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A central limit theorem of mixed type for a class of 1-dimensional transformations

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Contents

§0. Introduction.
§1. Central limit theorem of mixed type.
§2. Examples.
§3. Perron-Frobenius operators and invariant measures.
§4. Perturbed operators.
§5. Proofs of theorems.
§6. Remarks on limiting variances.

§0. Introduction

Let T be a nonsingular transformation on the interval $[0, 1]$. Many authors (cf. [2], [4], [5], [7], [12], [13], [14], [22], [25]) have investigated the following problem: under what conditions on T and f does the sequence of random variables $\{f(T^k x): k=0, 1,...\}$ satisfy a central limit theorem (c. l. t.)? Recently, J. Rousseau-Egele ([22]) obtained a c. l. t. for a class of transformations T and its rate of convergence, by estimating the asymptotic behavior of the characteristic function with the help of the Perron-Frobenius operator corresponding to T.

Generalizing his method, we can get central limit theorems of mixed type for a certain class of transformations, which are stated in §1. That is, under suitable assumptions on T, f and ν, the distribution function $\nu\{\sum_{k=0}^{n-1} f(T^k x)/n^{1/2}<z\}$ is asymptotically a mixed normal distribution function. Central limit theorems for β-transformations, α-continued fraction transformations, Wilkinson's piecewise linear transformations and unimodal linear transformations are given as corollaries to our theorems.

In §1 we intrdouce the notations and the assumption (A) under which our results are obtained. Then we state our theorems. We should remark that the rate of convergence given in (1.9), (1.12) and (1.14) is $O(1/n^{1/2})$ and it is best possible for c. l. t.

In §2 it is shown that the transformations, treated in the articles [2], [4], [5], [7], [12], [13], [14], [22] and [25], satisfy the condition (A). Therefore, the results in the above articles are given as corollaries to our Theorems 3 and 4. Moreover the unimodal linear transformations are discussed as the concrete examples which do not satisfy the ordinary c. l. t. but our c. l. t. of mixed type.

In §3 we shall prove that under the assumption (A), for some $m_0>0$, T^{m_0} has a finite number of weakly mixing invariant probability measures and the other invariant probability measures can be represented as convex combinations of them. And some spectral properties of Perron-Frobenius operators, which are used in the proofs of our results, are also studied.

In §4 we shall investigate some perturbed operators of Perron-Frobenius operators and show that the chraracteristic function of $\sum_{k=0}^{n-1} f(T^k x)$ can be written by iterations of them.

In §5 the theorems in §1 are proved with the help of Esseen's inequality and the preparations in §3 and §4.

In §6 a concrete sufficient condition to ensure the positivity of the limiting variance is given by a method similar to that in J. Rousseau-Egele's article.

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§1. Central limit theorem of mixed type

We denote by m the Lebesgue measure on the interval $[0, 1]$ and by $(L^1(m), \|\cdot\|_m)$ the Banach space of Lebesgue integrable functions. Let T be a nonsingular transformation from $[0, 1]$ into itself, namely $m(A)=0$ implies $m(T^{-1}A)=0$. Let us write T^n for the n-th iterate of T.

We shall begin by defining the Perron-Frobenius operator $\Phi\colon L^1(m)\to L^1(m)$ corresponding to T by

$$\int g\Phi f dm = \int f(x)g(Tx)dm$$

for all $g\in L^\infty(m)$, where $L^\infty(m)$ denotes the Banach space of m-essentially bounded functions. It is well known that the operator Φ is linear, positive and has the following properties:

(1.1) *Φ preserves integrals* $\int \Phi f dm = \int f dm, f\in L^1(m)$;

(1.2) $|\Phi f| \le \Phi|f|$ *m-a.e.*, $f\in L^1(m)$;

(1.3) $\|\Phi f\|_m \le \|f\|_m$;

(1.4) $\Phi^n = \Phi_{T^n}$ *(Φ_{T^n} stands for the Perron-Frobenius operator corresponding to T^n)*;

(1.5) $\overline{\Phi f} = \Phi\bar{f}$, $f\in L^1(m)$;

(1.6) $\Phi((g\circ T)f) = g\Phi(f)$, $g\in L^\infty(m)$, $f\in L^1(m)$;

(1.7) *$\Phi f = f$ iff the measure $d\mu = f dm$ is invariant under T, that is* $\mu(T^{-1}A)$

$=\mu(A)$ *for each measurable* A.

For f: $[0, 1]\to C$, we denote the total variation of f by $\mathrm{var}(f)$. Let V be the set of functions $f\in L^1(m)$ which have versions $\tilde{f}$ with $\mathrm{var}(\tilde{f})<\infty$. V is a subspace of $L^1(m)$, but not closed. Put

$$\|f\|_V \equiv \|f\|_m + v(f)$$

for $f\in V$, where $v(f)\equiv\inf\{\mathrm{var}(\tilde{f}): \tilde{f} \text{ is a version of } f\}$.

Then we can easily prove the following

LEMMA 1.1 ([4], [22]). *$(V, \|\cdot\|_V)$ is a Banach space and*

$$\|fg\|_V \leq 2\|f\|_V\|g\|_V$$

for $f\in V$ and $g\in V$.

Now we assume that T satisfies the following condition:

(A) *There exist a positive integer n_0 and real numbers $0<\alpha<1$, $0<\beta<\infty$ such that*

$$v(\Phi^{n_0}f) \leq \alpha v(f) + \beta\|f\|_m$$

for all $f\in V$.

Note that transformations of various types satisfy this condition, as is well known (cf. §2).

Under this assumption we can get the following proposition, which seems to be essentially known already. But we shall prove it in §3 for completeness.

PROPOSITION 1.2. *There exist positive integers m_0, M and nonnegative functions $g_1, g_2,\dots, g_M$, belonging to V, such that $\{g_i>0\}\cap\{g_j>0\}=\emptyset$ $(i\neq j)$, $d\mu_j=g_j dm$ $(j=1, 2,\dots, M)$ are invariant probability measures under T^{m_0} and all other T^{m_0}-invariant m-absolutely continuous probabilities are convex combinations of μ_j's. Moreover (T^{m_0}, μ_j) $(j=1, 2,\dots, M)$ are weakly mixing.*

In the sequel we shall use the following notations. For a function f we denote $S_n(f)=\sum_{k=0}^{n-1}f(T^kx)$ and $b_j=\mu_j(f)=\int f d\mu_j$ if it has the meaning for each $j=1, 2,\dots, M$. Since f and $b_j=\mu_j(f)$ appear at the same time, there will be no confusion.

LEMMA 1.3. *Under the condition* **(A)** *it holds that for any $f\in V$ the limit*

$$\lim_{n\to\infty}\int\left(\sum_{k=0}^{n} f(T^kx)-b_j\right)/n^{1/2})^2 d\mu_j = \sigma_j^2$$

exists for each $j=1, 2,\dots, M$.

This lemma will be proved in §5. We define

$$F(b, \sigma^2; y) = (1/\sigma(2\pi)^{1/2})\int_{-\infty}^{y} \exp\{-(x-b)^2/2\sigma^2\}dx$$

for $\sigma^2 > 0$ and

$$F(b, 0; y) = \begin{cases} 1 & (y > b) \\ 0 & (y \leq b). \end{cases}$$

Using these notations we give our results. Their proofs are deferred to §5.

THEOREM 1 (Central limit theorem of mixed type). *Let the condition* (A) *be satisfied and* ν *an m-absolutely continuous probability measure with* $d\nu/dm \in V$. *For a function* f, *suppose that* $S_{m_0}(f)$ *belongs to* V. *If* $\sigma_j^2 \neq 0$ *for all* $0 \leq j \leq M$, *then*

$$\text{(1.8)} \qquad \sup_y |\nu\{S_n(f)/n^{1/2} < y\} - \textstyle\sum_{j=1}^{M} a_j F(n^{1/2} b_j, \sigma_j^2; y)| \leq C/n^{1/4}$$

for some real numbers $a_1 \geq 0$, $a_2 \geq 0, \ldots, a_M \geq 0$ $(\sum_{k=1}^{M} a_k = 1)$ *and* $C > 0$. *Moreover, if* ν *is T-invariant, we can get*

$$\text{(1.9)} \qquad \sup_y |\nu\{S_n(f)/n^{1/2} < y\} - \textstyle\sum_{j=1}^{M} a_j F(n^{1/2} b_j, \sigma_j^2; y)| \leq C/n^{1/2}.$$

If we put a further assumption on T, we can get a simpler statement:

THEOREM 2. *Assume that the conditions on* T *and* ν *in Theorem* 1 *are satisfied. If* $\{I_j : j = 1, 2, \ldots, N\}$ *is a partition of* $[0, 1]$ *into disjoint intervals, i.e.* $[0, 1] = \cup_{j=1}^{N} I_j$, *and* $T|I_j$ *is monotonic, then for all* $f \in V$ *with* $\sigma_j^2 \neq 0$ *(all* j*) the conclusions of Theorem* 1 *remain valid.*

As corollaries to Theorem 1, ordinary central limit theorems for 1-dimensional transformations are obtained.

THEOREM 3. *If* T *satisfies the condition* (A) *and has a unique m-absolutely continuous invariant probability measure* μ, *and if* (T, μ) *is weakly mixing, then for any probability measure* ν *with* $d\nu/dm \in V$ *and any* $f \in V$ *there exist* $\sigma^2 \geq 0$ *and* b *such that*

$$\text{(1.10)} \qquad \lim_{n\to\infty} \nu\{S_n(f-b)/n^{1/2} < y\} = F(0, \sigma^2; y)$$

at any continuity point of F. *In case* $\sigma^2 \neq 0$,

$$\text{(1.11)} \qquad \sup_y |\nu\{S_n(f-b)/n^{1/2} < y\} - F(0, \sigma^2; y)| \leq C/n^{1/4}$$

and

$$\sup_y |\mu\{S_n(f-b)/n^{1/2} < y\} - F(0, \sigma^2; y)| \leq C/n^{1/2} \tag{1.12}$$

for some $C>0$.

