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On the Adams-Novikov spectral sequence and products of β-elements

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§1. Introduction

Let p be a given prime $\geqq 5$ and BP the Brown-Peterson spectrum at p; and consider the Hopf algebroid (cf. [2], [12])

$$(A, \Gamma) = (BP_*, BP_*BP) = (Z_{(p)}[v_1, v_2, \cdots], BP_*[t_1, t_2, \cdots]),$$

and the Γ-comodules A and $A/(p)$. Then, for the sphere spectrum S localized at p and the Moore spectrum M mod p, we have the Adams-Novikov spectral sequence (cf. [3], [12]):

$$E_2 = \mathrm{Ext}_\Gamma^*(A, A) \quad (\text{resp. } \mathrm{Ext}_\Gamma^*(A, A/(p))) \Longrightarrow \pi_*S \quad (\text{resp. } \pi_*M). \tag{1.1}$$

This is investigated by several authors to study the structure of the stable homotopy ring π_*S of spheres ([3], [6], [7]).

Now, for the Γ-comodules N_1^j and $M_1^j = v_{1+j}^{-1}N_1^j$ such that $N_1^0 = A/(p)$ and N_1^{j+1} is the cokernel of the localization map $N_1^j \to M_1^j$, we have the chromatic spectral sequence (cf. [3]):

$$E_2 = \mathrm{Ext}_\Gamma^*(A, M_1^*) \Longrightarrow \mathrm{Ext}_\Gamma^*(A, A/(p)). \tag{1.2}$$

In this paper, we are concerned with $\mathrm{Ext}_\Gamma^*(A, M_1^1)$ for $* \geqq 2$ by continuing the studies in [3] and [11] for $* = 0, 1$ to obtain the following

THEOREM A. *The $F_p[v_1]$-module $\mathrm{Ext}_\Gamma^*(A, M_1^1)$ is given by Theorem 4.4.*

Here, we note the following: Consider the spectrum N which is the cofiber of the localization map $M \to \alpha^{-1}M$ for the Adams map $\alpha \in [M, M]_*$. Then, by Ravenel's localization functor L_2 (see [10]), we have the spectrum L_2N with $BP \wedge L_2N = N \wedge v_2^{-1}BP$ and the Adams-Novikov spectral sequence:

$$E_2 = \mathrm{Ext}_\Gamma^*(A, M_1^1) \Longrightarrow \pi_*(L_2N). \tag{1.3}$$

Thus, Theorem A implies immediately the following

COROLLARY. *The spectral sequence (1.3) collapses, and $\pi_*(L_2N)$ is an $F_p[\alpha]$-module isomorphic to $\mathrm{Ext}_\Gamma^*(A, M_1^1)$ in Theorem 4.4 by sending α to v_1.*

As an application, we are concerned with the β-elements (see (2.1.7))

(1.4) $\beta_{tp/j}$ for $(j, t) \in \boldsymbol{B} = \{(j, t) | 1 \leqq j \leqq p,\ t \geqq 1,\ (j, t) \neq (p, 1)\}$,
β_s for $s \geqq 1$ satisfying $\beta_{tp} = \beta_{tp/1}$, and $\beta_{up^2/p,2}$ for $u \geqq 2$

in the p-component $\pi_* S$ of the stable homotopy ring of spheres, given by Toda [13] and Oka [4–6]. On the products of these elements in $\pi_* S$, [7] says that

(1.5) $\beta_s \beta_{tp/j} = 0$ unless a) $j+1 = p \not| t$ and $p | s+1$, b) $j = p \not| t$ $(\geqq 2)$;

and we prove in this paper the following

THEOREM B. (i) $\beta_s \beta_{tp/p}$ *(resp.* $\beta_s \beta_{tp^2/p,2}$*)* $(s \geqq 1, t \geqq 2)$ *is non-trivial in* $\pi_* S$ *if* $p \not| ts(s-1)$, *or* $s = rp + 1$ *and* $p \not| t(r+t)(r+t+1)$ *(resp.* $p \not| tr(r+1)$*).*

(ii) $\beta_{sp/i} \beta_{tp/j}$ $((i, s), (j, t) \in \boldsymbol{B})$ *is* 0 *if* $i + j \leqq p$ *and* $s + t \geqq 3$, *and is not* 0 *if*

$$p \,|\, s + t, \quad p^2 \not| t(s+t+p) \quad and \quad p + 3 \leqq i + j < 2p.$$

Here, we note that (i) for $p \not| ts(s-1)$ is proved in [7]. Furthermore, $\beta_s \beta_{tp/p}$ in the E_2-term of (1.1) is 0 in case a), or if $p | s$ in case b) of (1.5), and its pre-image in $\pi_* M$ is not 0 if $p | s-1$ in case b) (see [7] and [11]).

The triviality in Theorem B is in Theorem 2.2, which is an immediate consequence of the known results in [14] and [7].

Theorem 4.4 is proved by using the change of rings theorem [2];

$$\mathrm{Ext}_\Gamma^* (A, M_1^1) \cong \mathrm{Ext}_\Sigma^* (B, M_1^1 \otimes_A B)$$
$$\text{for } (B, \Sigma) = (\boldsymbol{Z}_{(p)}[v_1, v_2, v_2^{-1}], B[t_1, t_2, \cdots] \otimes_A B).$$

In § 3, we study the cobar complex $\Omega_\Sigma^* B$ and prove the key lemma (Proposition 3.7) which assures the existence of the nice elements $G_n \in \Omega_\Sigma^2 B$. Then, we can determine $\mathrm{Ext}_\Sigma^* (B, M_1^1 \otimes_A B)$ in Theorem 4.4 by using the results for $* \leqq 1$ obtained in [3] and [11] and by using the exact sequence associated to the short exact sequence $0 \to M_2^0 \xrightarrow{1/v_1} M_1^1 \xrightarrow{v_1} M_1^1 \to 0$.

The non-triviality in Theorem B is proved in Theorem 5.5 by expressing the β-elements in $\Omega_\Gamma^* A$ and by studying the images of their products under $H^4 A \xleftarrow{\cong} H^3 N_0^1 \xleftarrow{\cong} H^2 N_0^2 \to H^2 M_0^2 \to H^3 M_1^1$.

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§ 2. Triviality of some products of the β-elements

In this paper, we assume that p is a prime $\geqq 5$.

Let S be the sphere spectrum localized at p, and recall ([13], [4]) the Moore spectrum M mod p and the spectra $X(r)$ ($r \geqq 1$), defined by the cofiber sequences

(2.1.1) $S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{\pi} \Sigma S$

and $\Sigma^{rq} M \xrightarrow{\alpha^r} M \xrightarrow{i_r} X(r) \xrightarrow{\pi_r} \Sigma^{rq+1} M$ $(q=2p-2)$

for the map p of degree p and the Adams map α: $\sum^q M \to M$, and the maps

(2.1.2) β: $\Sigma^{(p+1)q} X(1) \longrightarrow X(1)$,

$R(r)$: $\Sigma^{(p^2+p)q} X(r) \longrightarrow X(r)$ and B: $X(r+1) \longrightarrow X(r)$

for $0 \leqq r < p$ ($X(0) = *$, the point spectrum), which are related by

(2.1.3) $R(1) = \beta^p$, $i_r = Bi_{r+1}$,

$\pi_r B = \alpha \pi_{r+1}$ and $BR(r) = R(r-1)B$ $(1 \leqq r < p)$.

Then, the β-elements in the homotopy ring $[M, M]_*$ are defined by

(2.1.4) $\beta_{(s)} = \pi_1 \beta^s i_1$ and $\beta_{(sp/r)} = \pi_r R(r)^s i_r$ for $s \geqq 1$ and $1 \leqq r < p$,

(cf. [7]), which satisfy the following by (2.1.3) and $\alpha^r \pi_r = 0$:

(2.1.5) $\beta_{(sp/1)} = \beta_{(sp)}$, $\beta_{(sp/r)} = \alpha^{p-1-r} \beta_{(sp/p-1)}$, $\alpha^r \beta_{(sp/r)} = 0$.

Furthermore, we know the following ([14], [7]):

(2.1.6) In $[M, M]_*$, $\alpha^2 \delta = 2\alpha\delta\alpha - \delta\alpha^2$ for $\delta = i\pi$, and

$\beta_{(tp/r)} = \beta_{(tp/p)} \alpha^{p-r}$ for some $\beta_{(tp/p)}$ if $t \geqq 2$ and $1 \leqq r < p$.

