,1}, \omega^{3k}b_{3,0}, \omega^{4k}b_{3,1}).$$

Furthermore, we put from Lemma 5.1

$$f = \frac{\partial C_1^0(b)}{\partial b_{3,0}}\bigg|_{b=0} \times b_{3,0} + \frac{\partial C_1^0(b)}{\partial b_{2,1}}\bigg|_{b=0} \times b_{2,1}.$$

Then the Jacobian determinant of the system (9.14) with respect to $\psi_{2,0}$, $\psi_{3,1}$ and f at $b=0$, $\psi=0$ (i.e. $f=0$) is given by

$$\left| \begin{matrix} \dfrac{\partial F(b,\psi)}{\partial b_{2,0}} & \dfrac{\partial F(b,\psi)}{\partial f} & \dfrac{\partial F(b,\psi)}{\partial b_{3,1}} \\[2ex] \dfrac{\partial \alpha_6(b+\psi)}{\partial b_{2,0}} & \dfrac{\partial \alpha_6(b+\psi)}{\partial f} & \dfrac{\partial \alpha_6(b+\psi)}{\partial b_{3,1}} \\[2ex] \dfrac{\partial \alpha_6(G(b+\psi))}{\partial b_{2,0}} & \dfrac{\partial \alpha_6(G(b+\psi))}{\partial f} & \dfrac{\partial \alpha_6(G(b+\psi))}{\partial b_{3,1}} \end{matrix} \right|_{\substack{b=0 \\ \psi=0}}$$

$$
= \begin{vmatrix} * & \dfrac{\omega^3-1}{C_1^0(0)} & * \\ \dfrac{1}{3} & 0 & -\dfrac{1}{3} \\ \dfrac{1}{3}\omega^2 & 0 & -\dfrac{1}{3}\omega^4 \end{vmatrix} = \frac{(\omega^3-1)(\omega^4-\omega^2)}{9\times C_1^0(0)} \neq 0.
$$

Therefore, we have determined $\psi_{2,0}(\varepsilon)$ and $\psi_{3,1}(\varepsilon)$. Next, we fix them and consider the following Jacobian determinant:

$$
\begin{vmatrix} \dfrac{\partial f}{\partial b_{3,0}} & \dfrac{\partial f}{\partial b_{2,1}} \\ \dfrac{\partial F^*(b,\psi)}{\partial b_{3,0}} & \dfrac{\partial F^*(b,\psi)}{\partial b_{2,1}} \end{vmatrix}_{b=0,\psi=0}
$$

$$
= \begin{vmatrix} \dfrac{\partial C_1^0(b)}{\partial b_{3,0}} & \dfrac{\partial C_1^0(b)}{\partial b_{2,1}} \\ \dfrac{\partial C_2^0(b)}{\partial b_{3,0}} C_1^0(0) + \dfrac{\partial C_1^0(b)}{\partial b_{3,0}} C_2^0(0)\omega^6 & \dfrac{\partial C_2^0(b)}{\partial b_{2,1}} C_1^0(0) + \dfrac{\partial C_1^0(b)}{\partial b_{2,1}} C_2^0(0)\omega^6 \end{vmatrix}_{\substack{b=0\\ \psi=0}}
$$

$$
= \begin{vmatrix} \dfrac{\partial C_1^0(b)}{\partial b_{3,0}} & \dfrac{\partial C_2^0(b)}{\partial b_{2,1}} \\ \dfrac{\partial C_1^0(b)}{\partial b_{3,0}} & \dfrac{\partial C_2^0(b)}{\partial b_{2,1}} \end{vmatrix} \times C_1^0(0).
$$

Therefore, by virtue of Lemma 6.1, this determinant is different from zero. Thus we have proved Lemma 9.1.