THEOREM 4. *If* T *satisfies the condition* (A) *and* μ *is an* m*-absolutly continuous* T*-invariant probability measure and if* (T, μ) *is weakly mixing, then for any* $f \in V$ *there exists* $\sigma^2 \geq 0$ *such that*

$$\lim_{n\to\infty} \mu\{S_n(f-b)/n^{1/2} < y\} = F(0, \sigma^2; y) \tag{1.13}$$

at any continuity point of F, *where* $b = \int f d\mu$. *In case* $\sigma^2 \neq 0$,

$$\sup_y |\mu\{S_n(f-b)/n^{1/2} < y\} - F(0, \sigma^2; y)| \leq C/n^{1/2} \tag{1.14}$$

for some $C>0$.

REMARK. The results obtained in [2], [5], [7], [12], [13], [14], [22] and [25] follow from (1.10) of Theorem 3, and the central limit theorem in [4] can be given as a corollary to (1.13) of Theorem 4.

§2. Examples

In this section we describe some examples which satisfy the condition (A).

(I) *Let* $\{I_j: j=1, 2,\dots, N\}$ *be a finite partition of* $[0, 1]$ *into intervals. Suppose that* T *satisfies the following*:

(2.1) $T|I_j$ *is monotonic and can be extended to a* C^2*-function on the closure* $\bar{I}_j$ *for all* $j=1, 2,\dots, N$.

(2.2) *There exists a positive integer* n_0 *such that*

$$\inf_x |d(T^{n_0})/dx| > 1.$$

Then the condition (A) in §1 is satisfied. See, for example, [15]. Various transformations have these properties.

β-transformation: Put $Tx = \beta x - [\beta x]$ for $0 \leq x \leq 1$, where $\beta > 1$ and $[x]$ denotes the integral part of x. This mapping T clearly has the above properties. It is already known that T has a unique m-absolutely continuous invariant measure μ and (T, μ) has weak Bernoulli property ([8]). Hence, this ensures the conclusions of Theorem 3, which generalize the central limit theorem obtained in [7].

Unimodal linear transformation: Let us define

$$Tx = \begin{cases} ax + (a+b-ab)/b & (0 \leq x \leq 1 - (1/b)) \\ -b(x-1) & (1-(1/b) \leq x \leq 1), \end{cases}$$

where $a>0$ and $b>1$. In [9] and [10], Sh. Ito, S. Tanaka and H. Nakada investigated in detail how the behavior of T depends on parameter values (a, b). The mapping in question does not always have the property (2.2). There exists the so-called window case, in which (2.2) is not satisfied. In this case there exists a unique periodic orbit and all points except the fixed points approach this orbit. Except the window case, T has a unique m-absolutely continuous invariant measure μ but (T, μ) is not always weakly mixing. It is shown in [10] that the result in Proposition 1.2 is valid for some integer $m_0=M$. For example, if $a=0.6$ and $b=3$, then it is shown that $m_0=M=3$. Let $(a, b)=(0.6, 3)$, $f(x)=x$ and $\nu=m$. The graph of the 11736 sample points of $S_{400}(f)$ is shown in Fig. 1 (due to I. Kubo).

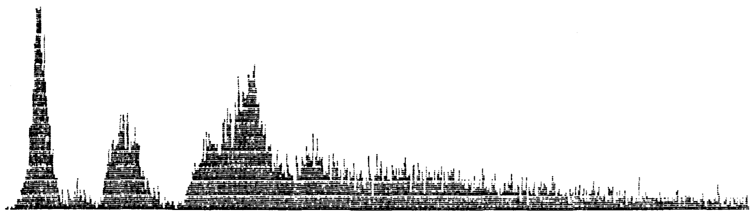

Fig. 1

Wilkinson's piecewise linear transformation: Wilkinson ([24]) studied the following transformations. For $0=a_0<a_1<\cdots<a_N=1$, $\beta_j>1$, $0\leq\alpha_j<1$ and $\beta_j(a_j-a_{j-1})+\alpha_j\leq 1$ $(j=1, 2,\dots, N)$, define

$$Tx = \beta_j(x-a_{j-1}) + \alpha_j, \quad a_{j-1} \leq x < a_j, \quad j = 1, 2,\dots, N.$$

He proved that these transformations satisfy the weak Bernoulli condition under an additional assumption. Since $\beta_j>1$, we get (2.1) and (2.2). Hence Theroem 2 in §1 can be applied.

(II) *Let* $\{I_j\colon j=1, 2,\dots\}$ *be a countable partition of* $[0, 1]$ *into intervals. Suppose that* T *satisfies the following* (2.3), (2.4) *and* (2.5):

(2.3) $T|I_j$ *is monotonic and can be extended to a* C^2*-function on* $\bar{I}_j$ *for all* $j=1, 2,\dots$.

(2.4) *There exists a positive integer* n_0 *such that*

$$\inf_x |d(T^{n_0})/dx| > 1.$$

(2.5) *The collection* $\{T(I_j)\colon j=1, 2,\dots\}$ *contains only a finite number of intervals.*

Then it was proved by J. Rousseau-Egele in [22] that T satisfies the condition

(A). Continued-fraction type transfomations have these properties (2.3), (2.4) and (2.5).

α-continued fraction transformation: In [19], [20] and [21], H. Nakada, Sh. Ito and S. Tanaka have defined an α-continued fraction transformation as follows. For $1/2 \leq \alpha \leq 1$, define T: $[\alpha-1, \alpha] \to [\alpha-1, \alpha]$ by

$$Tx = \begin{cases} |1/x| - [|1/x| + (1-\alpha)] & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Clearly, T is the classical continued fraction transformation if $\alpha=1$. They have proved that T has a unique m-absolutely continuous invariant measure μ and (T, μ) is weakly mixing. We can easily check that (2.3), (2.4) and (2.5) are valid for T. Hence, Theorem 3 gives us a central limit theorem and its rate of convergence for α-continued fraction transformations.

(III) *A. Lasota and J. A. Yorke treated another type of transformations, piecewise convex mappings* ([16]). *Let $0=a_0<a_1<\cdots<a_N=1$. Suppose that*

(2.6) *$T|[a_{j-1}, a_j)$ is continuous and convex for $j=1, 2,\dots, N$,*

(2.7) *$T(a_{j-1}) = 0$ and $T'(a_{j-1}) > 0$ for $j = 1, 2,\dots, N$,*

and

(2.8) $T'(0) > 1$.

Then the condition (A) is satisfied (cf. [11]). Using the results in [11] and the method of [7], M. Jabłonski and J. Malczak ([12]) obtained a central limit theorem, which is now given as a corollary to our Theorem 3.

§ 3. Perron-Frobenius operators and invariant measures

We shall show the outline of the proof of Proposition 1.2 and study some properties of Perron-Frobenius operators, which will be used in the following sections.

The ergodic theorem of C. Ionescu-Tulcea and G. Marinescu, given in [6], plays an essential role in what follows.

THEOREM 5 ([6]). *Let $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ and $(\mathscr{L}, \|\cdot\|_{\mathscr{L}})$ be Banach spaces with $\mathscr{V}$ dense in $\mathscr{L}$, and $\mathscr{T}: \mathscr{L} \to \mathscr{L}$ be a bounded linear map. Suppose further:*

(3.1) *If $f_n \in \mathscr{V}$ $(n=1, 2,\dots)$, $f \in \mathscr{L}$, $\lim_{n\to\infty} \|f_n - f\|_{\mathscr{L}}=0$ and $\|f_n\|_{\mathscr{V}} \leq C$ for all n, then $f \in \mathscr{V}$ and $\|f\|_{\mathscr{V}} \leq C$.*

(3.2) $\sup\{\|\mathscr{T}^n f\|_{\mathscr{L}}: f\in\mathscr{V},\ \|f\|_{\mathscr{L}}\leq 1,\ n\geq 0\} < \infty$.

(3.3) *There are a nonnegative integer n_0 and real numbers $0<\alpha<1$ and $0<\beta<\infty$ such that*

$$\|\mathscr{T}^{n_0}f\|_{\mathscr{V}} \leq \alpha\|f\|_{\mathscr{V}} + \beta\|f\|_{\mathscr{L}}$$

for all $f\in\mathscr{V}$.

(3.4) *If B is a bounded subset of $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$, then $\mathscr{T}^{n_0}B$ is relatively compact in $(\mathscr{L}, \|\cdot\|_{\mathscr{L}})$.*

Then we get the following:

(3.5) *$\mathscr{T}: \mathscr{L}\to\mathscr{L}$ has only a finite number of eigenvalues $\lambda_1, \lambda_2,\dots, \lambda_r$ of modulus* 1.

(3.6) *Set $\mathscr{E}(\lambda_j) = \{f\in\mathscr{L}: \mathscr{T}f = \lambda_j f\}$ for $1\leq j\leq r$. Then $\mathscr{E}(\lambda_j)\subset\mathscr{V}$ and* $\dim\mathscr{E}(\lambda_j) < \infty$.

(3.7) *The operator $\mathscr{T}$ can be represented as*

$$\mathscr{T} = \sum_{j=1}^{r}\lambda_j\mathscr{P}_j + \Psi$$

where $\mathscr{P}_j$ is the projection onto $\mathscr{E}(\lambda_j)$, $\|\mathscr{P}_j\|_{\mathscr{L}}\leq 1$, and Ψ is a linear operator on $\mathscr{L}$ with $\sup\{\|\Psi^n\|_{\mathscr{L}}: n\geq 1\}<\infty$. *Furthermore $\mathscr{P}_j\mathscr{P}_i=0$ $(i\neq j)$ and $\mathscr{P}_j\Psi=0$ (all j).*

(3.8) *$\Psi(\mathscr{V})\subset\mathscr{V}$ and, considered as a linear operator on $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$, Ψ satisfies $\|\Psi^n\|_{\mathscr{V}}\leq Hq^n$ $(n\geq 1)$ for some constants H and q with $H>0$ and $0<q<1$.*

Applying Theorem 5, we get the following properties of the Perron-Frobenius operator Φ corresponding to (T, m).