(2.1.7) In $\pi_* S$, we have the β-elements $\beta_s = \pi \beta_{(s)} i$ ($s \geqq 1$) and $\beta_{sp/r} = \pi \beta_{(sp/r)} i$ ($s \geqq 1$ and $1 \leqq r < p$, or $s \geqq 2$ and $r = p$), satisfying

$$\beta_{sp/1} = \beta_{sp} \quad \text{and} \quad \beta_{sp} \beta_{tp/p-1} = 0 \qquad \text{for} \quad s, t \geqq 1 .$$

THEOREM 2.2. *For $s, t \geqq 1$ and $1 \leqq r, u \leqq p$ with $s \geqq 2$ if $r = p$ and $t \geqq 2$ if $u = p$,*

$$\beta_{sp/r} \beta_{tp/u} = 0 \quad \textit{in} \quad \pi_* S \quad \textit{if} \quad r + u \leqq p \quad \textit{and} \quad s + t \geqq 3 .$$

PROOF. Assume that $r+u \leqq p$ and $s+t \geqq 3$. Then, we may assume $s \geqq 2$ since $\pi_* S$ is commutative. (2.1.6) implies $\alpha^n \delta = n\alpha\delta\alpha^{n-1} - (n-1)\delta\alpha^n$ in $[M, M]_*$ for $n \geqq 1$, and so $\beta_{sp/r} \beta_{tp/u}$ is equal to

$$-r\pi\beta_{(sp/p)} \alpha\delta\alpha^{p-1-r} \beta_{(tp/u)} i + (r+1)\pi\beta_{(sp/p)} \delta\alpha^{p-r} \beta_{(tp/u)} i .$$

This is 0 if $p-1-r \geqq u$ and $-r\beta_{sp/p-1}\beta_{tp}$ if $p-r=u$ by (2.1.5), and the latter is also 0 by (2.1.7). q.e.d.

§3. Key lemma on the cobar complex $\Omega_\Sigma^* B$

Let BP be the Brown-Peterson ring spectrum at p. Then

(3.1.1) $\pi_* BP = BP_* = Z_{(p)}[v_1, v_2, \cdots]$ and $BP_*BP = BP_*[t_1, t_2, \cdots]$ with $\deg v_n = \deg t_n = 2(p^n - 1)$ (cf. [8], [1]), and

(3.1.2) $(A, \Gamma) = (BP_*, BP_*BP)$ is the Hopf algebroid (whose left unit η_L is considered to be the inclusion $A \subset \Gamma$), with right unit η_R (denoted simply by η in this paper): $A \to \Gamma$ and diagonal $\Delta: \Gamma \to \Gamma \otimes_A \Gamma$ satisfying (cf. [11; (2.3.4–5)])

(3.1.3) $\eta v_1 = v_1 + pt_1$, $\eta v_2 \equiv v_2 + v_1 t_1^p + pt_2 - (p+1) v_1^p t_1 \bmod (p^2)$,

$$\Delta t_1 = \psi t_1,\ \Delta t_2 = \psi t_2 + t_1 \otimes t_1^p + v_1 T,$$

$$\Delta t_3 \equiv \psi t_3 + g + v_2 T^p \bmod (p, v_1),$$

where $\psi x = x \otimes 1 + 1 \otimes x$ ($x \in \Gamma$) and T, $g \in \Gamma \otimes_A \Gamma$ are given by

(3.1.4) $T = \{\psi(t_1^p) - \Delta t_1^p\}/p$, $g = t_1 \otimes t_2^p + t_2 \otimes t_1^{(2)}$

($u^{(n)}$ denotes u^{p^n} in this paper).

For any Hopf algebroid (A, Γ) and a Γ-comodule M, we consider the homology

(3.2.1) (cf. [2]). $\mathrm{Ext}_\Gamma^*(A, M)$ of the cobar complex $\Omega_\Gamma^* M$ with $\Omega_\Gamma^0 M = M$, $\Omega_\Gamma^n M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma$ (n copies of Γ) and differential $d_n: \Omega_\Gamma^n M \to \Omega_\Gamma^{n+1} M$ given by

$$d_n(m \otimes x) = \eta_M m \otimes x$$
$$+ \textstyle\sum_{i=1}^n (-1)^i m \otimes x_1 \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_n - (-1)^n m \otimes x \otimes 1 \quad (n \geq 0)$$

for $m \in M$, $x_i \in \Gamma$ and $x = x_1 \otimes \cdots \otimes x_n$, where $\eta_M: M \to M \otimes_A \Gamma$ is the coaction.

(3.2.2) Especially, for the Γ-comodule A with $\eta_A = \eta: A \to A \otimes_A \Gamma = \Gamma$, $d_0 u = \eta u - u$ ($u \in A = \Omega_\Gamma^0 A$); $d_1 x = \psi x - \Delta x$ ($x \in \Gamma = \Omega_\Gamma^1 A$).

Hereafter let (A, Γ) be (BP_*, BP_*BP) in (3.1.2). Recall (see [3; §3])

(3.3.1) the Γ-comodules N_i^j and M_i^j ($i, j \geq 0$) with coactions η induced from η for A, which are defined inductively by $N_i^0 = A/(v_0, \cdots, v_{i-1})$ ($v_0 = p$), $M_i^j = v_{i+j}^{-1} N_i^j$ and the short exact sequence $0 \to N_i^j \xrightarrow[\subset]{\lambda} M_i^j \to N_i^{j+1} \to 0$, such that

(3.3.2) $\qquad 0 \longrightarrow M_{i+1}^{j-1} \xrightarrow{1/v_i} M_i^j \xrightarrow{v_i} M_i^j \longrightarrow 0$ is exact.

(3.3.3) Here, by definition, we denote any element of M_i^j by a linear combination of fractions x/y of monomials

$$x = \prod_n v_n^{s_n} \text{ (finite product)} \in v_{i+j}^{-1}A \ (s_n \geqq 0 \text{ for } n \neq i+j,\ s_{i+j} \in \mathbf{Z})$$

and $y=\prod_{i\leqq n<i+j} v_n^{r_n}$ $(r_n>0)$, and $x/y \in N_i^j$ in case $x \in A$, i.e., $s_{i+j} \geqq 0$. We note that $x/y \neq 0$ in M_i^j if and only if $s_n=0$ for $n<i$ and $s_n<r_n$ for $i \leqq n < i+j$.

To study $\mathrm{Ext}_\Gamma^*(A, M_i^j)$ $(i+j=2)$, we use the change of rings theorem. Put

(3.4.1) $B = \mathbf{Z}_{(p)}[v_1, v_2, v_2^{-1}]$ and $\Sigma = B \otimes_A \Gamma \otimes_A B = B[t_1, t_2, \cdots] \otimes_A B$,

where v_n $(n \geqq 3)$ act trivially on B. Then, [2; §3] says the following:

(3.4.2) $(B, \Sigma)=(E(2)_*, E(2)_*E(2))$ is the Hopf algebroid so that the natural map $(A, \Gamma) \to (B, \Sigma)$ sending v_n $(n \geqq 3)$ to 0 is a map of Hopf algebroids, i.e., the structure maps of (B, Σ) satisfy (3.1.3) and $\eta(v_2^{-1})\eta v_2=1$ in Σ.

(3.4.3) For a Γ-comodule M, we have the induced Σ-comodule $M \otimes_A B$ and the natural map induces $\mathrm{Ext}_\Gamma^*(A, M) \to \mathrm{Ext}_\Sigma^*(B, M \otimes_A B)$, which is isomorphic if M is v_2-local (i.e. v_2 acts bijectively on M); and we identify as

(3.4.4) $H^*M = \mathrm{Ext}_\Gamma^*(A, M) = \mathrm{Ext}_\Sigma^*(B, M \otimes_A B)$ for $M = M_i^j$ $(i+j=2)$.

Now, we prepare some results on the cobar complex $\Omega_\Sigma^* B$, by considering the elements T, $g \in \Gamma \otimes_A \Gamma$ in (3.1.4) and V, $\tau \in \Gamma$, ζ, $\sigma \in \Sigma$ and $g_\varepsilon \in \Sigma \otimes_B \Sigma$ given by

$$\begin{aligned}(3.5.1)\quad V &= \{v_1^p t_1^{(2)} - v_1^{(2)} t_1^p + v_2^p - (v_1 t_1^p - v_1^p t_1 + v_2)^p\}/pv_1 \\ &\equiv - v_2^{p-1} t_1^p \bmod (p, v_1),\ \tau = t_1^{1+p} - t_2; \\ \zeta &= v_2^{-1} t_2 - v_2^{-p} \tau^p,\ \sigma = 2t_1 - v_1 \zeta^p;\ g_0 = v_2^{-p} g,\ g_1 = v_2^{-1} g_0^p.\end{aligned}$$

Here / is the division, and $\zeta \in v_2^{-1}\Gamma$ in [11; (2.5.3)] is the above ζ in Σ.