Let us put

$$
\begin{aligned}
(9.15) \qquad & \tilde{b}_{2,0} = \frac{d_1^3(\varepsilon)\gamma_0(\varepsilon)}{d_1^0(\varepsilon)\gamma_1(\varepsilon)} - \frac{C_1^0(G^3(b))}{C_1^0(b)}, \\
& \tilde{b}_{2,1} = \frac{d_2^0(\varepsilon)d_1^2(\varepsilon)}{\gamma_0(\varepsilon)} - C_2^0(b)C_1^0(G^2(b)), \\
& \tilde{b}_{3,0} = \phi_1(\varepsilon)/3, \quad \tilde{b}_{3,1} = \phi_2(\varepsilon),
\end{aligned}
$$

and

$$
\psi = g(b, \tilde{b}).
$$

Then, from Lemma 9.1, we can obtain (9.12). Furthermore, if we put

$$
(9.16) \qquad \beta_1(\varepsilon) = \frac{C_1^0(b(\varepsilon)+\psi(\varepsilon))}{d_1^0(\varepsilon)} \quad \text{and} \quad \beta_2(\varepsilon) = \frac{C_2^0(b(\varepsilon)+\psi(\varepsilon))}{d_2^0(\varepsilon)},
$$

then the functions $\beta_1(\varepsilon)$ and $\beta_2(\varepsilon)$ are holomorphic for (3.29) and

$$\beta_1(\varepsilon),\ \beta_2(\varepsilon) \simeq 1 \quad \text{as} \quad \mu \text{ tends to infinity in (3.29)}. \tag{9.17}$$

Here we used (8.5). From (9.16) and (9.12), we can derive

$$\begin{cases} \beta_1(\varepsilon)\dfrac{d_1^3(\varepsilon)\gamma_0(\varepsilon)}{\gamma_1(\varepsilon)} = C_1^0(G^3(b(\varepsilon) + \psi(\varepsilon))), \\ \dfrac{d_1^2(\varepsilon)}{\beta_2(\varepsilon)\gamma_0(\varepsilon)} = C_1^0(G^2(b(\varepsilon) + \psi(\varepsilon))), \\ \tilde{d}(\varepsilon) = \omega^{3\alpha_6(b(\varepsilon)+\psi(\varepsilon))-3} = C_3^0(b(\varepsilon)+\psi(\varepsilon)) = C_3^0(G^3(b(\varepsilon) + \psi(\varepsilon))), \\ \tilde{d}^*(\varepsilon) = \omega^{3\alpha_6(G(b(\varepsilon)+\psi(\varepsilon)))-3} = C_3^0(G(b(\varepsilon)+\psi(\varepsilon))) = C_3^0(G^4(b+\psi)), \\ \tilde{d}^{**}(\varepsilon) = \omega^{3\alpha_6(G^2(b(\varepsilon)+\psi(\varepsilon)))-3} = C_3^0(G^2(b(\varepsilon)+\psi(\varepsilon))) = C_3^0(G^5(b+\psi)). \end{cases} \tag{9.18}$$

Therefore, if we put

$$\begin{cases} \eta_0(t,\varepsilon) = \xi_0(t,\varepsilon), & \eta_3(t,\varepsilon) = \xi_3(t,\varepsilon), \\ \eta_1(t,\varepsilon) = \beta_1(\varepsilon)\xi_1(t,\varepsilon), & \eta_4(t,\varepsilon) = \beta_1(\varepsilon)\xi_4(t,\varepsilon), \\ \eta_2(t,\varepsilon) = \beta_2(\varepsilon)\xi_2(t,\varepsilon), & \eta_5(t,\varepsilon) = \beta_2(\varepsilon)\xi_5(t,\varepsilon), \end{cases}$$

