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A limit theorem for collision path of one-dimensional independent random motions

Norio Konno

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§ 1. Introduction

Let $\{x_k\}$, $k=\pm 1, \pm 2, \ldots$ be a Poisson point process on the real line with parameter 1, to which we add the origin $x_0=0$, i.e.

$$(1.1)\qquad \cdots < x_{-2} < x_{-1} < x_0 = 0 < x_1 < x_2 < \cdots .$$

Then $\{\zeta_k = x_k - x_{k-1}\}$, $-\infty < k < \infty$, is a sequence of i.i.d. random variables with exponential distributions of mean one. For each k, we consider a random motion $x_k(t)$ starting from x_k. Suppose that we are given a random process $x(t)$ satisfying $x(0)=0$, and that $\{x_k(t)-x_k\}$, $-\infty<k<\infty$, are independent copies of $x(t)$. Then we can define the collision path $y_0(t)$ as in [4] and [7],

$$(1.2)\qquad y_0(t) = \lim_{n\to\infty} \text{median of } \{x_k(t);\ -n \leqq k \leqq n\}, \quad \text{for} \quad t \geqq 0.$$

Our purpose of this paper is to investigate limiting behaviors of the collision path by taking a suitable space-time scaling.

In case that $x(t)$ is a standard Brownian motion, Harris showed in [4] that the distribution of $t^{-1/4}y_0(t)$ converges to a normal distribution as $t\to\infty$. On the other hand, Spitzer [7] treated a uniform motion case, and proved that $y_0(At)/\sqrt{A}$ converges to a Brownian motion as $A\to\infty$ in the sense of weak convergence of probability distributions on the path space. Also, in case that $x(t)$ is a symmetric stable process of index $1<\gamma\leqq 2$, Gisselquist [3] proved that $y_0(At)/A^{1/2\gamma}$ converges to a fractional Brownian motion with mean zero and covariance function

$$\sigma(t, s) = \text{const}\,(t^{1/\gamma} + s^{1/\gamma} - |t-s|^{1/\gamma}), \qquad t, s \geqq 0,$$

as $A\to\infty$ in the sense of finite dimensional distributions.

In the present paper we will discuss limiting behaviors of the collision path in more general setting. In particular it will be shown that, as far as the convergence of finite dimensional distributions, the limiting behavior of the collision path is completely determined by the asymptotic behavior of the first order absolute moments of increments of $x(t)$. Moreover for a certain class of the processes with stationary increments we will prove that the rescaled process of the collision path $y_0(t)$ converges to a fractional Brownian motion in the sense of weak convergence of probability distributions on the path space.

In §2 we will give some preliminaries and in §3 we will prove the convergence of finite dimensional distributions of the rescaled collision path. Finally, in §4 we will prove the tightness of the rescaled collision path under a restrictive situation.

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§2. Preliminaries

Consider a real-valued random process $x(t)$ with $x(0)=0$ such that the sample paths are right continuous and have left limit for any $t \geqq 0$. For our purpose it is convenient to define the independent system $\{x_k(t)\}$, $k \in \mathbf{Z}$, by making use of a Poisson random measure.

Let W be the set of sample paths of $x(t)$, equipped with the usual σ-field $\mathscr{F}$ generated by all cylindrical sets, and we denote by P_0 the probability distribution on W induced by $x(t)$.

Let M be the set of counting measures on $\mathbf{R} \times W$, equipped with the weak topology and let $\mathscr{B}_M$ be the topological Borel field of M. We denote by ν the Lebesgue measure on $\mathbf{R}$.

DEFINITION 2.1. Let μ be an $(M, \mathscr{B}_M)$-valued random variable defined on a probability space $(\Omega, \mathscr{B}, P)$. μ is called a *Poisson random measure* on $\Omega \times W$ with the intensity measure $\lambda = \nu \times P_0$ if

(i) for any $B \in \mathscr{B}_{\mathbf{R}} \times \mathscr{F}$,

$$P[\mu(B)=j] = \exp\{-\lambda(B)\}\{\lambda(B)\}^j/j! \qquad (j = 0, 1, 2,\dots),$$

and

(ii) if $B_1,\dots, B_n \in \mathscr{B}_{\mathbf{R}} \times \mathscr{F}$ are disjoint, then $\{\mu(B_1),\dots, \mu(B_n)\}$ are independent.