(3.5.2) [11; (3.2.1–5)] The following relations hold in Σ for $n \geqq 1$:

$$\begin{aligned} v_2^p t_1 - v_2 t_1^{(2)} - v_1 t_2^p &\equiv v_1^2 V - v_1^p t_1^{1+p^2} \bmod (p, v_1^{(2)}), \\ &\equiv - v_1^2 v_2^{p-1} t_1^p \bmod (p, v_1^3),\ v_2^{(n)} t_n \equiv v_2 t_n^{(2)} + v_1 t_{n+1}^p \bmod (p, v_1^2), \\ \zeta^{(n)} &\equiv \zeta^{(n-1)} \bmod (p, v_1^{(n-1)}),\ v_2^{(2)} T \equiv v_2^p T^{(2)} \bmod (p, v_1).\end{aligned}$$

(3.5.3) [11; (3.3.1–2), (3.4.2–8)] There are elements ξ_2, ξ_4 amd W_s, Z_s $(s \in \mathbf{Z})$ in Σ, with $v_1 W_s \equiv Z_s \equiv -v_1 v_2^{sp-p} V \bmod (p, v_1^{p-1})$, such that the differential $d_1\colon \Omega_\Sigma^1 B = \Sigma \to \Omega_\Sigma^2 B = \Sigma \otimes_B \Sigma$ satisfies

$$\begin{aligned} d_1 \xi_2 &\equiv 2v_2^{-p} t_1^{(2)} \otimes V - v_1 v_2^{p-1} g_1 \text{ and } d_1 \xi_4 \equiv v_2^{-2p} V \otimes \sigma - v_1 v_2^{-p} g_1 \bmod (p, v_1^2), \\ d_1 W_s &\equiv v_1^{p-1} W_s' \bmod (p, v_1^{p+2}) \text{ where } W_s' = v_2^{sp} g_1^p - (s-1) v_1^2 v_2^{sp-1} g_1/2, \\ d_1 Z_s &\equiv v_1^{p-1} v_2^{sp-p} t_1^{(2)} \otimes \sigma - (s+1) v_1^{p+2} v_2^{sp-1} g_1/2 \bmod (p, v_1^{p+3}).\end{aligned}$$

(3.5.4) [3; Prop. 3.18 c)] $d_1(\zeta^{(n)}) \equiv 0 \bmod (p, v_1^{(n)})$ in $\Omega_\Sigma^2 B$ for $n \geqq 0$.

For d_2: $\Omega_\Sigma^2 B = \Sigma \otimes_B \Sigma \to \Omega_\Sigma^3 B = \Sigma \otimes_B \Sigma \otimes_B \Sigma$, we see the following by (3.2.1–2), (3.1.3–4) and definition:

$$(3.5.5) \quad d_2 g_0 \equiv -v_1 v_2^{-p} T \otimes t_1^{(2)} \bmod (p, v_1^p),$$
$$d_2 g_1 \equiv -v_1 v_2^{-1} t_1^p \otimes g_1 \bmod (p, v_1^2).$$

(3.5.6) [9; Th. 3.2] $H^i M_2^0$ is 0 if $i \geqq 5$, and is the $F_p[v_2, v_2^{-1}]$-vector space with basis represented by the following cycles in $\Omega_\Sigma^i(M_2^0 \otimes_A B) = \Omega_\Sigma^i B/(p, v_1)$ for $i \leqq 4$:

$$1\ (i=0);\ h_0 = t_1,\ h_1 = v_2^{-1} t_1^p \text{ and } \zeta\ (i=1);$$
$$g_\varepsilon \text{ and } h_\varepsilon \otimes \zeta \text{ for } \varepsilon = 0, 1\ (i=2);$$
$$\theta_\varepsilon = g_\varepsilon \otimes \zeta \text{ for } \varepsilon = 0, 1 \text{ and } \rho = t_1 \otimes g_1\ (i=3);\ \rho \otimes \zeta\ (i=4),$$

where ρ^p is homologous to ρ.

We note that the above elements are all homogeneous, and

$$(3.5.7) \quad |v_n| = |t_n| = e'(n) = (p^n - 1)/(p-1),\ |T| = p,$$
$$|g| - 1 = |V| + 1 = p^2 + p,\ |\tau| = p + 1 = |v_2|,\ |\zeta| = |\rho| = 0,$$
$$|\sigma| = 1 = |v_1|,\ |h_\varepsilon| = |g_\varepsilon| = |\theta_\varepsilon| = (-1)^\varepsilon,\ |\xi_2| = p^2 - 1,$$
$$|\xi_4| = -p^2 - p,\ |W_s| + 1 = |W_s'| + p = |Z_s| = s(p^2 + p).$$

(3.5.8) Here, $|x| = m$ means that x is a homogeneous element of degree mq for $q = 2p - 2$.

LEMMA 3.6. *If $G \in \Omega_\Sigma^2 B$ and positive integers n and a satisfy*

$$(3.6.1) \quad |G| = -(p+1)e'(n) + a \text{ and } d_2 G \equiv v_1^a v_2^{-e'(n)} \rho \bmod (p, v_1^{1+a}),$$

then there is $F \in \Omega_\Sigma^2 B$ with $|F| = |G| - a - 1$ and

$$(3.6.2) \quad d_2(G - v_1^{1+a} F) \equiv v_1^a v_2^{-e'(n)} \{\rho + v_1(\phi + k\theta_1)\} \bmod (p, v_1^{2+a})$$

for some $k \in \boldsymbol{Z}$. Here,

$$(3.6.3) \quad \phi = v_2^{-1}(t_2 - \tau) \otimes g_1 \in \Omega_\Sigma^3 B, \text{ and } |\phi| = -1.$$

PROOF. By assumption, there is $\alpha \in \Omega_\Sigma^3 B$ ($|\alpha| = |G| - a - 1$) with

$$d_2 G \equiv v_1^a v_2^e \rho + v_1^{1+a} \alpha \text{ and so}$$
$$d_3(v_1^{1+a} \alpha) \equiv -d_3(v_1^a v_2^e \rho) \bmod (p, v_1^{2+a})\ (e = -e'(n)).$$

On the other hand, (3.2.1–2), (3.1.3–4) and (3.5.5) imply that

$$d_3(v_2^e\rho) \equiv v_1v_2^{e-1}(t_1^p\otimes t_1 - t_1\otimes t_1^p)\otimes g_1 \equiv -v_1d_3(v_2^e\phi) \bmod (p, v_1^2).$$

Thus $v_1^{1+a}d_3\alpha \equiv d_3(v_1^{1+a}\alpha) \equiv -d_3(v_1^qv_2^e\rho) \equiv v_1^{1+a}d_3(v_2^e\phi) \bmod (p, v_1^{2+a})$, which means that $\alpha - v_2^e\phi$ is a cycle in the range of the projection $\Omega_2^3B \to \Omega_2^3B/(p, v_1) = \Omega_2^3(M_2^0\otimes_A B)$. Therefore, by (3.5.6) for $H^3M_2^0$ and $|\alpha - v_2^e\phi| = (p+1)e - 1 = |v_2^e\theta_1|$, we have

$$\alpha - v_2^e\phi \equiv kv_2^e\theta_1 + d_2F \bmod (p, v_1) \text{ for some } k\in \boldsymbol{Z} \text{ and } F\in\Omega_2^2B.$$

These show the lemma. q.e.d.

By this lemma, we can prove the following key lemma, where

(3.6.4) $\varepsilon_0 = 1 = a_0,\ \varepsilon_n = \min\{n, 2\}$ and $a_n = p^n + p^{n-1} - 1\ (n\geqq 1)$.

PROPOSITION 3.7. *There are $G_n\in\Omega_2^2B$ for $n\geqq 0$ such that*

(3.7.1) $|G_0| = 1,\ |G_n| = -(p+1)e'(n-1) - 1\ (n\geqq 1)$,

$$G_0 \equiv g_0 \text{ and } G_n \equiv v_2^{-e'(n-1)}g_1 \bmod (p, v_1)\ (n\geqq 1),$$

$$d_2G_n \equiv \varepsilon_n v_1^{q_n}v_2^{-e'(n)}\rho \bmod (p, v_1^{1+a_n})\ (n\geqq 0).$$

PROOF. Put $G_0 = g_0 + v_1v_2^{-p^2-p}t_3^p\otimes t_1^{(2)} - v_1g_1\Delta t_1$. Then (3.7.1) holds for $n=0$ by (3.2.1–2), (3.1.3–4) and (3.5.1–7).