the connection formulas (9.10) for $\eta_k(t,\varepsilon)$ become

$$\begin{cases} \xi_0(t,\varepsilon) = e_1^0(\varepsilon)\xi_1(t,\varepsilon) + e_2^0(\varepsilon)\xi_2(t,\varepsilon) + e_3^0(\varepsilon)\xi_3(t,\varepsilon), \\ \xi_1(t,\varepsilon) = d_1^1(\varepsilon)\dfrac{\beta_2(\varepsilon)}{\beta_1(\varepsilon)}\xi_2(t,\varepsilon) + \dfrac{d_2^1(\varepsilon)}{\gamma_0(\varepsilon)\beta_1(\varepsilon)}\xi_3(t,\varepsilon) + e_3^1(\varepsilon)\xi_4(t,\varepsilon), \\ \xi_2(t,\varepsilon) = e_1^2(\varepsilon)\xi_3(t,\varepsilon) + \dfrac{d_2^2(\varepsilon)\beta_1(\varepsilon)}{\gamma_1(\varepsilon)\beta_2(\varepsilon)}\xi_4(t,\varepsilon) + e_3^2(\varepsilon)\xi_5(t,\varepsilon), \\ \xi_3(t,\varepsilon) = e_1^3(\varepsilon)\xi_4(t,\varepsilon) + \dfrac{\gamma_0(\varepsilon)d_2^3(\varepsilon)\beta_2(\varepsilon)}{\gamma_2(\varepsilon)}\xi_5(t,\varepsilon) + e_3^3(\varepsilon)\xi_0(t,\varepsilon), \\ \xi_4(t,\varepsilon) = \dfrac{\gamma_1(\varepsilon)d_1^4(\varepsilon)\beta_2(\varepsilon)}{\gamma_2(\varepsilon)\beta_1(\varepsilon)}\xi_5(t,\varepsilon) + \dfrac{\gamma_1(\varepsilon)d_2^4(\varepsilon)}{\beta_1(\varepsilon)}\xi_0(t,\varepsilon) + e_3^4(\varepsilon)\xi_1(t,\varepsilon), \\ \xi_5(t,\varepsilon) = \dfrac{\gamma_2(\varepsilon)d_1^5(\varepsilon)}{\beta_2(\varepsilon)}\xi_0(t,\varepsilon) + \dfrac{\gamma_2(\varepsilon)d_2^5(\varepsilon)\beta_1(\varepsilon)}{\beta_2(\varepsilon)}\xi_1(t,\varepsilon) + e_3^5(\varepsilon)\xi_2(t,\varepsilon), \end{cases} \tag{9.19}$$

We rewrite these connection formulas as

$$\xi_k(t,\varepsilon) = d_1^{*k}(\varepsilon)\xi_{k+1}(t,\varepsilon) + d_2^{*k}(\varepsilon)\xi_{k+2}(t,\varepsilon) + d_3^{*k}(\varepsilon)\xi_{k+3}(t,\varepsilon)$$
$$(k=0, 1, \dots, 5).$$

Then, using Lemma 7.2, we can derive

$$d_j^{*k}(\varepsilon) = e_j^k(\varepsilon) \qquad (k=0, 1, \dots, 5;\ j=1, 2, 3).$$

Thus we have obtained Theorem 1.3 for the case (II).

Case (III) $n=3$ *and* $q=4$; Since

$$C_n^k(b) = (-1)^{n-1}\exp[q(n-1)\pi i/2]\,\omega^{n\alpha_{n+q}(b)\exp[-2k\pi i/n]-(n-1)q/2}$$
$$(k=0, 1,\dots, n+q-1),$$

in this case, it follows that

$$C_3^k(b) = \omega^{3\alpha_7(b)\exp[-2k\pi i/3]-4} \qquad (k=0, 1,\dots, 6).$$

Then, using (8.14), we can easily derive

$$\omega^{3\alpha_7(b)\exp[-2k\pi i/3]} = 1 \qquad (k=0, 1,\dots, 6).$$

It follows from Lemma 8.1 that

$$C_3^k(b) = \omega^{-4} \simeq d_3^k(\varepsilon) \qquad (k=0, 1,\dots, 6). \tag{9.20}$$