LEMMA 3.1. *Under the assumption* (A) *we have the following*:

(3.9) *$\Phi: L^1(m)\to L^1(m)$ has only a finite number of eigenvalues $\lambda_1, \lambda_2,\dots, \lambda_p$ of modulus* 1.

(3.10) *Set $E(\lambda_j)=\{f\in L^1(m): \Phi f=\lambda_j f\}$ for $1\leq j\leq p$. Then $E(\lambda_j)\subset V$ and* $\dim E(\lambda_j) < \infty$.

(3.11) *The operator Φ can be represented as*

$$\Phi^n = \sum_{j=1}^{p}\lambda_j^n P_j + \Psi^n \quad (n\geq 1)$$

where P_j is the projection onto $E(\lambda_j)$, $\|P_j\|_m\leq 1$, and Ψ is a linear operator on $L^1(m)$ with $\sup\{\|\Psi^n\|_m: n\geq 1\}<\infty$. *Furthermore $P_iP_j=0$ $(i\neq j)$ and $P_j\Psi=\Psi P_j=0$ (all j).*

(3.12) *$\Psi(V)\subset V$ and, considered as a linear operator on $(V, \|\cdot\|_V)$, Ψ*

satisfies $\|\Psi^n\|_V \le Hq^n$ $(n \ge 1)$ *for some constants* $H>0$ *and* $0<q<1$.

(3.13) $\lambda_1 = 1$ *is an eigenvalue of* Φ; *and* P_1 *is a positive operator and* $\int P_1 f dm = \int f dm$ *for all* $f \in L^1(m)$.

PROOF. Let us put $(\mathscr{V}, \|\cdot\|_{\mathscr{V}}) = (V, \|\cdot\|_V)$, $(\mathscr{L}, \|\cdot\|_{\mathscr{L}}) = (L^1(m), \|\cdot\|_m)$ and $\mathscr{T} = \Phi$. From Lemma 5 in [4], it follows that for every $C>0$ the set $E = \{f \in L^1(m): \|f\|_V \le C\}$ is compact in $L^1(m)$. (3.1) follows immediately from this. (3.2), (3.3) and (3.4) follow from this and properties (1.1), (1.2) and (1.3) and the assumption (A). Hence we get (3.9), (3.10), (3.11) and (3.12) from Theorem 5. (3.13) can be obtained by the same method as in the proof of Theorem 1 in [15].

We denote $N = \dim E(1)$. The following lemma is a direct modification of a result in [17], so we shall omit its proof.

LEMMA 3.2. *There exists a base* $\{g_1, g_2, \dots, g_N\}$ *of* $E(1)$ *such that* $g_j \in V$, $g_j \ge 0$, $\int g_j dm = 1$ *(all* j*) and* $\{g_i > 0\} \cap \{g_j > 0\} = \emptyset$ $(i \ne j)$.

Let μ_j be a probability measure defined by $d\mu_j = g_j dm$, which is T-invariant. Let $(L^1(\mu_j), \|\cdot\|_{\mu_j})$ be the Banach space of μ_j-integrable functions and

$$V_j = \{f \in L^1(\mu_j): fg_j \in V\}. \tag{3.14}$$

for $j = 1, 2, \dots, N$. We define a norm $\|\cdot\|_j$ by

$$\|f\|_j = \|fg_j\|_V, \quad f \in V_j. \tag{3.15}$$

Then we have

LEMMA 3.3. $(V_j, \|\cdot\|_j)$ *is a Banach space and* $V \subset V_j$ *for every* $j = 1, 2, \dots, N$.

PROOF. Let $\{f_n\}$ be a Cauchy sequence in $(V_j, \|\cdot\|_j)$. Then $\{f_n g_j: n = 1, 2, \dots\}$ is a Cauchy sequence in $(V, \|\cdot\|_V)$. Hence Lemma 1.1 shows that there exists a function $F \in V$ with $\lim_{n\to\infty} \|f_n g_j - F\|_V = 0$. The inequality $\|\cdot\|_m \le \|\cdot\|_V$ clearly holds, so there is a subsequence $\{f_{n_k} g_j\}$ of $\{f_n g_j\}$ which is m-a.e. convergent to F. If we set $f(x) = F(x)/g_j(x)$ for $g_j(x) > 0$ and $f(x) = 0$ for $g_j(x) = 0$, then we get $fg_j = F$ and $\lim_{n\to\infty} \|f_n - f\|_j = 0$. This shows that the normed space $(V_j, \|\cdot\|_j)$ is complete. The relation $V \subset V_j$ immediately follows from Lemma 1.1. So we get the desired results.

For each $j = 1, 2, \dots, N$, we consider the Perron-Frobenius operator Φ_j: $L^1(\mu_j) \to L^1(\mu_j)$ corresponding to (T, μ_j) defined by

$$\int h\Phi_j f d\mu_j = \int f(x) h(Tx) d\mu_j \tag{3.16}$$

for all $h \in L^\infty(\mu_j)$, where $L^\infty(\mu_j)$ stands for the set of all μ_j-essentially bounded functions. The next lemma will provide a description of Φ_j in terms of Φ.

Lemma 3.4. *For $f \in L^1(\mu_j)$ we have*

$$\Phi_j f = \Phi(fg_j)/g_j \qquad (\mu_j\text{-a.e.}).$$

Proof. It is easily seen that $T^{-1}\{g \neq 0\} \supset \{g \neq 0\}$ holds for $g \in E(1)$. Therefore we have

$$\begin{aligned}
\int h\Phi_j f d\mu_j &= \int f(x)h(Tx)d\mu_j = \int f(x)h(Tx)g_j(x)dm \\
&= \int f(x)g_j(x)h(Tx)I_{T^{-1}\{g_j>0\}}\, dm \\
&= \int f(x)g_j(x)h(Tx)I_{\{g_j>0\}}\,(Tx)dm \\
&= \int \Phi(fg_j)hI_{\{g_j>0\}}\, dm \\
&= \int (\Phi(fg_j)/g_j)hd\mu_j
\end{aligned}$$

for all $h \in L^\infty(\mu_j)$, where I_A stands for the indicator function of A. This implies the desired result.

The operator Φ_j has properties similar to the properties (1.1) – (1.7) of Φ.

Lemma 3.5. *The operator Φ_j is positive, bounded and linear, and has the following properties*:

(3.17) *Φ_j preserves integrals* $\int \Phi_j f d\mu_j = \int f d\mu_j$, $f \in L^1(\mu_j)$;

(3.18) $|\Phi_j f| \leq \Phi_j|f|$ *μ_j-a.e.*, $f \in L^1(\mu_j)$;

(3.19) $\|\Phi_j f\|_{\mu_j} \leq \|f\|_{\mu_j}$;

(3.20) $\Phi_j^n = \Phi_{j,T^n}$ *(Φ_{j,T^n} stands for the Perron-Frobenius operator corresponding to (T^n, μ_j))*;

(3.21) $\overline{\Phi_j f} = \Phi_j \bar{f}$, $f \in L^1(\mu_j)$;

(3.22) $\Phi_j((g \circ T)f) = g\Phi_j(f)$, $g \in L^\infty(\mu_j)$, $f \in L^1(\mu_j)$;

(3.23) $\Phi_j f = f$ ($f \in L^1(\mu_j)$) *if and only if f is μ_j-a.e. equal to a constant.*

Proof. The results (3.17) – (3.22) are immediately derived from (1.1) – (1.6), (3.16) and Lemma 3.4. If $\Phi_j f = f$, then Lemma 3.4 implies that $fg_j \in E(1)$. On the other hand, it is trivial that $\{fg_j\neq 0\}\subset\{g_j\neq 0\}$. Therefore Lemma 3.2 asserts that f is equal to a constant μ_j-almost everywhere. Since the converse is trivial, this completes the proof.

The relationship between the eigenvalues of modulus 1 of Φ and those of Φ_j can be described as follows.

LEMMA 3.6. *If $\Phi f=\lambda f$ for $|\lambda|=1$ and a nonzero element f of $L^1(m)$, then there exists an integer j $(1\leq j\leq N)$ such that f/g_j is a nonzero element of $L^1(\mu_j)$ and $\Phi_j(f/g_j)=\lambda(f/g_j)$.*

PROOF. Combining the assumption $\Phi f=\lambda f$ and the property (1.2), we can get $|f|=|\lambda f|=|\Phi f|\leq\Phi|f|$. Hence, the property (1.1) of Φ shows that $|f|=\Phi|f|$ and $|f|\in E(1)$. On the other hand, since $\{g_1, g_2,\dots, g_N\}$ is a base of $E(1)$, $|f|=\sum_{j=1}^N a_jg_j$ for some $(a_1, a_2,\dots, a_N)\neq(0, 0,\dots, 0)$ and $\{f\neq 0\}\subset\{\sum_{j=1}^N g_j>0\}$. This allows us to define

$$\tilde{f}(x) = \begin{cases} f(x)/\sum_{j=1}^N g_j(x), & \text{if } \sum_{j=1}^N g_j(x) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Noticing (1.6) and that $T^{-1}\{h\neq 0\}\supset\{h\neq 0\}$ holds for $h\in E(1)$, we can get

$$\begin{aligned}\lambda\tilde{f}\textstyle\sum_{j=1}^N g_j &= \lambda f = \Phi f = \Phi(\tilde{f}\textstyle\sum_{j=1}^N g_j) = \textstyle\sum_{j=1}^N\Phi(\tilde{f}g_j) \\ &= \textstyle\sum_{j=1}^N \Phi(\tilde{f}g_jI_{T^{-1}\{g_j>0\}}) = \textstyle\sum_{j=1}^N \Phi(\tilde{f}g_j)I_{\{g_j>0\}}\,.\end{aligned}$$

Since $\{g_i>0\}\cap\{g_j>0\}=\emptyset$, we have $\lambda\tilde{f}g_j=\Phi(\tilde{f}g_j)I_{\{g_j>0\}}$. This and Lemma 3.4 imply that $\Phi_j(f/g_j)=\lambda(f/g_j)$, because $f/g_j=\tilde{f}$ (μ_j-a.e.). Since $|f|=\sum_{j=1}^N a_jg_j$ for some $(a_1, a_2,\dots, a_N)\neq(0, 0,\dots, 0)$, the proof is completed.