We can construct a Poisson random measure μ with the intensity measure $\lambda = \nu \times P_0$ and a random process $x_0(t)$ whose probability law is identical with $x(t)$, on the same probability space $(\Omega, \mathscr{B}, P)$, so that μ and $x_0(t)$ are independent. We write

$$\mu = \sum_{k \in \mathbf{Z} \setminus \{0\}} \delta_{(x_k, w_k)},$$

where $\cdots < x_{-k} < \cdots < x_{-1} < 0 < x_1 < x_2 < \cdots < x_k < \cdots$ and $\delta_{(x,w)}$ stands for the δ-measure at $(x, w) \in \mathbf{R} \times W$. Then defining random processes $\{x_k(t)\}$, $k \in \mathbf{Z} \setminus \{0\}$, by

$$x_k(t) = x_k + w_k(t), \qquad k \in \mathbf{Z} \setminus \{0\}, \tag{2.1}$$

where $w_k(t)$ is the value of w_k at time t, we can easily see that $\{x_k\}$, $k\in\mathbf{Z}\setminus\{0\}$, is a Poisson point process on $\mathbf{R}$ with parameter 1, $\{x_k(t)-x_k\}$, $k\in\mathbf{Z}$, is independent of $\{x_k\}$, $k\in\mathbf{Z}\setminus\{0\}$, and $\{x_k(t)-x_k\}$, $k\in\mathbf{Z}$, are independent copies of $x(t)$.

By using this construction and a formula of the characteristic functional for a Poisson random measure (see [5] p. 57), we get

LEMMA 2.2. *For any ξ, α and $x\in\mathbf{R}$,*

$$E[\exp\{i\xi\textstyle\sum_{k=1}^{\infty}\chi[x_k(t)\leqq\alpha]\}] = \exp\Big[-\displaystyle\int_0^{\infty}\{1-\phi(x)\}dx\Big],$$

$$E[\exp\{i\xi\textstyle\sum_{k=-\infty}^{-1}\chi[x_k(t)>\alpha]\}] = \exp\Big[-\displaystyle\int_{-\infty}^{0}\{1-\tilde{\phi}(x)\}dx\Big],$$

where

$$\phi(x) = \int_W \exp\{i\xi\chi[x+w(t)\leqq\alpha]\}P_0(dw),$$

$$\tilde{\phi}(x) = \int_W \exp\{i\xi\chi[x+w(t)>\alpha]\}P_0(dw),$$

and $\chi[B]$ stands for the indicator function of B.

Let $\theta_t w(\cdot)=w(\cdot+t)-w(t)$ be the shifted path, and set

$$\mu_t = \textstyle\sum_{k\neq 0}\delta_{(x_k+w_k(t),\,\theta_t w_k)} \qquad \text{for} \quad \mu = \sum_{k\neq 0}\delta_{(x_k,w_k)}.$$

Then μ_t also is a Poisson random measure on $\mathbf{R}\times W$ with the intensity measure $\lambda_t=\nu\times\theta_t\circ P_0$, where $\theta_t\circ P_0$ is the image measure of P_0 by θ_t. Therefore if $x(t)$ has stationary increments (equivalently $\theta_t\circ P_0=P_0$ for any $t\geqq 0$), μ_t is a stationary process taking values in M. This fact reflects the following lemma, which will be used in §4.

LEMMA 2.3. *Suppose that $x(t)$ has stationary increments. Then the collision path $y_0(t)$ also has stationary increments.*

PROOF. Similarly to the collision path $y_0(t)$ we can define

$$y_k(t) = \lim_{n\to\infty}\text{median of }\{x_{k-n}(t),\dots,x_k(t),\dots,x_{k+n}(t)\} \quad (k\in\mathbf{Z}),$$

Let any $t>0$ be fixed, and set $L_k(t)=\sum_{i<k}\chi[x_i(t)>x_k(t)]$ and $R_k(t)=\sum_{i>k}\cdot\chi[x_i(t)<x_k(t)]$. Then we see that

$$x_k(t) = y_{k-L_k(t)+R_k(t)}(t) \qquad \text{a.s.} \quad (k\in\mathbf{Z}). \tag{2.2}$$