If we put

$$\gamma_k(\varepsilon) = d_3^k(\varepsilon)/\omega^{-4} \qquad (k=0, 1,\dots, 6), \tag{9.21}$$

the quantities $\gamma_k(\varepsilon)$ are holomorphic in (3.29) and

$$\gamma_k(\varepsilon) \simeq 1 \quad \text{as} \quad \mu \text{ tends to infinity in (3.29)}. \tag{9.22}$$

From (8.14) and (9.21), it holds that

$$\gamma_0(\varepsilon)\gamma_1(\varepsilon)\gamma_2(\varepsilon)\gamma_3(\varepsilon)\gamma_4(\varepsilon)\gamma_5(\varepsilon)\gamma_6(\varepsilon) = 1. \tag{9.23}$$

Furthermore, let us put

$$\begin{cases} \xi_0(t, \varepsilon) = z_0(t, \varepsilon), \\ \xi_1(t, \varepsilon) = \gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)\gamma_5(\varepsilon)z_1(t, \varepsilon), \\ \xi_2(t, \varepsilon) = \gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)z_2(t, \varepsilon), \\ \xi_3(t, \varepsilon) = \gamma_0(\varepsilon)z_3(t, \varepsilon), \\ \xi_4(t, \varepsilon) = \gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)\gamma_5(\varepsilon)\gamma_1(\varepsilon)z_4(t, \varepsilon), \\ \xi_5(t, \varepsilon) = \gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)z_5(t, \varepsilon), \\ \xi_6(t, \varepsilon) = \gamma_0(\varepsilon)\gamma_3(\varepsilon)z_6(t, \varepsilon). \end{cases} \tag{9.24}$$

Then, from (9.23), the connection formulas for $z_k(t, \varepsilon)$ become

$$\begin{cases} \xi_0(t,\varepsilon) = [d_1^0(\varepsilon)/\gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)\gamma_5(\varepsilon)]\xi_1(t,\varepsilon) \\ \qquad + [d_2^0(\varepsilon)/\gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)]\xi_2(t,\varepsilon) + \omega^{-4}\xi_3(t,\varepsilon), \\ \xi_1(t,\varepsilon) = \gamma_5(\varepsilon)\gamma_2(\varepsilon)d_1^1(\varepsilon)\xi_2(t,\varepsilon) + \gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_1(\varepsilon)d_2^1(\varepsilon)\xi_3(t,\varepsilon) + \omega^{-4}\xi_4(t,\varepsilon), \\ \xi_2(t,\varepsilon) = \gamma_3(\varepsilon)\gamma_6(\varepsilon)d_1^2(\varepsilon)\xi_3(t,\varepsilon) + [d_2^2(\varepsilon)/\gamma_5(\varepsilon)\gamma_2(\varepsilon)\gamma_1(\varepsilon)]\xi_4(t,\varepsilon) + \omega^{-4}\xi_5(t,\varepsilon), \\ \xi_3(t,\varepsilon) = [d_1^3(\varepsilon)/\gamma_5(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)\gamma_1(\varepsilon)]\xi_4(t,\varepsilon) \\ \qquad + [d_2^3(\varepsilon)/\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)]\xi_5(t,\varepsilon) + \omega^{-4}\xi_6(t,\varepsilon), \\ \xi_4(t,\varepsilon) = \gamma_5(\varepsilon)\gamma_1(\varepsilon)d_1^4(\varepsilon)\xi_5(t,\varepsilon) + \gamma_5(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)\gamma_1(\varepsilon)d_2^4(\varepsilon)\xi_6(t,\varepsilon) + \omega^{-4}\xi_0(t,\varepsilon), \\ \xi_5(t,\varepsilon) = \gamma_6(\varepsilon)\gamma_2(\varepsilon)d_1^5(\varepsilon)\xi_6(t,\varepsilon) + \gamma_0(\varepsilon)\gamma_3(\varepsilon)\gamma_6(\varepsilon)\gamma_2(\varepsilon)d_2^5(\varepsilon)\xi_0(t,\varepsilon) + \omega^{-4}\xi_1(t,\varepsilon), \\ \xi_6(t,\varepsilon) = \gamma_3(\varepsilon)\gamma_0(\varepsilon)d_1^6(\varepsilon)\xi_0(t,\varepsilon) + [d_2^6(\varepsilon)/\gamma_2(\varepsilon)\gamma_5(\varepsilon)\gamma_6(\varepsilon)]\xi_1(t,\varepsilon) + \omega^{-4}\xi_2(t,\varepsilon). \end{cases}$$