For $n=1$, consider $\sigma' = 2t_1 - v_1\zeta^{(2)}\in\Sigma$ and $\gamma = -Z_{-1}\otimes\sigma' - v_1d_1\xi_4\in\Omega_2^2B$. Then

$$\sigma' \equiv \sigma \bmod (p, v_1^{1+p}) \text{ and } \gamma \equiv v_1^2v_2^{-p}g_1 \bmod (p, v_1^3) \text{ by (2.5.2–3)},$$

$$d_1\sigma' \equiv 0 \text{ and } d_2\gamma \equiv -v_1^{p-1}v_2^{-2p}t_1^{(2)}\otimes\sigma\otimes\sigma \equiv -v_1^2v_2^{-p}d_2\gamma' \bmod (p, v_1^{3+p})$$

for $\gamma' = v_1^{p-3}v_2^{-p}t_1^{(2)}\otimes\sigma^2/2$ by (3.5.3–4). Therefore, we have the element

$$G_1 = v_2^p\gamma/v_1^2 + \gamma' \equiv g_1 \bmod (p, v_1) \text{ in } \Omega_2^2B \text{ with } |G_1| = -1$$

and $d_2(v_1^2G_1) \equiv d_0v_2^p\otimes\gamma + v_2^p\otimes d_2\gamma + d_2(v_1^2\gamma') \equiv v_1^pt_1^{(2)}\otimes v_1^2v_2^{-p}g_1 \equiv v_1^{p+2}v_2^{-1}t_1\otimes g_1$ (by (3.5.2)) $= v_1^{p+2}v_2^{-1}\rho \bmod (p, v_1^{3+p})$, which implies $d_2G_1 \equiv v_1^pv_2^{-1}\rho \bmod (p, v_1^{1+p})$ ($e'(1) = 1$) as desired, since $v_1\colon \Omega_2^*B/(p) \to \Omega_2^*B/(p)$ is monomorphic.

Now, assume inductively that G_n ($n\geqq 1$) satisfies (3.7.1). Then, for $a = a_n$, $e = -e'(n)$ and $f = (p+1)e$, $|G_n| = f + a$ and we can apply Lemma 3.6 to obtain

(3.7.2) $$d_2(G_n - v_1^{1+a}F) \equiv \varepsilon_nv_1^qv_2^e\{\rho + v_1(\phi + k\theta_1)\} \bmod (p, v_1^{2+a})$$

for some $k\in\boldsymbol{Z}$ and $F\in\Omega_2^2B$ with $|F| = f - 1$. We consider the element

$$\gamma = \{W'_{e_1} - (G_n - v_1^{1+a}F)^p\}/\varepsilon_n + kv_1^{ap+1}W_e\otimes\zeta^p\ (e_1 = -e'(n-1) = (e+1)/p)$$

by (3.5.3). Then (3.7.1–2), (3.5.3–4) and (3.5.7) show that $|\gamma|=f+1$,

$$\gamma \equiv v_1^2 v_2^e g_1/2 \bmod (p, v_1^3) \text{ and } d_2\gamma \equiv -v_1^{ap}v_2^{ep}(\rho+v_1\phi)^p \bmod (p, v_1^{3+a'})$$

($a'=ap+p-1=a_{n+1}$), since $-(e_1-1)/\varepsilon_n \equiv 1 \bmod p$. Consider the elements

$$\gamma' = v_2^{2+e'}g_1^p - 3v_1t_1 \otimes W_e, \; E \text{ with } |E| = 0 \text{ and } d_2E \equiv \rho - \rho^p \bmod (p, v_1),$$
$$G_{n+1} = 2\gamma/v_1^2 + 2v_1^{a'-p-2}(\gamma' - 2v_1^{2+p}v_2^{e'}E) \equiv v_2^e g_1 \bmod (p, v_1)$$

($e'=ep-1=-e'(n+1)$), where E exists by (3.5.6). Then

$$d_2\gamma' \equiv v_2^{ep+p}d_0(v_2^{1-p}) \otimes g_1^p + v_2^{2+e'}d_2(g_1^p) + 3v_1t_1 \otimes d_1W_e \bmod (p, v_1^{(2)})$$
$$\equiv v_1v_2^{ep}(\rho+v_1\phi)^p + v_1^{2+p}v_2^{e'}(3\rho - 2\rho^p) \bmod (p, v_1^{3+p})$$

by (3.5.2–4) and $d_0(v_2^{1-p}) \equiv v_1v_2^{-p}t_1^p - v_1^pv_2^{-p}t_1 - v_1^pv_2^{-2p}(v_2+v_1t_1^p)t_1^{(2)} \bmod (p, v_1^{2p})$. These imply $d_2G_{n+1} \equiv 2v_1^{a'}v_2^{e'}\rho \bmod (p, v_1^{1+a'})$ in (3.7.1), and $|G_{n+1}|=f-1$ is clear. Thus, the proposition is proved by induction on n. q.e.d.

§4. Determination of $H^*M_1^1$

In this section, we study $H^*M_1^1$ in (3.4.4) by using the exact sequence

(4.1.1) $$\cdots \longrightarrow H^{n-1}M_1^1 \xrightarrow{\delta_{n-1}} H^nM_2^0 \xrightarrow{f_n} H^nM_1^1 \xrightarrow{v_1} H^nM_1^1 \xrightarrow{\delta_n} H^{n+1}M_2^0 \longrightarrow \cdots$$

($f_n=(1/v_1)_*$) for $n \geqq 1$ associated to the short exact sequence in (3.3.2) for $i=j=1$.

Hereafter, for M_i^j ($i+j=2$), we use the following notations:

(4.1.2) An element $(x/y)\otimes\gamma$ in the cobar complex $\Omega_\Sigma^*(M_i^j \otimes_A B) = M_i^j \otimes_A \Omega_\Sigma^* B$ for $x/y \in M_i^j$ (see (3.3.3)) and $\gamma \in \Omega_\Sigma^* B$ is denoted by $x\otimes\gamma/y$, and if it is a cycle, then its homology class in $H^*M_i^j$ is denoted by the same letter;

(4.1.3) $\boldsymbol{F}_p\{\alpha_j\}$ denotes the $\boldsymbol{F}_p$-submodule of $H^*M_1^1$ generated by $\{\alpha_j | j \geqq 1\}$ with $v_1\alpha_{j+1}=\alpha_j$ such that the $\boldsymbol{F}_p[v_1]$-submodule $\boldsymbol{F}_p\{\alpha_j\}$ is isomorphic to $\boldsymbol{F}_p[v_1, v_1^{-1}]/\boldsymbol{F}_p[v_1]$; and $\boldsymbol{F}_p[v_1]\langle\alpha\rangle$ denotes the cyclic $\boldsymbol{F}_p[v_1]$-submodule of $H^*M_1^1$ generated by $\alpha=\alpha'/v_1^n$ such that it is isomorphic to $\boldsymbol{F}_p[v_1]/(v_1^n)$.

(4.1.4) [11; Lemma 3.9] In (4.1.1), assume that a submodule $K \supset \operatorname{Im} f_n$ of $H^nM_1^1$ is the direct sum of $\boldsymbol{F}_p\{\alpha_{\lambda,j}\}$ ($\lambda \in \Lambda$) and $\boldsymbol{F}_p[v_1]\langle k_\mu\rangle$ ($\mu \in M$) such that $\{\delta_nk_\mu | \mu \in M\}$ is linearly independent. Then $K=H^nM_1^1$.

$H^0M_1^1$ and $H^1M_1^1$ are given as $\boldsymbol{F}_p[v_1]$-modules by

(4.1.5) [3; §5] $H^0M_1^1$ is the direct sum of $\boldsymbol{F}_p\{1/v_1^j\}$ and

$F_p[v_1]\langle x_n^s/v_1^{a_n}\rangle$ for $n \geqq 0$ and $s \in Z - pZ$, (see (3.6.4) for a_n),

where $x_n \in v_2^{-1}A$, $|x_n| = p^n(p+1)$ and $x_n = v_2^{(n)}$ in $M_2^0 = v_2^{-1}A/(v_0, v_1)$.