Moreover, denoting $\sigma(k)=k-L_k(t)+R_k(t)$ for each k, we see that σ is a bijective map from $\mathbf{Z}$ onto itself. Let τ be the inverse map of σ and set

$$\tilde{\mu}_t = \sum_{k\neq 0} \delta_{(y_k(t)-y_0(t),\theta_t w_\tau(k))}\,.$$

To complete the proof of this lemma it suffices to show that $\tilde{\mu}_t$ also is a Poisson random measure on $\mathbf{R}\times W$ with the intensity measure $\lambda=\nu\times P_0$, because $\theta_t y_0(\cdot)=y_0(\cdot+t)-y_0(t)$ is constructed by $\tilde{\mu}_t$ in the same procedure as $y_0(\cdot)$ is done from the Poisson random measure μ. By (2.2) we see that

$$\begin{aligned}
&P[\tilde{\mu}_t(B) = j] \\
&= \textstyle\sum_{p\in\mathbf{Z}} P[\sum_{k\in\mathbf{Z}\setminus\{0\}} \chi[(y_k(t),\, \theta_t w_{\tau(k)}) \in B + x_p(t)] = j,\ y_0(t) = x_p(t)] \\
&= \textstyle\sum_{p\in\mathbf{Z}} P[\sum_{n\neq 0} \chi[(x_n(t),\, \theta_t w_n) \in B + x_p(t)] = j,\ L_p(t) - R_p(t) = p] \\
&= \textstyle\sum_{p\in\mathbf{Z}} P[\sum_{n\neq 0} \chi[(x_n(t),\, \theta_t w_n) \in B + x_0(t)] = j,\ L_0(t) - R_0(t) = p] \\
&= P[\textstyle\sum_{n\neq 0} \chi[(x_n(t),\, \theta_t w_n) \in B + x_0(t)] = j] \\
&= P[\mu_t(B) = j], \quad \text{where} \quad B + x_p = \{(x + x_p,\, w);\, (x,\, w) \in B\}\,.
\end{aligned}$$

Here we used the stationarity of μ_t and the equivalence between $\{x_n(t)-x_p(t)\}$, $n\in\mathbf{Z}$, and $\{x_n(t)-x_{p+1}(t)\}$, $n\in\mathbf{Z}$. Since it is clear that $\tilde{\mu}_t(B_1),\dots,\tilde{\mu}_t(B_n)$ are independent if $B_1,\dots,B_n$ are disjoint, $\tilde{\mu}_t$ is a Poisson random measure on $\mathbf{R}\times W$ with the intensity measure $\lambda=\nu\times P_0$. Thus we complete the proof of Lemma 2.3.

Finally we quote an estimate concerning i.i.d. random variables, which will be used in §4.

LEMMA 2.4. *Let $\{X_n\}$, $n=1, 2,\dots$ be a sequence of i.i.d. random variables. Suppose that*

$$EX_1 = 0, \quad \textit{and} \quad E|X_1|^{2p} < \infty, \qquad \textit{for an integer} \quad p \geqq 1.$$

Then we have some constant $C_p>0$ such that

$$P[\max_{1\leqq n\leqq N} |\textstyle\sum_{k=1}^{n} X_k| \geqq \lambda] \leqq C_p N^p \lambda^{-2p},$$

for any $\lambda>0$ and $N\geqq 1$.

The proof follows immediately from a maximal inequality and moment estimates of martingales (e.g. see [6] p. 28 and [2] Theorem 9).

§3. Convergence of finite dimensional distributions

Noting that $\{x_k(t)-x_k\}$, $k\in\mathbf{Z}$, is an independent system of random processes having the identical probability law with $x(t)$, we impose the following assumptions

(3.1) $\qquad E[x(t)] = 0, \qquad$ for any $\ t \geqq 0, \qquad (x(0) = 0)$,

(3.2) there exist an increasing function $\gamma(A)$ defined on $(0, \infty)$ and a function