These connection formulas are written as

$$\xi_k(t,\varepsilon) = \hat{d}_1^k(\varepsilon)\xi_{k+1}(t,\varepsilon) + \hat{d}_2^k(\varepsilon)\xi_{k+2}(t,\varepsilon) + \omega^{-4}\xi_{k+3}(t,\varepsilon), \tag{9.25}$$

where $\hat{d}_1^k(\varepsilon)$ and $\hat{d}_1^k(\varepsilon)$ $(k=0, 1,\dots 6)$ are holomorphic in (3.29) and

$$\hat{d}_1^k(\varepsilon) \simeq C_1^k(b(\varepsilon)), \quad \hat{d}_2^k(\varepsilon) \simeq C_2^k(b(\varepsilon)). \tag{9.26}$$

We shall construct functions $\psi_{p,r}(\varepsilon)$ $(p=2, 3;\ r=0, 1, 2)$ so that

(a) each $\psi_{p,r}(\varepsilon)$ is holomorphic for (3.29);

(b) $\psi_{p,r}(\varepsilon) \simeq 0$ as μ tends to infinity in (3.29).

(c) $C_1^k(b(\varepsilon)+\psi(\varepsilon)) = \hat{d}_1^k(\varepsilon) \qquad (k=0, 1, -1(=6), -2(=5)),$

$C_2^j(b(\varepsilon)+\psi(\varepsilon)) = \hat{d}_2^j(\varepsilon) \qquad (j=0, 1).$

To do this, we need the following lemma.

LEMMA 9.2. *Let*

$$b = (b_{2,0}, b_{3,0}, b_{2,1}, b_{3,1}, b_{2,2}, b_{3,2}),$$

$$\tilde{b} = (\tilde{b}_{2,0}, \tilde{b}_{3,0}, \tilde{b}_{2,1}, \tilde{b}_{3,1}, \tilde{b}_{2,2}, \tilde{b}_{3,2}),$$

$$\psi = (\psi_{2,0}, \psi_{3,0}, \psi_{2,1}, \psi_{3,1}, \psi_{2,2}, \psi_{3,2})$$

and

$$F_k(b,\psi) = C_1^0(G^k(b+\psi)) - C_1^0(G^k(b)) \qquad (k=0, 1, -1, -2),$$

$$F_j^*(b,\psi) = C_2^0(G^j(b+\psi)) - C_2^0(G^j(b)) \qquad (j=0, 1).$$

Assume that ε_1 and ε_2 are sufficiently small positive numbers. Then, if

$$(9.27)\qquad \begin{aligned} &|b_{2,0}| + |b_{3,0}| + |b_{2,1}| + |b_{3,1}| + |b_{2,2}| + |b_{3,2}| \leqq \varepsilon_1, \\ &|\tilde{b}_{2,0}| + |\tilde{b}_{3,0}| + |\tilde{b}_{2,1}| + |\tilde{b}_{3,1}| + |\tilde{b}_{2,2}| + |\tilde{b}_{3,2}| \leqq \varepsilon_2, \end{aligned}$$

there exists a unique solution

$$\psi = g(b, \tilde{b}) = (g_{2,0}(b, \tilde{b}), \ldots, g_{3,2}(b, \tilde{b}))$$