Furthermore, $\delta_0(1/v_1^j) = 0$ and

$$\delta_0(x_0^s/v_1) = sv_2^s h_1,\ \delta_0(x_n^s/v_1^{a_n}) = \varepsilon_n s v_2^{c(n,s)} h_0 \text{ for } n \geqq 1\ (h_0 = t_1,\ h_1 = v_2^{-1}t_1^p)$$

in $H^1M_2^0$ (see (3.5.6)), where $\varepsilon_n = \min\{n, 2\}$ and $c(n, s) = sp^n - p^{n-1}$.

(4.1.6) [11; §3] $H^1M_1^1$ is the direct sum of $F_p\{h_0/v_1^j\}$, $F_p\{\zeta^{(j)}/v_1^j\}$ and

$F_p[v_1]\langle y_m/v_1^{A(m)}\rangle$

for $m \in \Lambda_0 = \{sp^n | n \geqq 0,\ s \in Z$ with $p \nmid s(s+1)$ or $p^2 | s+1\}$,

$F_p[v_1]\langle v_2^t V/v_1^{p-1}\rangle$ for $t \in pZ$ and

$F_p[v_1]\langle x_n^s \zeta^{(n+1)}/v_1^{a_n}\rangle$ for $n \geqq 0$, $s \in Z - pZ$.

Here, $A(m) = 2 + \varepsilon(s)p^n(p^2-1) + (p+1)e'(n)$ for $m = sp^n$, $\varepsilon(s) = 0$ if $p^2 \nmid s+1$ and $\varepsilon(s) = 1$ if $p^2 | s+1$; and the generators satisfy $y_m \in \Sigma$, $|y_m| = m(p+1)+1$, and

$$y_m = v_2^m h_0,\ v_2^t V = -v_2^{t+p}h_1 \text{ and } x_n^s\zeta^{(n+1)} = v_2^{sp^n}\zeta$$
$$\text{in } \Omega_\Sigma^1(M_2^0 \otimes_A B) = \Sigma/(v_0, v_1).$$

Furthermore, in $H^2M_2^0$ (see (3.5.6)), $\delta_1(h_0/v_1^j) = \delta_1(\zeta^{(j)}/v_1^j) = 0$ and

$\delta_1(y_m/v_1^{A(m)}) = -s_m v_2^{e(m)} g_1$ where $s_m \not\equiv 0 \bmod p$ and

$e(m) = m - \varepsilon(s)p^n(p-1) - e'(n) = m - (A(m)-2)/(p+1)$ for $m = sp^n \in \Lambda_0$,

$\delta_1(v_2^t V/v_1^{p-1}) = -v_2^{t+p-1}g_0$ for $t \in pZ$,

$$\delta_1(x_n^s\zeta^{(n+1)}/v_1^{a_n}) = \begin{cases} sv_2^s h_1 \otimes \zeta & \text{if } n = 0, \\ \varepsilon_n s v_2^{c(n,s)} h_0 \otimes \zeta & \text{if } n \geqq 1, \end{cases} \quad \text{for } s \in Z - pZ.$$

LEMMA 4.2. *$\mathrm{Im} f_2 \cong \mathrm{Coker}\,\delta_1$ in (4.1.1) is the F_p-vector space spanned by*

$v_2^s g_0/v_1$ and $v_2^{e(n,sp)}g_1/v_1$ for $s+1 \in Z - pZ$ and $n \geqq 0$,

$v_2^t h_1 \otimes \zeta/v_1$ for $t \in pZ$, $h_0 \otimes \zeta/v_1$, and $v_2^m h_0 \otimes \zeta/v_1$ for $m \in \Lambda_0$,

where $e(n, r) = rp^n - e'(n)$.

PROOF. Each $e \in Z$ is written as $e = e(n, r)$ with $n \geqq 0$ and $p \nmid r+1$. Then, by the definitions of Λ_0 and $e(m)$, we see that $e \neq e(m)$ for any $m \in \Lambda_0$ if and only if $p | r$ and $p^2 \nmid r+p$. Also, $m \neq c(n, s) = (sp-1)p^{n-1}$ for any $n \geqq 1$ and $s \in Z - pZ$ if and only if $m \in \Lambda_0$. Therefore we see the lemma by (3.5.6) and (4.1.6). q. e. d.

Proposition 3.7 and (4.1.4–5) imply the following

PROPOSITION 4.3. (i) *$H^2M_1^1$ contains the elements represented by the cycles*

(4.3.1) a) $y_m \otimes \zeta^{(n+3)}/v_1^j$ $(m=sp^n \in \Lambda_0,\ 1 \leqq j \leqq A(m))$,

b) $x_n^s G_n/v_1^j$ $(n \geqq 0,\ s \in \boldsymbol{Z},\ 1 \leqq j \leqq a_n)$,

c) $v_2^t V \otimes \zeta^p/v_1^j$ $(t \in p\boldsymbol{Z},\ 1 \leqq j < p)$, *and* d) $h_0 \otimes \zeta^{(j)}/v_1^j$ $(j \geqq 1)$;

and δ_2: $H^2M_1^1 \to H^3M_2^0$ in (4.1.1) *maps these elements to*

a) $-s_m v_2^{e(m)}\theta_1$ *when* $j = A(m)$, b) $(s+1)\varepsilon_n v_2^{e(n,s)}\rho$ *when* $j = a_n$,

c) $v_2^{t-1}\theta_0$ *when* $j=p-1$, *respectively, and* 0 *otherwise.*

(ii) *$H^3M_1^1$ contains the elements represented by the cycles*

(4.3.2) $$x_n^s G_n \otimes \zeta^{(n+1)}/v_1^j \quad (n \geqq 0,\ s \in \boldsymbol{Z},\ 1 \leqq j \leqq a_n),$$

and δ_3: $H^3M_1^1 \to H^4M_2^0$ maps these elements to

$$(s+1)\varepsilon_n v_2^{e(n,s)}\rho \otimes \zeta \text{ when } j = a_n, \text{ and } 0 \text{ otherwise.}$$

(4.3.3) *For z in* a)–d) *of* (4.3.1), *$|z|$ is given by*

a) $m(p+1) + 1 - j$,

b) $e(n-1, sp)(p+1) - 1 - j$ $(n \geqq 1)$, $s(p+1)$ $(n=0$ *and so* $j=1)$,

c) $(t+p)(p+1) - 1 - j$, *and* d) $1 - j$, *respectively.*

PROOF. Let $\alpha \in \Omega_\Sigma^n B$. Then, we see the following:

(4.3.4) $d_n(\alpha/v_1^a) = d_n(\alpha)/v_1^a$ in $\Omega_\Sigma^*(M_1^1 \otimes_A B) = M_1^1 \otimes_A \Omega_\Sigma^* B$ for $a \geqq 1$.

(4.3.5) $\alpha/v_1^a \in H^nM_1^1$ if and only if $d_n\alpha \equiv v_1^a\beta \bmod (p, v_1^{1+a})$ for some $\beta \in \Omega_\Sigma^{n+1}B$, and then $\delta_n(\alpha/v_1^a) = f^{-1}d_n(\alpha/v_1^{1+a}) = \beta$ in $H^{n+1}M_2^0$.

(4.3.6) If $\alpha/v_1^a \in H^nM_1^1$ and $\alpha'/v_1^a \in H^mM_1^1$ for $a \geqq 1$, then $\alpha \otimes \alpha'/v_1^a \in H^{n+m}M_1^1$ and

$$\delta_{n+m}(\alpha \otimes \alpha'/v_1^a) = \delta_n(\alpha/v_1^a) \otimes \alpha' + (-1)^n \alpha \otimes \delta_m(\alpha'/v_1^a) \text{ in } H^{n+m+1}M_2^0.$$

In fact, (4.3.4) is valid since the canonical map $B \to B/(v_0, v_1^a) \xrightarrow{1/v_1^a} M_1^1 \otimes_A B$ is a map of Σ-comodules by [3; Lemma 3.7]; and (4.3.4) shows (4.3.5–6) by definition.

Therefore, we see (i) by (3.5.2–4), (4.1.5–6), Proposition 3.7 and definition, and (ii) by (i) and (3.5.2–4). q.e.d.