It is clear that the converse statement of this lemma is also valid.

LEMMA 3.7. *If $\Phi_jf=\lambda f$ for $|\lambda|=1$ and a nonzero element $f\in L^1(\mu_j)$, then we have $\Phi(fg_j)=\lambda fg_j$.*

Let us write U_j for the isometric operator on $L^1(\mu_j)$ defined by

$$(U_jf)(x) = f(Tx)\,. \tag{3.24}$$

Then we have the following

LEMMA 3.8. *The following* (i) *and* (ii) *are equivalent.*

(i) *$\Phi_jf=\lambda f$ for $|\lambda|=1$ and a nonzero element $f\in L^1(\mu_j)$.*

(ii) *$U_jf=\bar{\lambda}f$ for a nonzero element $f\in L^1(\mu_j)$.*

PROOF. (This proof is due to T. Morita (private communication and [18]).)

Let $\mathscr{B}$ be the Borel σ-field on [0, 1] and $\mathscr{B}_n = T^{-n}\mathscr{B}$. Then we have

$$U_j^n \Phi_j^n(f) = E_{\mu_j}[f|\mathscr{B}_n] \qquad (\mu_j\text{-a.e.}) \tag{3.25}$$

for all $f \in L^1(\mu_j)$, where $E_{\mu_j}[f|\mathscr{F}]$ denotes the conditional expectation of f relative to the σ-field $\mathscr{F}$ by the probability μ_j. In fact we have

$$\begin{aligned}\int_{T^{-n}A} U_j^n \Phi_j^n(f)\, d\mu_j &= \int U_j^n \Phi_j^n(f) \cdot I_{T^{-n}A} d\mu_j \\ &= \int U_j^n(\Phi_j^n(f) \cdot I_A) d\mu_j \\ &= \int \Phi_j^n(f) \cdot I_A d\mu_j \\ &= \int f \cdot I_{T^{-n}A} d\mu_j = \int_{T^{-n}A} f d\mu_j\end{aligned}$$

for all $A \in \mathscr{B}$. Therefore, the property (i) implies

$$E_{\mu_j}[f|\mathscr{B}_n] = U_j^n \Phi_j^n(f) = U_j^n(\lambda^n f) = \lambda^n U_j^n(f).$$

On the other hand, using Doob's theorem, we can get

$$\lim_{n\to\infty} E_{\mu_j}[f|\mathscr{B}_n] = f^*$$

for some $f^* \in L^1(\mu_j)$ in the sense of $L^1(\mu_j)$ and also μ_j-almost everywhere. Hence, we get

$$\begin{aligned}f^* \circ T &= \lim_{n\to\infty} U_j^{n+1} \Phi_j^n(f) \\ &= \lim_{n\to\infty} \lambda^n U_j^{n+1}(f) \\ &= \bar{\lambda} \lim_{n\to\infty} \lambda^{n+1} U_j^{n+1}(f) \\ &= \bar{\lambda} \lim_{n\to\infty} E_{\mu_j}[f|\mathscr{B}_{n+1}] = \bar{\lambda} f^*.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int |f - f^*| d\mu_j &= \int |U_j^n(f) - U_j^n(f^*)| d\mu_j \\ &= \int |\lambda^n U_j^n(f) - \lambda^n U_j^n(f^*)| d\mu_j \\ &= \int |E_{\mu_j}[f|\mathscr{B}_n] - f^*| d\mu_j\end{aligned}$$

for all n. Hence, $\int |f - f^*| d\mu_j = \lim_{n\to\infty} \int |E_{\mu_j}[f|\mathscr{B}_n] - f^*| d\mu_j = 0$. Thus the statement (i) implies (ii).

Conversely, if we assume $U_j f=\lambda f$ for some $f\in L^1(\mu_j)$, then we have $f=f\Phi_j(1)=\Phi_j(f\circ T)=\Phi_j(U_j f)=\Phi_j(\lambda f)=\lambda\Phi_j(f)$, using the property (3.22). Because U_j is an isometric operator, we can get $|\lambda|=1$. The proof is therefore completed.

Now we can describe the outline of the proof of Proposition 1.2. It is well known that the set Λ_j of eigenvalues of U_j is a subgroup of S^1 and hence it consists of roots of unity. Combining Lemmas 3.1, 3.7 and 3.8, we get that Λ_j is a finite set and so $\Lambda_j=\{1, \eta_j, \eta_j^2,\dots, \eta_j^{m_j-1}\}$ $(\eta_j^{m_j}=1)$. Now using also Lemma 3.6, we see that Φ^{m_0} has only the unique eigenvalue 1 of modulus 1 for $m_0=m_1m_2\cdots m_N$. Notice that T^{m_0} also satisfies the condition (**A**). For simplicity of notations, we shall assume $m_0=1$ and use the same notations as in the case of T in what follows, where no danger of confusion exists. Then Lemma 3.1 can be modified as follows:

(3.26) *Φ has only the unique eigenvalue* 1 *of modulus* 1.

(3.27) *Set* $E(1)=\{f\in L^1(m)\colon \Phi f=f\}$. *Then* $E(1)\subset V$ *and* $\dim E(1)=M<\infty$.

(3.28) *The operator Φ^n can be represented as*

$$\Phi^n = P_1 + \Psi^n \qquad (n\geq 1),$$

where P_1 is the projection onto $E(1)$, $\|P_1\|_m\leq 1$, and Ψ is a linear operator on $L^1(m)$ with $\sup\{\|\Psi^n\|_m\colon n\geq 1\}<\infty$ *and* $P_1\Psi=\Psi P_1=0$.

(3.29) *$\Psi(V)\subset V$ and, considered as a linear operator on $(V, \|\cdot\|_V)$, Ψ satisfies $\|\Psi^n\|_V\leq Hq^n$ $(n\geq 1)$ for some constants $H>0$ and $0<q<1$.*

(3.30) *$\int P_1 f dm=\int f dm$ and $\int \Psi f dm=0$ for all $f\in L^1(m)$; and $f\geq 0$ implies that $P_1 f\geq 0$.*

Now Lemma 3.2 applies and we have the following:

(3.31) *There is a base $\{g_1, g_2,\dots, g_M\}$ of $E(1)$ such that $g_j\geq 0$, $\int g_j dm=1$ (all j) and $\{g_i>0\}\cap\{g_j>0\}=\emptyset$ $(i\neq j)$.*

For each j $(j=1, 2,\dots, M)$ we define μ_j, Φ_j and etc. in the same manner as before. Applying Theorem 5 and the above results, we can get results analogous to (3.26)-(3.30).

LEMMA 3.9. *Under the above assumptions and notations we have the following*:

(3.32) *Φ_j has only the unique eigenvalue* 1 *of modulus* 1.

(3.33) *Set* $E_j(1)=\{f\in L^1(\mu_j)\colon \Phi_j f=f\}$. *Then* $E_j(1)\subset V_j$ *and* $\dim E_j(1)=1$.

(3.34) *The operator Φ_j^n can be represented as*

$$\Phi_j^n = \mu_j + \Psi_j^n \qquad (n \geq 1)$$

where μ_j *is the projection onto* $E_j(1)$, $\mu_j(f) = \int f d\mu_j$ *and* Ψ_j *is a linear operator on* $L^1(\mu_j)$ *with* $\sup\{\|\Psi_j^n\|_{\mu_j}: n \geq 1\} < \infty$. *Furthermore* $\mu_j \Psi_j = \Psi_j \mu_j = 0$.

(3.35) $\Psi_j(V_j) \subset V_j$ *and, considered as a linear operator on* $(V_j, \|\cdot\|_j)$, Ψ_j *satisfies* $\|\Psi_j^n\|_j \leq \tilde{H}\tilde{q}^n$ $(n \geq 1)$ *for some constants* $\tilde{H} > 0$ *and* $0 < \tilde{q} < 1$.

(3.36) $\int \Psi_j f d\mu_j = 0$ *for all* $f \in L^1(\mu_j)$.