$\rho(t, s)$ defined on $(0, \infty)\times(0, \infty)$ such that $\lim_{A\to\infty}\gamma(A)=+\infty$, and for any $t, s\geqq 0$,

$$\lim_{A\to\infty}\gamma(A)^{-1}E|x(At)-x(As)|=\rho(t, s).$$

THEOREM 3.1. *Under the assumptions* (3.1) *and* (3.2), *the process* $\gamma(A)^{-1/2}y_0(At)$ *converges in the sense of finite dimensional distributions, as* $A\to\infty$, *to a centered Gaussian process of which covariance function* $\sigma(t, s)$ *is given by*

$$\sigma(t, s)=(1/2)\{\rho(t, 0)+\rho(s, 0)-\rho(t, s)\},\quad t, s\geqq 0. \tag{3.3}$$

Thus, as far as the convergence of finite dimensional distributions of the rescaled collision path we establish a quite general result. It is easily seen that the limiting process $y^\infty(t)$ enjoys a self-similarity, i.e., there exists $\gamma\geqq 0$ such that $y^\infty(ct)$ and $c^\gamma y^\infty(t)$ have the identical probability law for any $c>0$. Furthermore it is obvious that any self-similar centered Gaussian process can appear as the limiting process of a rescaled collision path.

PROOF OF THEOREM 3.1. Following Harris [4] and Spitzer [7] we introduce three vector random variables;

$$U(t, \alpha)=(u(t_i, \alpha_i))_{i=1}^n=(\textstyle\sum_{k=1}^\infty \chi[x_k(t_i)\leqq\alpha_i])_{i=1}^n,$$
$$V(t, \alpha)=(-v(t_i, \alpha_i))_{i=1}^n=(-\textstyle\sum_{k=-\infty}^{-1}\chi[x_k(t_i)>\alpha_i])_{i=1}^n,$$
$$W(t, \alpha)=(w(t_i, \alpha_i))_{i=1}^n=(\chi[x_0(t_i)\leqq\alpha_i])_{i=1}^n,$$

where $\quad t=(t_1,\dots, t_n),\ 0<t_1<\cdots<t_n,\ \alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{R}^n.$

Let $Z(t, \alpha)=U(t, \alpha)+V(t, \alpha)+W(t, \alpha)$.

We now use the equivalence of the following two events observed by Harris ([4] Theorem 5.1):

$$\{y_0(t_i)<\alpha_i;\ i=1, 2,\dots, n\}=\{Z(t, \alpha)\geqq 1\}, \tag{3.4}$$

where $1=(1,\dots, 1)$ and we denote $\alpha\geqq\beta$ when $\alpha_i\geqq\beta_i$ for all components. Let $Z_A=Z(At, \gamma(A)^{1/2}\alpha)$. We will later show that

$$\lim_{A\to\infty}E[\exp\{i\xi\cdot\gamma(A)^{-1/2}Z_A\}]=\phi(\xi, t)\exp(i\xi\cdot\alpha), \tag{3.5}$$

where

$$\phi(\xi, t)=\exp\{-(1/2)\textstyle\sum_{j=1}^n\sum_{k=1}^n\xi_j\xi_k\sigma(t_j, t_k)\}. \tag{3.6}$$

Once we obtain (3.5), Theorem 3.1 follows immediately. Because by (3.5) we see easily that

(3.7) $$\lim_{A\to\infty} P[Z_A \geqq 1] = \lim_{A\to\infty} P[Z_A \geqq 0],$$

and it follows from (3.4)$\sim$(3.7) that

$$\begin{aligned}
&\lim_{A\to\infty} P[\gamma(A)^{-1/2} y_0(At_i) < \alpha_i;\ i = 1, 2,\dots, n] \\
&= \lim_{A\to\infty} P[Z_A \geqq 0] \\
&= \lim_{A\to\infty} P[\gamma(A)^{-1/2} Z_A - \alpha \geqq -\alpha] \\
&= \int_{-\alpha_1}^{\infty} dx_1 \cdots \int_{-\alpha_n}^{\infty} dx_n P(x_1,\dots, x_n) \\
&= \int_{-\infty}^{\alpha_1} dx_1 \cdots \int_{-\infty}^{\alpha_n} dx_n P(x_1,\dots, x_n),
\end{aligned}$$

where $P(x_1,\dots. x_n)$ denotes the probability density of the normal distribution with mean vector 0 and covariance matrix $\{\sigma(t_j, t_k)\}_{1\leqq j,k\leqq n}$.