Now, we can prove the main result in this section:

THEOREM 4.4. *$H^*M_1^1 = \mathrm{Ext}_\Gamma^*(A, M_1^1) = \mathrm{Ext}_\Sigma^*(B, M_1^1 \otimes_A B)$ in* (3.4.4) *is given as $F_p[v_1]$-modules by* (4.1.5–6) *for $* \leqq 1$ and the following for $* \geqq 2$:*

(i) *$H^2M_1^1$ is the direct sum of $F_p\{h_0 \otimes \zeta^{(j)}/v_1^j\}$ and*

$F_p[v_1]\langle y_m \otimes \zeta^{(n+3)}/v_1^{A(m)}\rangle$ *for* $m = sp^n \in A_0$,

$F_p[v_1]\langle x_n^s G_n/v_1^{a_n}\rangle$ *for* $s+1 \in Z - pZ$ *and* $n \geqq 0$ *and*

$F_p[v_1]\langle v_2^t V \otimes \zeta^p/v_1^{p-1}\rangle$ *for* $t \in pZ$.

(ii) *$H^3M_1^1$ is the direct sum of $F_p[v_1]\langle x_n^s G_n \otimes \zeta^{(n+1)}/v_1^{a_n}\rangle$ for $s+1 \in Z - pZ$ and $n \geqq 0$.*

(iii) *$H^nM_1^1 = 0$ for $n \geqq 4$.*

PROOF. The direct sum K in (i) satisfies the assumption in (4.1.4) by Lemma 4.2, (4.1.5–6), (3.7.1) and Proposition 4.3. Thus (4.1.4) implies (i).

In the same way as Lemma 4.2, (i) and Proposition 4.3 show that

(4.4.1) $\mathrm{Im} f_3 \cong \mathrm{Coker}\, \delta_2$ in (4.1.1) is the F_p-vector space spanned by

$$v_2^s\theta_0/v_1 \text{ and } v_2^{e(n,sp)}\theta_1/v_1 \text{ for } s+1 \in Z - pZ \text{ and } n \geqq 0.$$

Thus (4.1.4) implies (ii) in the same way as (i). Also, we see (iii) since $\mathrm{Im} f_n = 0$ by (ii) and Proposition 4.3 for $n=4$ and by (3.5.6) for $n \geqq 5$. q.e.d.

In the rest of this section, we consider the short exact sequence

(4.5.1) $$0 \longrightarrow M_1^1 \xrightarrow{f'} M_0^2 \xrightarrow{v_0} M_0^2 \longrightarrow 0, \; f'x = x/v_0 \; (v_0 = p),$$

in (3.3.2) and the associated exact sequence

(4.5.2) $$\cdots \longrightarrow H^nM_1^1 \xrightarrow{f_n'} H^nM_0^2 \xrightarrow{v_0} H^nM_0^2 \xrightarrow{\delta_n'} H^{n+1}M_1^1 \longrightarrow \cdots.$$

Here, we notice the following (4.5.3–5) for any element

$$\alpha = \alpha'/v_0^i v_1^j \in \Omega_\Sigma^n(M_0^2 \otimes_A B) = M_0^2 \otimes_A \Omega_\Sigma^n B \text{ with } \alpha' \in \Omega_\Sigma^n B:$$

(4.5.3) $d_n\alpha = d_n(v_1^{k-j}\alpha')/v_0^i v_1^k$ in $\Omega_\Sigma^{n+1}(M_0^2 \otimes_A B)$ for $p^{i-1} | k \geqq j$.

(4.5.4) $\alpha \in H^nM_0^2$ if and only if $d_n(v_1^{k-j}\alpha') \equiv v_0^i\beta \bmod (v_0^{i+1}, v_1^k)$ for some k with $p^i | k \geqq j$ and $\beta \in \Omega_\Sigma^{n+1}B$, and then $\delta_n'\alpha = \beta/v_1^k \in H^{n+1}M_1^1$.

(4.5.5) If $\alpha \in H^nM_0^2$ and $l = mp^{i-1}$, then $v_1^l\alpha = \alpha'/v_0^i v_1^{j-l} \in H^nM_0^2$ and

$$\delta_n'(v_1^l\alpha) = v_1^l\delta_n'\alpha + mt_1 \otimes \alpha'/v_1^{j-l+1} \text{ in } H^{n+1}M_1^1.$$

In fact, let $p^{i-1}|k$. Then $\eta v_1^k \equiv v_1^k \bmod (v_0^i)$ by (3.1.3), and so the canonical map $B \to B/(v_0^i, v_1^k) \xrightarrow{1/y} M_0^2 \otimes_A B$ $(y = v_0^i v_1^k)$ is a map of Σ-comodules by [3; Lemma 3.7]. Thus, we have (4.5.3), which implies (4.5.4–5) by definition and by noticing $d_0(v_1^l)/v_0^{i+1}v_1^j = mt_1/v_0 v_1^{j-l+1}$ for $l = mp^{i-1}$.

LEMMA 4.6. *There are elements $\zeta' \in v_1^{-1}\Sigma$ and $\xi' \in \Sigma$ with $|\zeta'| = 0$, $|\xi'| = -p^2$, $\zeta' \equiv \zeta^{(3)} \bmod (v_0)$ and $d_1\zeta' \equiv v_1^{(2)}\xi' \bmod (v_0^3)$.*

PROOF. We have $\alpha_n = \zeta^{(n)}/v_0 v_1^{(n)} \in H^1 M_0^2$ for any $n \geqq 0$ by (3.5.4), and

$$\delta_1'\alpha_3 = v_1^c \delta_1'\alpha_n \text{ in } H^2M_1^1 \text{ for } c = p^n - p^3 \text{ by (4.5.5), since } \alpha_3 = v_1^c\alpha_n \text{ by (3.5.2).}$$

Therefore, Theorem 4.4 (i) shows that $\delta_1'\alpha_3 = ah_0 \otimes \zeta^{(4)}/v_1^{1+p^3}$ for some $a \in \boldsymbol{F}_p$, since $|\alpha_3| = -p^3 = |h_0 \otimes \zeta^{(4)}/v_1^{1+p^3}|$. Here

$$ah_0 \otimes \zeta^{(4)}/v_1 = v_1^{(3)}\delta_1'\alpha_3 = \delta_1'(v_1^{(3)}\alpha_3) = 0, \text{ and so } a = 0,$$

which shows $\delta_1'\alpha_3 = 0$. Hence, by definition of δ_1', we have

$$d_1(\zeta^{(3)})/v_0^2 v_1^{(3)} = (d_1\omega)/v_0 \text{ in } \Omega_\Sigma^2(M_0^2 \otimes_A B) \text{ for some } \omega \in \Omega_\Sigma^1(M_1^1 \otimes_A B).$$

Thus, $v_1^{(3)}\omega$ is a cycle in $H^1M_1^1$ of degree 0 and so $v_1^{(3)}\omega = a'h_0/v_1$ for some $a' \in \boldsymbol{F}_p$ by (4.1.6).

Put $\alpha = \zeta^{(3)}/v_0^2 v_1^{(3)} - \omega/v_0$ and $\alpha' = v_1^{c'}\alpha$ $(c' = p^3 - p^2)$. Then, $|\alpha'| = -p^2$,

$$\delta_1'\alpha' = v_1^{c'}\delta_1'\alpha \text{ and } v_1^{(2)}\delta_1'\alpha' = \delta_1'(v_1^{(3)}\alpha) = \delta_1'(-a'h_0/v_0v_1) = a'h_0 \otimes h_0/v_1^2 = 0$$

by (4.5.5), since $d_1(h_0^2/2v_1^2) = -h_0 \otimes h_0/v_1^2$ by (3.2.2) and (3.5.6). On the other hand, we see an element $z \in H^2M_1^1$ with $|z| = -p^2$ is a linear combination of $y_{-1}/v_1^{p^2-p}$, y_{1-p}/v_1^2, $G_2/v_1^{p^2-p-2}$ and $h_0 \otimes \zeta^{(3)}/v_1^{1+p^2}$ by Theorem 4.4 (i) together with (4.3.3). Therefore, $\delta_1'\alpha' = 0$, and so by (4.5.4), we have

$$d_1(\zeta^{(3)} - v_0 v_1^{(3)}\bar{\omega}) \equiv v_1^{(2)}\xi' + v_0^2 d_1\omega' \bmod (v_0^3) \text{ for some } \xi' \in \Sigma \text{ and } \omega' \in v_1^{-1}\Sigma,$$

where $\bar{\omega} \in v_1^{-1}\Sigma$ is an element mapped to ω under the canonical map $v_1^{-1}\Sigma \to \Omega_\Sigma^1(M_1^1 \otimes_A \Sigma) = \Sigma/(v_0, v_1^\infty)$. Then, $\zeta' = \zeta^{(3)} - v_0 v_1^{(3)}\bar{\omega} - v_0^2\omega'$ stisfies the conditions of the lemma. q. e. d.