PROOF. The property (3.32) follows from (3.26) and Lemmas 3.6 and 3.7. If $\Phi_j h = h$ holds for some $h \in L^1(\mu_j)$, Lemma 3.4 shows that $\Phi(hg_j) = hg_j$ (μ-a.e.). On the other hand, (3.31) implies that $hg_j = a_j g_j$ (m-a.e.) for some $a_j \neq 0$. Hence, the property $\dim E_j(1) = 1$ follows. In order to prove (3.33)-(3.35), we apply Theorem 5. Remember the definitions (3.14) and (3.15). If $f_n \in V_j$ $(n = 1, 2, \ldots)$, $f \in L^1(\mu_j)$, $\lim_{n\to\infty} \|f_n - f\|_{\mu_j} = 0$ and $\|f_n\|_j \leq C$ for all n then $f_n g_j \in V$ $(n = 1, 2, \ldots)$, $fg_j \in L^1(m)$, $\lim_{n\to\infty} \|f_n g_j - fg_j\|_m = 0$ and $\|f_n g_j\|_V \leq C$. We can prove $fg_j \in V$ and $\|fg_j\|_V \leq C$ as in the proof of Lemma 3.1, and hence we have $f \in V_j$ and $\|f\|_j \leq C$. This implies that the condition (3.1) of Theorem 5 is satisfied for $\mathscr{V} = V_j$ and $\mathscr{L} = L^1(\mu_j)$. The condition (3.2) is an immediate consequence of (3.19). To show (3.3) we first remark that $g_j \Phi_j^n(f) = \Phi^n(fg_j)$ (m-a.e.) for all n. In fact Lemma 3.4 implies that $g_j \Phi_j^n(f) = \Phi^n(fg_j)$ (μ_j-a.e.). So it is sufficient to show $\{\Phi^n(fg_j) \neq 0\} \subset \{g_j \neq 0\}$ (m-a.e.). Remember that $g_j \in E(1)$ implies $T^{-1}\{g_j \neq 0\} \supset \{g_j \neq 0\}$. Then we have

$$\begin{aligned}\int \Phi^n(fg_j) I_{\{g_j \neq 0\}} h dm &= \int f(x) g_j(x) I_{T^{-n}\{g_j \neq 0\}}(x) h(T^n x) dm \\ &= \int f(x) g_j(x) h(T^n x) dm \\ &= \int \Phi^n(fg_j) h dm\end{aligned}$$

for all $h \in L^\infty(m)$. This shows that $\{\Phi^n(fg_j) \neq 0\} \subset \{g_j \neq 0\}$ (m-a.e.) and hence $g_j \Phi_j^n(f) = \Phi^n(fg_j)$ (m-a.e.). Therefore we have

$$\begin{aligned}\|\Phi_j^{n_0}(f)\|_j &= \|g_j \Phi_j^{n_0}(f)\|_V = \|\Phi^{n_0}(fg_j)\|_V \\ &= v(\Phi^{n_0}(fg_j)) + \|\Phi^{n_0}(fg_j)\|_m \\ &\leq \alpha v(fg_j) + (\beta + 1)\|fg_j\|_m \\ &= \alpha \|fg_j\|_V + (\beta + 1 - \alpha)\|fg_j\|_m \\ &= \alpha \|f\|_j + (\beta + 1 - \alpha)\|f\|_{\mu_j}.\end{aligned}$$

This implies the condition (3.3). Because (3.4) can be proved by the same method as in Lemma 3.1, Theorem 5 shows that (3.33)-(3.35) are valid. Since (3.36) immediately follows from (3.34), the proof is completed.

The results (3.26) and (3.27) imply (3.31) with the help of Lemma 3.2. For each j ($j=1, 2,\dots, M$), (3.32) and (3.33) show that Φ_j has the unique and simple eigenvalue 1 of modulus 1. Hence, it follows from Lemma 3.8 that 1 is the unique and simple eigenvalue of U_j and that (T, μ_j) is weakly mixing for each $1 \leq j \leq M$.

§4. Perturbed operators

In this section we shall investigate the perturbed operators of Φ and Φ_j, which play an essential role to prove our theorems in the next section. We can follow very closely to the technique of Rousseau-Egele's article [22]. Remember that we have assumed $m_0=1$. First we define the perturbed operator $\Phi(\theta; f)$ of Φ by

$$\Phi(\theta; f)(g) = \Phi(g \cdot \exp\{i\theta f\}) \tag{4.1}$$

for $f \in L^1(m)$, $g \in L^1(m)$ and $\theta \in \boldsymbol{R}$. Then we can get the following

Lemma 4.1. *For all $\theta \in \boldsymbol{R}$ and $n \geq 0$ we have*

$$\Phi^n(\theta; f)(g) = \Phi^n(g \cdot \exp\{i\theta S_n(f)\}) \qquad (m\text{-a.e.})$$

and hence

$$\int \Phi^n(\theta; f)(g)dm = \int g \cdot \exp\{i\theta S_n(f)\}dm.$$

Proof. Using the property (1.6), we have

$$\begin{aligned}\Phi^n(g \cdot \exp\{i\theta S_n(f)\}) &= \Phi(\Phi^{n-1}(g \cdot \exp\{i\theta f \circ T^{n-1}\} \cdot \exp\{i\theta S_{n-1}(f)\})) \\ &= \Phi(\exp\{i\theta f\} \cdot \Phi^{n-1}(g \cdot \exp\{i\theta S_{n-1}(f)\})) \\ &= \Phi(\theta; f)(\Phi^{n-1}(g \cdot \exp\{i\theta S_{n-1}(f)\})).\end{aligned}$$

This implies the desired results.

Thus the behavior of the characteristic function $\int g \cdot \exp\{i\theta S_n(f)\}dm$ can be described by the iteration of $\Phi(\theta; f)$ and hence by the spectrum of $\Phi(\theta; f)$. We can also get the following lemmas for $\Phi(\theta; f)$ as an operator on V.

Lemma 4.2. *If $f \in V$, then $\Phi(\theta; f)$: $V \to V$ is a bounded linear operator and analytic in θ.*

Proof. We get that $g \cdot \exp\{i\theta f\} \in V$ for $f \in V$ and $g \in V$ by means of Lemma 1.1. Noticing also that Φ: $V \to V$ is a bounded linear operator, we have

$$\|\Phi(\theta;f)(g)\|_V \leq \|\Phi\|_V\|g\cdot\exp\{i\theta f\}\|_V$$
$$\leq 2\|\Phi\|_V\|g\|_V\|\exp\{i\theta f\}\|_V$$

Hence $\Phi(\theta;f)$ is a bounded linear operator on V. Clearly, Lemma 1.1 implies that $\sum_{n=0}^{\infty}\{(i\theta)^n/n!\}\Phi(f^n g)$ converges to $\Phi(\theta;f)g$ in the sense of $\|\cdot\|_V$-norm. Therefore, $\Phi(\theta;f)$ is analytic in θ.

Now we have the following key lemma.

LEMMA 4.3. *For $f\in V$ there exists $d>0$ such that if $|\theta|<d$, then we have the following*:

(i) *For all $g\in V$ and $n\geq 1$*

$$\Phi^n(\theta;f)g = \Phi^n(\theta;f)P_1(\theta)g + \Psi^n(\theta)g$$

holds, where $P_1(\theta)$ is the projection onto the M-dimensional subspace of V with $P_1(0)=P_1$ and $\Phi(\theta;f)P_1(\theta)V\subset P_1(\theta)V$, and $\Psi(\theta)$ is a bounded linear operator on V with

$$\limsup_{n\to\infty}(\|\Psi^n(\theta)\|_V)^{1/n} \leq (1+2q)/3 < 1$$

and $\Psi(\theta)P_1(\theta)=P_1(\theta)\Psi(\theta)=0$. Here q is the constant given in (3.29) *for Ψ.*

(ii) *$P_1(\theta)$ and $\Psi(\theta)$ are analytic in θ.*

(iii) *There exists $C>0$ such that*

$$\left|\int\Psi^n(\theta)gdm\right| \leq C|\theta|\{(1+2q)/3\}^n\|g\|_V$$

for all $g\in V$ and $n\geq 1$.

PROOF. Let $R(z)$ be the resolvent operator of Φ. Noticing (3.28), we have for $|z|>q$ and $z\neq 1$

(4.2) $$R(z) = (zI-\Phi)^{-1} = \{P_1/z(z-1)\} + \sum_{n=0}^{\infty}(\Psi^n/z^{n+1}).$$

It is easy to see that

(4.3) $$R(\theta;\, z) = R(z)\sum_{n=0}^{\infty}((\Phi(\theta;f) - \Phi)R(z))^n$$

converges in the sense of $\|\cdot\|_V$-norm for small $|\theta|$ with $\|\Phi(\theta;f)-\Phi\|_V<\|R(z)\|_V^{-1}$; and that $R(\theta;\, z)$ is the resolvent operator of $\Phi(\theta;f)$ and analytic in θ. Let I_1 be the circle of center 1 and radius $\rho_1=(1-q)/3$; let I_2 be the one of center 0 and radius $\rho_2=(1+2q)/3$. Choose $\delta>0$ such that $\delta<\rho_1$ and $q+\delta<\rho_2$. Let us define $M(\delta)=\sup\{\|R(z)\|_V\colon |z|>q+\delta,\ |z-1|>\delta\}$. Then we can get $0<M(\delta)<\infty$ from (4.2). If $|\theta|$ is sufficiently small, then $\|\Phi(\theta;f)-\Phi\|_V<1/M(\delta)$, and hence I_1 and

I_2 are contained in the resolvent set of $\Phi(\theta; f)$. Define the projections

(4.4) $$P_1(\theta) = (1/2\pi i)\int_{I_1} R(\theta;\, z)dz$$

(4.5) $$Q(\theta) = (1/2\pi i)\int_{I_2} R(\theta;\, z)dz.$$

Because $R(\theta;\, z)$ is analytic in θ, $P_1(\theta)$ and $Q(\theta)$ are also analytic. Therefore, for any θ with sufficiently small $|\theta|$, we have $\|P_1(\theta)-P_1\|_V<1$ and hence $\dim P_1(\theta)V=\dim P_1 V$ (cf. Chap. VII, [1]). We get for all $n\geq 1$

$$\begin{aligned}\Phi^n(\theta; f) &= \Phi^n(\theta; f)P_1(\theta) + \Phi^n(\theta; f)Q(\theta)\\ &= \Phi^n(\theta; f)P_1(\theta) + \Psi^n(\theta),\end{aligned}$$

putting

(4.6) $$\Psi^n(\theta) = (1/2\pi i)\int_{I_2} z^n R(\theta;\, z)dz.$$

Since $R(\theta;\, z)$ is analytic in θ, we have the expansion

$$R(\theta;\, z) = R(z) + \theta\cdot R^{(1)}(\theta;\, z)$$

for some bounded linear operator $R^{(1)}(\theta;\, z)$. Hence, from (4.6), we get

$$\begin{aligned}\Psi^n(\theta)g &= (1/2\pi i)\int_{I_2} z^n R(z)gdz + (\theta/2\pi i)\int_{I_2} z^n R^{(1)}(\theta;\, z)gdz\\ &= \Psi^n g + (\theta/2\pi i)\int_{I_2} z^n R^{(1)}(\theta;\, z)gdz.\end{aligned}$$

Therefore, the property (3.30) shows that

$$\left|\int \Psi^n(\theta)gdm\right| \leq C|\theta|\,\{(1+2q)/3\}^n\|g\|_V$$

holds for

$$C = \sup\,\{\|R^{(1)}(\theta;\, z)\|_V/2\pi\colon z\in I_2,\ |\theta| < d\}.$$

This completes the proof.