of the systems of equations

$$(9.28)\qquad F_0(b, \psi) = \tilde{b}_{2,0}, \quad F_1(b, \psi) = \tilde{b}_{3,0}, \quad F_{-1}(b, \psi) = \tilde{b}_{2,1}, \quad F_{-2}(b, \psi) = \tilde{b}_{3,1},$$

$$(9.29)\qquad F_0^*(b, \psi) = \tilde{b}_{2,2}, \quad F_1^*(b, \psi) = \tilde{b}_{3,2},$$

such that $g_{p,r}(b, \tilde{b})$ *are holomorphic in the domain* (9.27) *and*

$$g_{p,r}(b, 0) = 0 \qquad (p=2, 3;\ r=0, 1, 2).$$

PROOF. Using Lemma 5.1, we put

$$f_1 = \frac{\partial C_1^0(b)}{\partial b_{3,0}}\bigg|_{b=0} \times b_{3,0} + \frac{\partial C_1^0(b)}{\partial b_{2,1}}\bigg|_{b=0} \times b_{2,1}$$

and

$$f_2 = \frac{\partial C_1^0(b)}{\partial b_{3,1}}\bigg|_{b=0} \times b_{3,1} + \frac{\partial C_1^0(b)}{\partial b_{2,2}}\bigg|_{b=0} \times b_{2,2}.$$

Since we defined $G^k(b)$ $(k=0, 1, \ldots, 6)$ as

$$G^k(b) = (\omega^{2k} b_{2,0}, \omega^{3k} b_{3,0}, \omega^{3k} b_{2,1}, \omega^{4k} b_{3,1}, \omega^{4k} b_{2,2}, \omega^{5k} b_{3,2}),$$

we can easily obtain

$$\begin{vmatrix} \dfrac{\partial F_0(b, \psi)}{\partial b_{2,0}} & \dfrac{\partial F_0(b, \psi)}{\partial f_1} & \dfrac{\partial F_0(b, \psi)}{\partial f_2} & \dfrac{\partial F_0(b, \psi)}{\partial b_{3,2}} \\ \dfrac{\partial F_1(b, \psi)}{\partial b_{2,0}} & \dfrac{\partial F_1(b, \psi)}{\partial f_1} & \dfrac{\partial F_1(b, \psi)}{\partial f_2} & \dfrac{\partial F_1(b, \psi)}{\partial b_{3,2}} \\ \dfrac{\partial F_{-1}(b, \psi)}{\partial b_{2,0}} & \dfrac{\partial F_{-1}(b, \psi)}{\partial f_1} & \dfrac{\partial F_{-1}(b, \psi)}{\partial f_2} & \dfrac{\partial F_{-1}(b, \psi)}{\partial b_{3,2}} \\ \dfrac{\partial F_{-2}(b, \psi)}{\partial b_{2,0}} & \dfrac{\partial F_{-2}(b, \psi)}{\partial f_1} & \dfrac{\partial F_{-2}(b, \psi)}{\partial f_2} & \dfrac{\partial F_{-2}(b, \psi)}{\partial b_{3,2}} \end{vmatrix}_{b=0, \psi=0}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} \\ \omega^{-4} & \omega^{-6} & \omega^{-8} & \omega^{-10} \end{vmatrix} \frac{\partial C_1^0(b)}{\partial b_{2,0}}\bigg|_{b=0} \times \frac{\partial C_1^0(b)}{\partial b_{3,2}}\bigg|_{b=0}$$