LEMMA 4.7. *For $m = sp^n \in \boldsymbol{Z}$ with $n \geqq 0$ and $p \nmid s$, let*

$$z_m = 2v_2^m z_n' - sv_2^{m-1}t_1\tau/v_0v_1, \quad z_n' = \textstyle\sum_{i=1}^{n+2}((-1)^{i-1}/i)t_1^i/v_0^{n+3-i}v_1^i.$$

Then $d_1 z_m = sv_2^m(g_0 - t_1 \otimes \zeta)/v_0 v_1$.

PROOF. $d_1 z_n' = 0$ for $n \geqq 0$ by [3; Th. 4.2 b)]; and (3.1.3), (3.2.2) and (3.5.2) imply that

$$d_0(v_2^m) \equiv mv_2^{m-1}(v_1t_1^p+v_0t_2) \bmod (v_0, v_1)^{n+2} \qquad \text{for} \quad m = sp^n,$$

$$d_1(t_1\tau) \equiv v_2t_1 \otimes \zeta - t_1^p \otimes t_1^2 - 2\tau \otimes t_1 - v_2g_0 \bmod (v_0, v_1).$$

Therefore, we see the lemma, since

$$d_1z_m = 2d_0(v_2^m) \otimes z_n' - sv_2^{m-1}d_1(t_1\dot{\tau}/v_0v_1) \qquad \text{for} \quad m = sp^n. \qquad \text{q.e.d.}$$

LEMMA 4.8. *$v_2^{sp}g_0\otimes\zeta/v_1 = x_0^{sp}G_0\otimes\zeta^p/v_1$ with $p\nmid s(s+1)$ in $H^3M_1^1$ (see Theorem 4.4 (ii)) does not belong to $\mathrm{Im}(\delta_2'f_2')$ in (4.5.2).*

PROOF. $v_2^{sp}g_0\otimes\zeta/v_1 \notin v_1H^3M_1^1$ by Theorem 4.4 (ii) and the elements in (4.3.1) form an F_p-basis of $H^2M_1^1$ by Theorem 4.4 (i). Therefore, we see the lemma by the following:

(4.8.1) The $\delta_2'f_2'$-images of $y_m\otimes\zeta^{(n+3)}/v_1^j$ for $j\leqq A(m)-2$, $x_n^sG_n/v_1^j$ for $n\geqq 1$ and $j\leqq a_n-2$, and $h_0\otimes\zeta^{(j)}/v_1^j$ are all contained in $v_1H^3M_1^1$;

(4.8.2) $\delta_2'f_2'(x_0^sG_0/v_1) = 0$ if $p\,|\,s$ and $p^2\nmid s$; and

(4.8.3) the degrees of the other elements in (4.3.1) is not equal to that $sp(p+1)q$ of $v_2^{sp}g_0\otimes\zeta/v_1$.

We see immediately (4.8.1) by (4.5.5) and (4.8.3) by (4.3.3). By Lemma 4.6–7, $f_2'(x_0^sG_0/v_1)=v_2^st_1\otimes\zeta'/v_0v_1$ if $p|s$ and $p^2\nmid s$. Then, by (3.1.3), Lemma 4.6 and definition, we have (4.8.2). q.e.d.

Now, we have the following proposition, which implies the non-triviality in §5:

PROPOSITION 4.9. *$v_2^mt_1\otimes\zeta/v_0v_1\neq 0$ in $H^2M_0^2$ for $m=sp^n$ with $n=0, 1$ and $p\nmid s(s+1)$.*

PROOF. By Lemmas 4.6–7, we have

$$a_m = 2v_2^mt_1 \otimes \zeta/v_0v_1 = v_0a_m' \text{ for } a_m' = v_0^nz_m \otimes \zeta' \text{ in } H^2M_0^2; \text{ and}$$

$$\delta_2'a_m = mv_2^mg_0 \otimes \zeta/v_1 \text{ for } p\nmid m \text{ and}$$

$$\delta_2'a_m' = sv_2^mg_0 \otimes \zeta/v_1 \text{ for } m = sp \text{ with } p\nmid s,$$

since $v_2^mt_1\otimes\zeta\otimes\zeta/v_1=0$. Assume that $p\nmid s(s+1)$. Then, the first equality shows $a_m\neq 0$ for $m=s$ by Theorem 4.4 (ii); and the second one shows $\delta_2'a_m' \notin \mathrm{Im}(\delta_2'f_2')$ for $m=sp$ by Lemma 4.8, which implies $a_m'\notin \mathrm{Im}f_2'$ and $a_m=v_0a_m'\neq 0$, as desired. q.e.d.

§5. Non-triviality of the products of β-elements

Consider the boundary and induced homomorphisms

(5.1.1) $$H^nN_0^2 \xrightarrow{\delta} H^{n+1}N_0^1 \xrightarrow{\delta'} H^{n+2}A \text{ and } \lambda_*: H^nN_0^2 \longrightarrow H^nM_0^2$$

associated to the short exact sequences in (3.3.1). Then, we have the β-elements

(5.1.2) (cf. [3; p. 483]) $\beta_s=\delta'\delta(v_2^s/v_0v_1)$ and $\beta_{sp/i}=\delta'\delta(v_2^{sp}/v_0v_1^i)$ in H^2A for $s\geqq 1$ and $1\leqq i\leqq p$, where $\beta_{sp/1}=\beta_{sp}$; and

(5.1.3) these elements except for $\beta_{p/p}$ converge in the Adams-Novikov spectral sequence (1.1) to the β-elements in π_*S with same notation given in (2.1.7).

LEMMA 5.2 (cf. [7; Lemma 4.4]). *$\beta_{sp/i}\equiv sB_{s-1,i} \bmod (v_0, v_1^{2p-i-1})$ in the cobar complex Ω_Γ^2A, where*

(5.2.1) $$B_{s,i} = v_1^{p-i-1}v_2^{sp-p}(-iv_2^pt_1\otimes t_1^{(2)}-sv_1^2V\otimes t_1^{(2)}+v_1v_2^pT^p).$$

PROOF. By the definition of δ and (3.1.1–3), we see that

$$\delta(v_2^{sp}/v_0v_1^i) = d_0(v_1^{-i}v_2^{sp}/v_0) = (sv_1^{p-i}v_2^{sp-p}t_1^{(2)}+v_1^{2p-i}X)/v_0 \in H^1N_0^1$$

for some $X\in\Gamma$. Furthermore, in $\Omega_\Gamma^2M_0^0=\Omega_\Gamma^2v_0^{-1}A$,

$$\begin{aligned} d_1(v_0^{-1}v_1^{2p-i}X) &= v_0^{-1}\{d_0(v_1^{2p-i})\otimes X + v_1^{2p-i}d_1X\} \\ &= -\, iv_1^{2p-i-1}t_1\otimes X + v_0Y + v_1^{2p-i}Z \end{aligned}$$

for some $Y\in\Omega_\Gamma^2A$ and $Z\in\Omega_\Gamma^2M_0^0$. Thus we see the lemma by definition and

(5.2.2) $$d_1(v_1^{p-i}v_2^{sp}t_1^{(2)}) \equiv v_0B_{s,i} \bmod (v_0^2, v_1^{2p-i-1}) \quad (s\geqq 0),$$

which is proved directly from (3.1.1–3) and (3.2.2). q.e.d.

LEMMA 5.3. *Let s and t be positive integers. Then,*

$$\beta_s\beta_{tp/p} = t\delta'\delta b \quad \text{and} \quad \lambda_*b = -\, v_2^{s+tp-1}t_1\otimes\zeta/v_0v_1$$

for $b=v_2^sB_{t-1,p}/v_0v_1=v_2^{s+tp-p}T^p/v_0v_1\in H^2N_0^2$.

PROOF. (3.1.1–4) and Lemma 4.7 show that

$$d_1(v_2^{m-p}t_3/v_0v_1+(1/s)z_m) = -\, v_2^m(v_2^{1-p}T^p+t_1\otimes\zeta)/v_0v_1 \quad \text{in} \quad \Omega_\Sigma^2(M_0^2\otimes_AB)$$

for $m=sp^n$ with $p\nmid s$. Therefore, we see the lemma by Lemma 5.2 and definition. q.e.d.