Now we define the perturbed operator $\Phi_j(\theta; f)$ of Φ_j for each $j=1, 2,\ldots, M$ as follows:

(4.7) $$\Phi_j(\theta; f) = \Phi_j(g\cdot\exp\{i\theta f\})$$

for $f\in V$ and $g\in V_j$. Then we have a lemma analogous to Lemma 4.2.

LEMMA 4.4. *For $f \in V$, $\Phi_j(\theta; f)$: $V_j \to V_j$ is a bounded linear operator and analytic in θ.*

PROOF. If we remark the fact that $h \cdot g \in V_j$ holds for $h \in V$ and $g \in V_j$ and that $\|g\|_j = \|g \cdot g_j\|_V$, then the proof can be copied from the proof of Lemma 4.2.

Using Lemma 3.9 and remembering that Φ_j has only the unique and simple eigenvalue 1 of modulus 1, we get stronger results than those of Lemma 4.3.

LEMMA 4.5. *For $f \in V$ there exists $d > 0$ such that for $|\theta| < d$ we have the following*:

(i) *For all $g \in V_j$ and $n \geq 1$*

$$\Phi_j^n(\theta; f)g = \lambda_j^n(\theta) M_j(\theta) g + \Psi_j^n(\theta) g$$

holds, where $\lambda_j(\theta)$ is the unique eigenvalue of $\Phi_j(\theta; f)$ with the maximum absolute value and $|\lambda_j(\theta)| > (2 + \tilde{q})/3$; $M_j(\theta)$ is the projection onto the 1-dimensional eigenspace corresponding to $\lambda_j(\theta)$ with $M_j(0) = \mu_j$; and $\Psi_j(\theta)$ is a bounded linear operator on V_j with

$$\limsup_{n \to \infty} (\|\Psi_j^n(\theta)\|_j)^{1/n} \leq (1 + 2\tilde{q})/3 < 1$$

and $\Psi_j(\theta) M_j(\theta) = M_j(\theta) \Psi_j(\theta) = 0$.

(ii) *$\lambda_j(\theta)$, $M_j(\theta)$ and $\Psi_j(\theta)$ are analytic in θ.*

(iii) *There exists $C > 0$ such that*

$$\left| \int \Psi_j^n(\theta) g d\mu_j \right| \leq C|\theta| \, \{(1 + 2\tilde{q})/3\}^n \|g\|_j$$

for all $g \in V_j$ and $n \geq 1$.

PROOF. If we use the properties (3.32) – (3.36) instead of (3.26) – (3.30), this lemma can be proved similarly to the proof of Lemma 4.3. We have to prove only that $\Phi_j^n(\theta; f) M_j(\theta) = \lambda_j^n(\theta) M_j(\theta)$ for some $\lambda_j(\theta) \in \boldsymbol{C}$ and $\lambda_j(\theta)$ is analytic. In fact $\dim M_j(\theta) V_j = \dim \mu_j V_j = 1$ holds, if $|\theta|$ is sufficiently small. And hence there is $\lambda_j(\theta) \in \boldsymbol{C}$ such that

$$\Phi_j(\theta; f) g = \Phi_j(\theta; f) M_j(\theta) g = \lambda_j(\theta) g$$

holds for all $g \in M_j(\theta) V_j$. Because $\Phi_j(\theta; f)|_{M_j(\theta) V_j} = \lambda_j(\theta)$, $\lambda_j(\theta)$ is analytic in θ (cf. Chap. VIII, 8.5, [1]).

The relationship between $\Phi(\theta; f)$ and $\Phi_j(\theta; f)$ can be described as follows.

LEMMA 4.6. *For each $j = 1, 2, \ldots, M$, all n and $\theta \in \boldsymbol{R}$ we have*

$$\Phi^n(\theta; f)g_j = g_j\Phi_j^n(\theta; f)1 \qquad (m\text{-}a.e.).$$

PROOF. Lemma 3.4 applies and we have $\Phi(h\cdot g_j)=g_j\Phi_j(h)$ (m-a.e.) for $h\in L^1(\mu_j)$. So we get

$$\begin{aligned}\Phi^n(\theta; f)(g_j) &= \Phi^{n-1}(\theta; f)\Phi(g_j\exp\{i\theta f\})\\ &= \Phi^{n-1}(\theta; f)(g_j\Phi_j(\theta; f)(1))\\ &= \Phi^{n-2}(\theta; f)\Phi(g_j\exp\{i\theta f\}\Phi_j(\theta; f)(1))\\ &= \Phi^{n-2}(\theta; f)(g_j\Phi_j^2(\theta; f)(1))\\ &= \cdots\\ &= g_j\Phi_j^n(\theta; f)(1).\end{aligned}$$

From this lemma we can get the following

LEMMA 4.7. *For any $g\in V$ with $g\geq 0$ and $\int g dm=1$, there exist $a_1\geq 0$, $a_2\geq 0,\dots, a_M\geq 0$ ($\sum_{j=1}^M a_j=1$) for which we have*

$$\left|\int \Phi^n(t/n^{1/2}; f)(g)dm - \textstyle\sum_{j=1}^M a_j\displaystyle\int \Phi_j^n(t/n^{1/2}; f)(1)d\mu_j\right| \leq C|t/n^{1/2}|\cdot\|g\|_V$$

for some constant $C>0$.

PROOF. Lemma 4.3 applies and we have the expansion

$$P_1(t/n^{1/2}) = P_1(0) + tP_1'(0)/n^{1/2} + o(t^2/n)$$

and hence we get the inequality

$$\begin{aligned}&\left|\int \Phi^n(t/n^{1/2}; f)(g)dm - \int \Phi^n(t/n^{1/2}; f)P_1(0)(g)dm\right|\\ &\quad\leq (|t/n^{1/2}|\cdot\|P_1'(0)\|_V + C|t/n^{1/2}|\cdot\{(1+2q)/3\}^n + C|t^2/n|)\|g\|_V.\end{aligned}$$

On the other hand $P_1(0)g=P_1g\in E(1)$ can be represented as $P_1(0)g=\sum_{j=1}^M a_jg_j$ for some $a_1, a_2,\dots, a_M$. Since $g\geq 0$, we have $P_1(0)g\geq 0$ from (3.30). Therefore, remarking (3.31), we get that $a_j\geq 0$ for all j. The assumption $\int g dm=1$ shows that $\sum_{j=1}^M a_j=1$, because $\int g_j dm=1$ for all $j=1, 2,\dots, M$. Thus Lemma 4.6 applies and we have

$$\int \Phi^n(t/n^{1/2}; f)P_1(0)(g)dm = \textstyle\sum_{j=1}^M a_j\displaystyle\int \Phi_j^n(t/n^{1/2}; f)(1)d\mu_j.$$

The proof is therefore completed.

§ 5. Proofs of theorems

In this section we shall prove our theorems using the previous preparations. The ergodic theorem implies the following

LEMMA 5.1. *The equality* $\lambda'_j(0)=i\int f d\mu_j$ *holds for* $f\in V$ *and each* $j=1, 2,\dots, M$.

PROOF. We have for all $n\geq 1$

$$\int \Phi^n_j(t/n; f)(1)d\mu_j = \int \Phi^n(t/n; f)(g_j)dm$$

$$= \int \exp\{(it/n)S_n(f)\}d\mu_j.$$

Lemma 4.5 applies and we get the equality

$$\int \Phi^n_j(t/n; f)(1)d\mu_j = \lambda^n_j(t/n)\int M_j(t/n)(1)d\mu_j + \int \Psi^n_j(t/n)(1)d\mu_j \tag{5.1}$$

and

$$\left|\int \Psi^n_j(t/n)(1)d\mu_j\right| \leq C|t/n|\{(1+2\tilde{q})/3\}^n.$$

On the other hand, we have the expansion for $|\theta|<d$

$$M_j(\theta) = \mu_j + \theta M'_j(0) + (\theta^2/2)M''_j(0) + \theta^2 M^*_j(\theta), \tag{5.2}$$

where $M'_j(0)$, $M''_j(0)$ and $M^*_j(\theta)$ are operators on V with $\lim_{\theta\to 0}\|M^*_j(\theta)\|_j=0$. Therefore we get

$$\lim_{n\to\infty}\int M_j(t/n)(1)d\mu_j = 1.$$

Analogously we have the expansion

$$\lambda_j(\theta) = 1 + \theta\lambda'_j(0) + (\theta^2/2)\lambda''_j(0) + \theta^2\lambda^*_j(\theta), \tag{5.3}$$

where $\lim_{\theta\to 0}\lambda^*_j(\theta)=0$; and so we get

$$\lim_{n\to\infty}\lambda^n_j(t/n) = \exp\{t\lambda'_j(0)\}.$$

Since $\lim_{n\to\infty}(1/n)S_n(f)=\int f d\mu_j$ (μ_j-a.e.), we can derive for all $t\in \boldsymbol{R}$

$$\exp\{t\lambda'_j(0)\} = \lim_{n\to\infty}\int \exp\{(it/n)S_n(f)\}d\mu_j = \exp\left\{it\int f d\mu_j\right\};$$

and hence we have the desired result $\lambda_j'(0)=i\int f d\mu_j$.