and

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial b_{3,0}} & \dfrac{\partial f_1}{\partial b_{2,1}} & \dfrac{\partial f_1}{\partial b_{3,1}} & \dfrac{\partial f_1}{\partial b_{2,2}} \\ \dfrac{\partial f_2}{\partial b_{3,0}} & \dfrac{\partial f_2}{\partial b_{2,1}} & \dfrac{\partial f_2}{\partial b_{3,1}} & \dfrac{\partial f_2}{\partial b_{2,2}} \\ \dfrac{\partial F_0^*(b,\psi)}{\partial b_{3,0}} & \dfrac{\partial F_0^*(b,\psi)}{\partial b_{2,1}} & \dfrac{\partial F_0^*(b,\psi)}{\partial b_{3,1}} & \dfrac{\partial F_0^*(b,\psi)}{\partial b_{2,2}} \\ \dfrac{\partial F_0^*(b,\psi)}{\partial b_{3,0}} & \dfrac{\partial F_0^*(b,\psi)}{\partial b_{2,1}} & \dfrac{\partial F_0^*(b,\psi)}{\partial b_{3,1}} & \dfrac{\partial F_0^*(b,\psi)}{\partial b_{2,2}} \end{vmatrix}_{b=0,\psi=0}$$

$$= \begin{vmatrix} \dfrac{\partial C_1^0(b)}{\partial b_{3,0}} & \dfrac{\partial C_1^0(b)}{\partial b_{2,1}} & 0 & 0 \\ 0 & 0 & \dfrac{\partial C_0^1(b)}{\partial b_{3,1}} & \dfrac{\partial C_1^0(b)}{\partial b_{2,2}} \\ \dfrac{\partial C_2^0(b)}{\partial b_{3,0}} & \dfrac{\partial C_2^0(b)}{\partial b_{2,1}} & \dfrac{\partial C_2^0(b)}{\partial b_{3,1}} & \dfrac{\partial C_2^0(b)}{\partial b_{2,2}} \\ \dfrac{\partial C_2^0(b)}{\partial b_{3,0}}\omega^3 & \dfrac{\partial C_2^0(b)}{\partial b_{2,1}}\omega^3 & \dfrac{\partial C_2^0(b)}{\partial b_{3,1}}\omega^4 & \dfrac{\partial C_2^0(b)}{\partial b_{2,2}}\omega^4 \end{vmatrix}_{b=0}$$

Therefore, by virtue of Lemmas 5.1 and 6.1, these Jacobian determinants are different from zero. Thus we have proved Lemma 9.2.

If we put

$$\begin{cases} \tilde{b}_{2,0} = \hat{a}_1^0(\varepsilon) - C_1^0(b(\varepsilon)), & \tilde{b}_{3,0} = \hat{a}_1^1(\varepsilon) - C_1^0(G(b(\varepsilon))), \\ \tilde{b}_{2,1} = \hat{a}_1^{-1}(\varepsilon) - C_1^0(G^{-1}(b(\varepsilon))), & \tilde{b}_{3,1} = \hat{a}_1^{-2}(\varepsilon) - C_1^0(G^{-2}(b(\varepsilon))), \\ \tilde{b}_{2,2} = \hat{a}_2^0(\varepsilon) - C_2^0(b(\varepsilon)), & \tilde{b}_{3,2} = \hat{a}_2^1(\varepsilon) - C_2^0(G(\varepsilon))), \end{cases}$$

then we can obtain from Lemma 9.2 that

$$\hat{a}_1^k(\varepsilon) = C_1^k(b(\varepsilon) + \psi(\varepsilon)) = e_1^k(\varepsilon) \qquad (k=0,\ 1,\ -1,\ -2) \tag{9.30}$$

and

$$\hat{a}_2^j(\varepsilon) = C_2^j(b(\varepsilon) + \psi(\varepsilon)) = e_2^j(\varepsilon) \qquad (j=0,\ 1). \tag{9.31}$$

Using Lemma 7.3, we derive from (9.30) and (9.31)

$$\hat{a}_j^k(\varepsilon) = e_j^k(\varepsilon) \qquad (k=0,\ 1,\dots,\ 6;\ j=1,\ 2,\ 3).$$

Thus we have proved Theorem 1.3 for case (III).

The proof of Theorem 1.3 for cases (IV), (V), (VI) is quite similar to that of case (III). Therefore, we have finished the proof of the main theorem on full uniform simplification of the turning point in this paper.

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