LEMMA 5.4. *Let s, $t\geqq 1$ and $1\leqq i, j\leqq p$ satisfy $p|u=s+t$ and $3\leqq k=i+j-p<p$. Then, $c=v_2^{sp}B_{t-1,j}/v_0v_1^i\in H^2N_0^2$ satisfies*

$$\beta_{sp/i}\beta_{tp/j} = t\delta'\delta c \text{ and } \lambda_*c = sv_2^{up-p}t_1^{(2)}\otimes\zeta^p/v_0v_1^k + Y/v_0v_1^{k-4} \text{ for some } Y\in\Omega_\Sigma^2B.$$

PROOF. By Lemma 5.2 and (5.1.2), c satisfies the first equality and

$(*)\quad c = B_{u-1,k}/v_0v_1^p + iv_2^{up-p}t_1 \otimes t_1^{(2)}/v_0v_1^{k+1} + sv_2^{up-2p}V \otimes t_1^{(2)}/v_0v_1^{k-1}$ in $H^2M_0^2$.

The first term is 0 because $d_1(v_1^{p-k}v_2^{up-p}t_1^{(2)}/v_0^2v_1^p) = B_{u-1,k}/v_0v_1^p$ by (5.2.2).

Consider the following elements in $\Omega_\Sigma^1(M_0^2 \otimes_A B)$:

$$(5.4.1)\qquad \chi_{k,l} = v_2^{lp}t_1^{(2)}/v_0^2v_1^k - v_2^{lp}V/v_0v_1^{k+p-1} - lv_2^{lp-p}Vt_1^{(2)}/v_0v_1^{k-1},$$

$$\omega_{k,u} = \chi_{k,u-1} - (k/2)Z_u/v_0v_1^{k+p} - kv_2^{up-p}t_1^{1+p^2}/v_0v_1^{k+1}.$$

Then, (3.1.1–3), (3.5.1–3) and direct calculations show that $d_1\chi_{k,l} = -kv_2^{lp}t_1 \otimes t_1^{(2)}/v_0v_1^{k+1}$ and so the second term in $(*)$ is 0 since $k<p$. Furthermore, we see that

$$(5.4.2)\qquad d_1\omega_{k,u} = (k/2)v_2^{up-p}t_1^{(2)} \otimes \zeta^p/v_0v_1^k + (k/4)v_2^{up-1}g_1/v_0v_1^{k-2} + X_1/v_0v_1^{k-3},$$

$$d_1(v_2^{up-p}(\xi_2 + 2v_2^{-p}t_1^{(2)}V)/v_0v_1^{k-1})$$

$$= -2v_2^{up-2p}V \otimes t_1^{(2)}/v_0v_1^{k-1} - v_2^{up-1}g_1/v_0v_1^{k-2} + X_2/v_0v_1^{k-3}$$

for some X_1 and $X_2 \in \Omega_\Sigma^2 B$. These together with $(*)$ show that

$$\lambda_* c = sv_2^{up-p}t_1^{(2)} \otimes \zeta^p/v_0v_1^k + X_3/v_0v_1^{k-3} \text{ for some } X_3 \in \Omega_\Sigma^2 B.$$

Since $\lambda_* c$ and the first term are in $H^2M_0^2$, so is the second term, which shows $X_3/v_0v_1 \in H^2M_0^2$ by (4.5.5). Then $X_3/v_1 \in H^2M_1^1$ and $|X_3/v_1| \equiv -3 \bmod p+1$, since $|c| \equiv |B_{t-1,j}| - i \equiv 1-k \bmod p+1$ by definition. On the other hand, there is no nonzero element z in $H^2M_1^1$ with $v_1z=0$ and $|z| \equiv -3 \bmod p+1$ by Theorem 4.4 (i). Therefore $X_3/v_1 = 0$ and there is an element $Y \in \Omega_\Sigma^2 B$ such that $X_3 = v_1 Y$. q.e.d.

By using these results together with Proposition 4.9, we can prove the following non-triviality theorem:

THEOREM 5.5. *On the products of the β-elements in π_*S given in* (1.4).

(i) $\beta_r\beta_{tp/p} \neq 0 \neq \beta_r\beta_{tp^2/p,2}$ *if* $p \nmid tr(r-1)$, *for* $r \geqq 1$, $t \geqq 2$;

$\beta_{sp+1}\beta_{tp/p} \neq 0$ *if* $p \nmid t(s+t)(s+t+1)$ *and*

$\beta_{sp+1}\beta_{tp^2/p,2} \neq 0$ *if* $p \nmid ts(s+1)$, *for* $s \geqq 0$, $t \geqq 2$;

(ii) $\beta_{sp/i}\beta_{tp/j} \neq 0$ *if* $p \nmid t$, $p \mid s+t$, $p^2 \nmid s+t+p$ *and* $p+3 \leqq i+j < 2p$,

for $s, t \geqq 1$ *and* $1 \leqq i, j \leqq p$ *with* $(i, s) \neq (p, 1) \neq (j, t)$.

PROOF. By (5.1.3) and the sparseness of the spectral sequence (1.1), it is sufficient to show the non-triviality in its E_2-term H^4A.

Consider the homomorphisms (5.1.1) for $n=2$, where δ and δ' are isomorphic because $H^nM_0^j = 0$ for $j=0, 1$ and $n \geqq j+1$ by [3; Th. 3.16, 4.2]. Then (i) is seen by Lemma 5.3, Proposition 4.9 and the equality $\beta_r\beta_{tp^2/p,2} = \beta_{r+t(p^2-p)}\beta_{tp/p}$ in H^4A [7; Prop. 6.1].

Now, let i, j, s and t satisfy the assumption in (ii), and put $k=i+j-p$ and $u=s+t$. Then, by Lemmas 5.4, 4.6, (5.4.1–2), Proposition 3.7 and (4.1.5), we have

$$\beta_{sp/i}\beta_{tp/j} = t\delta'\delta c,\ \lambda_* c - Y/v_0 v_1^{k-4} = s v_2^{up-p} t_1^{(2)} \otimes \zeta^p/v_0 v_1^k = s v_0 \omega_{k,u} \otimes \zeta',$$
$$\delta'_2 \lambda_* c = (ks/4) x_2^{u/p} G_2 \otimes \zeta/v_1^{k-2} + X/v_1^{k-3} \text{ for some } X \in \Omega_\Sigma^3 B,$$

because $v_2^{up-p} t_1^{(2)} \otimes \zeta^p \otimes \zeta^p / v_1^k = 0$. Thus $\delta'_2 \lambda_* c \neq 0$ in $H^3 M_1^1$ by Theorem 4.4 (ii), which shows (ii) since δ and δ' are isomorphic. q.e.d.

REMARK. The non-triviality of the other products of the β-elements in the E_2-term stated in [7; Th. 5.6 (ii)] can be seen by [7; Lemma 4.4] and Proposition 4.9 immediately.

References

[1] M. Hazewinkel, Constructing formal groups III: Application to complex cobordism and Brown-Peterson cohomology, J. Pure Appl. Algebra **10** (1977), 1–18.
[2] H. R. Miller and D. C. Ravenel, Morava stabilizer algebra and the localization of Novikov's E_2-term, Duke Math. J. **44** (1977), 433–447.
[3] H. R. Miller, D. C. Ravenel and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.
[4] S. Oka, A new family in the stable homotopy groups of spheres, Hiroshima Math. J. **5** (1975), 87–114.
[5] S. Oka, A new family in the stable homotopy groups of spheres II, Hiroshima Math. J. **6** (1976), 331–342.
[6] S. Oka, Realizing some cyclic BP_* modules and applications to stable homotopy of spheres, Hiroshima Math. J. **7** (1977), 427–447.
[7] S. Oka and K. Shimomura, On products of the β-elements in the stable homotopy of spheres, Hiroshima Math. J. **12** (1982), 611–626.
[8] D. G. Quillen, On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.
[9] D. C. Ravenel, The cohomology of the Morava stabilizer algebras, Math. Z. **152** (1977), 287–297.
[10] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. **106** (1984), 351–414.
[11] K. Shimomura and H. Tamura, Non-triviality of some compositions of β-elements in the stable homotopy of the Moore spaces, Hiroshima Math. J. **16** (1986), 121–133.
[12] R. M. Switzer, *Algebraic Topology —Homology and Homotopy—*, Springer, 1975.
[13] H. Toda, Algebra of stable homotopy of Z_p-spaces and applications, J. Math. Kyoto Univ. **11** (1971), 197–251.
[14] N. Yamamoto, Algebra of stable homotopy of Moore spaces, J. Math. Osaka City Univ. **14** (1963), 45–67.

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