In the sequel we shall denote $\tilde{\lambda}_j(\theta)=\lambda_j(\theta)\exp\{-i\theta b_j\}$ with $b_j=\int f d\mu_j$.

LEMMA 5.2. *For $f\in V$ and each $j=1, 2,\dots, M$ we have*

$$\tilde{\lambda}_j'(0) = 0$$

and

$$\tilde{\lambda}_j''(0) = -\lim_{n\to\infty}\int (S_n(f-b_j)/n^{1/2})^2 d\mu_j .$$

PROOF. It is easy to see that $\tilde{\lambda}_j'(0)=0$ follows from Lemma 5.1. Let us remark that the equality

$$\partial^2/\partial t^2\left(\int \exp\{(it/n^{1/2})S_n(f-b_j)\}d\mu_j\right)|_{t=0} = -\int (S_n(f-b_j)/n^{1/2})^2 d\mu_j$$

holds for all $n\geq 1$. From Lemmas 4.1, 4.5, and 4.6 we can derive

$$\int \exp\{(it/n^{1/2})S_n(f-b_j)\}d\mu_j = \tilde{\lambda}_j^n(t/n^{1/2})\int M_j(t/n^{1/2})(1)d\mu_j$$
$$+ \exp\{-itb_j n^{1/2}\}\int \Psi_j^n(t/n^{1/2})(1)d\mu_j .$$

Writing $R_j(\theta; z)$ for the resolvent operator of $\Phi_j(\theta; f)$ and $\tilde{I}_2$ for the circle in $\boldsymbol{C}$ with center 0 and radius $(1+2\tilde{q})/3$, we have

$$\Psi_j^n(t/n^{1/2})(1) = (1/2\pi i)\int_{\tilde{I}_2} z^n R_j(t/n^{1/2}; z)(1)dz$$

analogously to (4.6). Since $R_j(\theta; z)$ is also analytic in θ, we have the expansion

$$R_j(t/n^{1/2}; z) = R_j(z) + (t/n^{1/2})R_j^{(1)}(z) + (t^2/2n)R_j^{(2)}(z) + (t^2/n)R_j^*(t/n^{1/2}; z),$$

where $R_j(z)$ stands for the resolvent operator of Φ_j, and $R_j^{(1)}(z)$, $R_j^{(2)}(z)$ and $R_j^*(\theta; z)$ are bounded linear operators on V_j with

$$\lim_{\theta\to 0}\|R_j^*(\theta; z)\|_j = 0.$$

Hence, by means of elementary computations, we can get

$$(\partial^2/\partial t^2)(\exp\{-itb_j n^{1/2}\}\int \Psi_j^n(t/n^{1/2})(1)d\mu_j)|_{t=0} = O(n\{(1+2\tilde{q})/3\}^n).$$

With the help of the expansions (5.2) and the property $\tilde{\lambda}_j'(0)=0$, we obtain also that

$$\lim_{n\to\infty}(\partial^2/\partial t^2)(\tilde{\lambda}_j^n(t/n^{1/2})\int M_j(t/n^{1/2})(1)d\mu_j)|_{t=0}=\tilde{\lambda}_j''(0).$$

Therefore the limit of $\int(S_n(f-b_j)/n^{1/2})^2d\mu_j$ exists and is equal to $-\tilde{\lambda}_j''(0)$. The proof is therefore completed.

Let us denote $\sigma_j^2=-\tilde{\lambda}_j''(0)$. Then we have the following estimation.

LEMMA 5.3. *If $\sigma_j^2>0$ for $f\in V$, then there exists $d>0$ such that for all $|t|<dn^{1/2}$ we have*

$$\left|\int\Phi_j^n(t/n^{1/2};f)(1)d\mu_j-\exp\{itb_jn^{1/2}-t^2\sigma_j^2/2\}\right|$$

$$\leq\exp\{-t^2\sigma_j^2/4\}(\{A|t|^3+B|t|\}/n^{1/2})+(C|t|/n^{1/2})\rho^n$$

for some constants $A>0$, $B>0$, $C>0$, and $0<\rho<1$.

PROOF. Let d be so small that for $|\theta|<d$ Lemma 4.5 can apply. Then we have

$$\begin{aligned}\Phi_j^n(t/n^{1/2};f)(1)&=\lambda_j^n(t/n^{1/2})M_j(t/n^{1/2})(1)+\Psi_j^n(t/n^{1/2})(1)\\&=\exp\{itb_jn^{1/2}\}\tilde{\lambda}_j^n(t/n^{1/2})M_j(t/n^{1/2})(1)+\Psi_j^n(t/n^{1/2})(1)\end{aligned}$$

for $|t|<dn^{1/2}$. The property (iii) of Lemma 4.5 shows that

$$\left|\int\Psi_j^n(t/n^{1/2})(1)d\mu_j\right|\leq(C|t|/n^{1/2})\rho^n$$

holds for some $C>0$ and $0<\rho<1$. Using Lemmas 4.5 and 5.2, we obtain the expansion

$$\tilde{\lambda}_j(t/n^{1/2})=1-t^2\sigma_j^2/2n+t^3\tilde{\lambda}_j'''(0)/6n^{3/2}+t^3\hat{\lambda}_j^*(t/n^{1/2}),$$

where $\lim_{n\to\infty}\hat{\lambda}_j^*(t/n^{1/2})=0$. If we substitute $t/n^{1/2}$ for θ in the expansion (5.2), we get

$$M_j(t/n^{1/2})=\mu_j+(t/n^{1/2})M_j'(0)+(t^2/2n)M_j''(0)+(t^2/n)M_j^*(t/n^{1/2}),$$

where $\lim_{n\to\infty}\|M_j^*(t/n^{1/2})\|_j=0$. Therefore, the inequality

$$\left|\int\Phi_j^n(t/n^{1/2};f)(1)d\mu_j-\exp\{itb_jn^{1/2}-t^2\sigma_j^2/2\}\right|$$

$$\leq A_n(t)+B_n(t)+(C|t|/n^{1/2})\rho^n$$

holds, where

$$A_n(t)=|\tilde{\lambda}_j^n(t/n^{1/2})-\exp\{-t^2\sigma_j^2/2\}|$$

and

$$B_n(t) = \|\tilde{\lambda}_j^n(t/n^{1/2})\{(t/n^{1/2})M_j'(0) + (t^2/2n)M_j''(0) + (t^2/n)M_j^*(t/n^{1/2})\}\|_j.$$

Since $(1+z)^n = \exp\{n(z+\mathrm{Log}(1+z)-z\}$ and $\mathrm{Log}(1+z)-z = -z^2/2+o(z^2)$ as $|z|\to 0$, we obtain

$$\tilde{\lambda}_j^n(t/n^{1/2}) = \exp\{-t^2\sigma_j^2/2\}\exp\{C(n;\,t)\}$$

for $|t|<dn^{1/2}$, where $C(n;\,t)=t^3\tilde{\lambda}_j'''(0)/6n^{1/2}+t^3D(t/n^{1/2})/n^{1/2}$ with $\sup_{|\theta|\le d}|D(\theta)|<\infty$. If we choose so small $|d|$ that $|C(n;\,t)|\le t^2\sigma_j^2/4$ holds for $|t|\le dn^{1/2}$, then we get

$$\begin{aligned}|A_n(t)| &\le |\exp\{C(n;\,t)\} - 1|\cdot\exp\{-t^2\sigma_j^2/2\}\\ &\le |C(n;\,t)|\cdot\exp\{|C(n;\,t)|\}\cdot\exp\{-t^2\sigma_j^2/2\}\\ &\le (A|t|^3/n^{1/2})\exp\{-t^2\sigma_j^2/4\}\end{aligned}$$

for some $A>0$. Analogously, we have

$$\begin{aligned}|B_n(t)| &\le (B|t|/n^{1/2})\cdot|\tilde{\lambda}_j^n(t/n^{1/2})|\\ &\le (B|t|/n^{1/2})\cdot\exp\{-t^2\sigma_j^2/2\}\cdot\exp\{|C(n;\,t)|\}\\ &\le (B|t|/n^{1/2})\cdot\exp\{-t^2\sigma_j^2/4\}\end{aligned}$$

for some $B>0$. The proof is therefore completed.

We can now prove our theorems.

PROOF OF THEOREM 1. If we regard T and f as T^{m_0} and $S_{m_0}(f)$ respectively, it is enough for us to prove Theorem 1 in the case of $m_0=1$. Let T and ν satisfy the assumptions in Theorem 1, and f be a function of bounded variation. Esseen's inequality (cf. §39, [3]) shows that

$$\begin{aligned}(5.4)\qquad &\sup_{y\in R}|\nu\{S_n(f)/n^{1/2}\le y\} - \textstyle\sum_{j=1}^M a_jF(b_jn^{1/2},\,\sigma_j^2;\,y)| \le K/U + (1/\pi)\\ &\times\int_{-U}^{U}(1/|t|)\left|\int\exp\{(it/n^{1/2})S_n(f)\}d\nu - \textstyle\sum_{j=1}^M a_j\exp\{itb_jn^{1/2} - t^2\sigma_j^2/2\}\right|dt\end{aligned}$$

for all $U>0$ and $n\ge 1$. From Lemmas 4.1 and 4.7 we obtain

$$\begin{aligned}(5.5)\qquad &\left|\int\exp\{(it/n^{1/2})S_n(f)\}d\nu - \textstyle\sum_{j=1}^M a_j\int\Phi_j^n(t/n^{1/2};\,f)(1)d\mu_j\right|\\ &\le C|t/n^{1/2}|\cdot\|h\|_V,\end{aligned}$$

where h denotes the density function of ν with respect to the Lebesgue measure m.

If we put $U=dn^{1/4}$ in (5.4) and combine Lemma 5.3 and the inequality (5.5), we get the inequality

$$\begin{aligned}&\sup_{y\in\mathbf{R}}|\nu\{S_n(f)/n^{1/2}\le y\}-\textstyle\sum_{j=1}^{M}a_jF(b_jn^{1/2},\sigma_j^2;y)|\\ &\le K/dn^{1/4}+(1/\pi)\int_{-dn^{1/4}}^{dn^{1/4}}(C\|h\|_V/n^{1/2}\\ &\quad+\textstyle\sum_{j=1}^{M}a_j\exp\{-t^2\sigma_j^2/4\}\cdot(A|t|^3+B|t|)/n^{1/2}+(C|t|/n^{1/2})\rho^n)dt.\end{aligned}$$

And hence the result (1.8) is obtained. If we assume further that ν is T-invariant, we get for some $a_1\ge 0$, $a_2\ge 0,\dots, a_M\ge 0$ ($\sum_{j=1}^{M}a_j=1$)

$$\begin{aligned}\int\exp\{(it/n^{1/2})S_n(f)\}d\nu&=\int\Phi^n(t/n^{1/2};f)(h)dm\\ &=\textstyle\sum_{j=1}^{M}a_j\int\Phi^n(t/n^{1/2};f)(g_j)dm\\ &=\textstyle\sum_{j=1}^{M}a_j\int\Phi_j^n(t/n^{1/2};f)(1)d\mu_j\end{aligned}$$

from Lemmas 4.1 and 4.6. Hence, putting $U=dn^{1/2}$ in (5.4), we obtain the result (1.9).

Proof of Theorem 2. If we assume the assumptions of Theorem 2, then it is clear that $v(S_{m_0}(f))\le N^{m_0}\cdot v(f)$. Therefore, Theorem 2 is immediately derived from Theorem 1.

Proof of Theorem 3. The arguments in §3 shows that $m_0=M=1$ under the assumptions of Theorem 3. Therefore, the results (1.11) and (1.12) are corollaries to those of Theorem 1. In order to show (1.10), we first remark that

$$\begin{aligned}&\int\exp\{(it/n^{1/2})S_n(f-b)\}d\nu\\ &=\exp\{-itbn^{1/2}\}\int\Phi^n(t/n^{1/2};f)(h)dm\\ &=\exp\{-itbn^{1/2}\}\int\Phi^n(t/n^{1/2};f)(g_1)dm+O(1/n^{1/2})\\ &=\exp\{-itbn^{1/2}\}\int\Phi_1^n(t/n^{1/2};f)(1)d\mu_1+O(1/n^{1/2})\end{aligned}$$

follows from Lemma 4.7, because $\mu_1=\mu$ is the unique T-invariant measure. Substitute $t/n^{1/2}$ for θ in the equalities (5.2) and (5.3). Then we can get

$$\lim_{n\to\infty}\int\exp\{(it/n^{1/2})S_n(f-b)\}d\nu$$

$$= \lim_{n\to\infty} \tilde{\lambda}_1^n(t/n^{1/2}) \int M_1(t/n^{1/2})(1)d\mu_1$$

$$+ \lim_{n\to\infty} \exp\{-ibtn^{1/2}\} \int \Psi_1^n(t/n^{1/2})(1)d\mu_1$$

$$= \lim_{n\to\infty} \tilde{\lambda}_1^n(t/n^{1/2}) = \exp\{-t^2\sigma_1^2/2\}$$

from Lemmas 4.5 and 5.2. This completes the proof.

PROOF OF THEOREM 4. We define the Perron-Frobenius operator Φ_μ: $L^1(\mu)\to L^1(\mu)$ corresponding to (T, μ) by

$$\int h\Phi_\mu(f)d\mu = \int f(x)h(Tx)d\mu$$

similarly to (3.16), and define the perturbed operator $\Phi_\mu(\theta; f)$ in the same manner as in (4.7). Then it is very easy to check that the arguments in §4 and §5 remain valid for Φ_μ and $\Phi_\mu(\theta; f)$. Therefore, we get the results of Theorem 4 by the same method as in the proof of Theorem 3.

§6. Remarks on limiting variances

In this section we treat the following problem: in which case are the variances σ_j^2 in our theorems strictly positive? It may be one of the most difficult problems in the theory of central limit theorem for dependent variables. In [22] J. Rousseau-Egele got a concrete sufficient condition to ensure the positivity of σ_j^2. We can follow his arguments also in our case.

LEMMA 6.1. *For $f\in V$ and each $j=1, 2,\dots, M$ the limit*

$$\lim_{n\to\infty} \int (S_n(f-b_j)/n^{1/2})^2 d\mu_j = \sigma_j^2$$

exists and

$$\sigma_j^2 = \int (g^2-(\Phi_j g)^2)d\mu_j$$

holds, where $g=(I-\Phi_j)^{-1}(f-b_j)$.

PROOF. The existence of the limit was proved in Lemma 5.2. From Lemma 3.9 and the definition (3.16), we have

$$\left|\int (f(x)-b_j)\cdot(f(T^k x)-b_j)d\mu_j\right| = \left|\int (f-b_j)\cdot\Phi_j^k(f-b_j)d\mu_j\right|$$

$$= \left|\int (f-b_j)\cdot\Psi_j^k(f-b_j)d\mu_j\right|$$

$$\le \tilde{H}\cdot\tilde{q}^k\cdot\|f-b_j\|_j^2$$

for $k \geq 1$. Therefore, the series

$$\sum_{k=-\infty}^{\infty}\int (f(x)-b_j)(f(T^{|k|}x)-b_j)d\mu_j$$

absolutely converges to

$$\lim_{n\to\infty}\int (S_n(f-b_j)/n^{1/2})^2 d\mu_j\,;$$

and hence we have

$$\begin{aligned}\sigma_j^2 &= \lim_{n\to\infty}\int (S_n(f-b_j)/n^{1/2})^2 d\mu_j \\ &= \sum_{k=-\infty}^{\infty}\int (f-b_j)\cdot \Phi_j^{|k|}(f-b_j)d\mu_j \\ &= \int (f-b_j)\cdot \sum_{k=-\infty}^{\infty}\Phi_j^{|k|}(f-b_j)d\mu_j \\ &= \int (f-b_j)\cdot(2g-(f-b_j))d\mu_j \\ &= \int (g-\Phi_j g)\cdot(g+\Phi_j g)d\mu_j,\end{aligned}$$

remarking $\sum_{k=0}^{\infty}\Phi_j^k=(I-\Phi_j)^{-1}$. The proof is therefore completed.

Using this lemma we can get the following

LEMMA 6.2. *For $f\in V$, $\sigma_j^2=0$ if and only if the equation*

$$f(x) = b_j + \varphi(Tx) - \varphi(x) \qquad (\mu_j\text{-a.e.}) \tag{6.1}$$

has a solution φ in $L^2(\mu_j)$.

PROOF. First of all we remark that $\Phi_j\colon L^2(\mu_j)\to L^2(\mu_j)$ and hence $g=\sum_{n=0}^{\infty}\Phi_j^n(f-b_j)\in L^2(\mu_j)$. We get

$$\begin{aligned}&(g - U_j\Phi_j g,\ g - U_j\Phi_j g)_{L^2(\mu_j)} \\ &\quad = (g,g)_{L^2(\mu_j)} - (U_j\Phi_j g,\ g)_{L^2(\mu_j)} \\ &\qquad - (g,\ U_j\Phi_j g)_{L^2(\mu_j)} + (U_j\Phi_j g,\ U_j\Phi_j g)_{L^2(\mu_j)} \\ &\quad = (g,g)_{L^2(\mu_j)} - (\Phi_j g,\ \Phi_j g)_{L^2(\mu_j)}.\end{aligned}$$

If we assume $\sigma_j^2=0$, then $U_j\Phi_j g=g$ follows from Lemma 6.1 and the above. Putting $\varphi=g-f+b_j$, we have

$$\begin{aligned} U_j\varphi - \varphi &= U_j g - U_j(f-b_j) - g + f - b_j \\ &= U_j g - U_j(I-\Phi_j)g - g + f - b_j \\ &= f - b_j \end{aligned}$$

and so φ is a solution of (6.1). Conversely, let $\varphi \in L^2(\mu_j)$ be a solution of (6.1). Then we easily see that

$$S_n(f-b_j)/n^{1/2} = \{\varphi(T^n x) - \varphi(x)\}/n^{1/2}$$

and hence that

$$\int \{S_n(f-b_j)/n^{1/2}\}^2 d\mu_j \leq (4/n)\int \varphi^2 d\mu_j .$$

This implies that $\sigma_j^2 = 0$.

Using this lemma we can get the following concrete result.

PROPOSITION 6.3. *If A is a measurable set with $0<\mu_j(A)<1$ and $I_A(x) \in V$, then we have $\sigma_j^2>0$ for $f=I_A$.*

PROOF. If we suppose $\sigma_j^2=0$, then we get from Lemma 6.2 that there is $\varphi \in L^2(\mu_j)$ which satisfies the equation

$$I_A(x) = \mu_j(A) + \varphi(Tx) - \varphi(x) \qquad (\mu_j\text{-a.e.}).$$

Then we have

$$\exp\{2\pi i \varphi\circ T\} = \exp\{-2\pi i \mu_j(A)\}\exp\{2\pi i\varphi\} \qquad (\mu_j\text{-a.e.})$$

and hence

$$\exp\{2\pi i \varphi\circ T^{m_0}\} = \exp\{-2\pi i m_0 \mu_j(A)\}\exp\{2\pi i\varphi\} \qquad (\mu_j\text{-a.e.}).$$

Since (T^{m_0}, μ_j) is weakly mixing, we have $\exp\{2\pi i\varphi(x)\}=1$ (μ-a.e.) and hence φ is μ_j-a.e. integer-valued. However, this contradicts the fact that $I_A(x) = \mu_j(A) + \varphi(Tx) - \varphi(x)$ is equal to 0 or 1. The proof is therefore completed.

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