In order to prove (3.5), it is sufficient to show that

(3.8) $$\begin{aligned}
&\lim_{A\to\infty} E[\exp\{i\xi\cdot(U(At, \gamma(A)^{1/2}\alpha) + V(At, \gamma(A)^{1/2}\alpha))/\gamma(A)^{1/2}\}] \\
&= \phi(\xi, t)\exp(i\xi\cdot\alpha)
\end{aligned}$$

By Lemma 2.2,

(3.9) $$\begin{aligned}
&-\log E[\exp\{i\xi\cdot(U(At, \gamma(A)^{1/2}\alpha) + V(At, \gamma(A)^{1/2}\alpha)/\gamma(A)^{1/2}\}] \\
&= \int_0^{\infty}\int_W \{1 - \exp(i\textstyle\sum_{j=1}^n \xi_j \gamma(A)^{-1/2} I_j)\} dP dx \\
&\quad + \int_0^{\infty}\int_W \{1 - \exp(i\textstyle\sum_{j=1}^n \xi_j \gamma(A)^{-1/2} J_j)\} dP dx \\
&= M(A, \xi),
\end{aligned}$$

where

$$\begin{aligned}
I_j &= \chi[x(At_j) \leqq \gamma(A)^{1/2}\alpha_j - x], \\
J_j &= \chi[x(At_j) > \gamma(A)^{1/2}\alpha_j + x] \qquad (j = 1, 2,\dots, n).
\end{aligned}$$

To evaluate (3.9), we use the following simple formulae which are easily checked.

LEMMA 3.2. *Let X and Y be random variables with $E|X|<\infty$ and $E|Y|<\infty$. Then*

(i) $$\int_0^{\infty} P[X > x]dx = E[X^+] \qquad (X^+ = \max(X, 0)),$$

(ii) $$\int_0^{\infty} (P[X > x] - P[X < -x])dx = E[X],$$

(iii) $\int_0^\infty (P[X>x]+P[X<-x])dx=E|X|,$

(iv) $\int_0^\infty P[X>x,\ Y>x]dx=E[\min(X,\ Y)],$

(v) $\int_0^\infty (P[X>x,\ Y>x]+P[X<-x,\ Y<-x])dx$

$= (1/2)(E|X|+E|Y|-E|X-Y|).$

Now we decompose $M(A,\ \xi)$ of (3.9) as follows:

(3.10) $M(A,\ \xi) = M_1(A,\ \xi) + M_2(A,\ \xi) + M_3(A,\ \xi),$

where

$$M_1(A,\ \xi) = (-i)\int_0^\infty \int_W \sum_{j=1}^n \xi_j\gamma(A)^{-1/2}(I_j-J_j)dPdx,$$

$$M_2(A,\ \xi) = (1/2)\int_0^\infty \int_W \{(\sum_{j=1}^n \xi_j\gamma(A)^{-1/2}I_j)^2 + (\sum_{j=1}^n \xi_j\gamma(A)^{-1/2}J_j)^2\}dPdx,$$

$M_3(A,\ \xi) =$ the remainder term.

Using the inequality $|1+i\alpha-\alpha^2/2-\exp(i\alpha)|\leqq|\alpha|^3/6$ and Lemma 3.2 (iii), we can estimate $M_3(A,\ \xi)$ as follows:

$$\begin{aligned}|M_3(A,\ \xi)| &\leqq (1/6)\int_0^\infty \int_W (|\sum_{j=1}^n \xi_j\gamma(A)^{-1/2}I_j|^3 + |\sum_{j=1}^n \xi_j\gamma(A)^{-1/2}J_j|^3)dPdx \\ &\leqq (1/6)n^2\|\xi\|^{3/2}\gamma(A)^{-3/2} \sum_{j=1}^n \int_0^\infty \int_W (I_j+J_j)dPdx \\ &\leqq (1/6)n^2\|\xi\|^{3/2}\gamma(A)^{-3/2} \sum_{j=1}^n E|x(At_j) - \gamma(A)^{1/2}\alpha_j|.\end{aligned}$$

Therefore from the assumption (3.2) it follows that $M_3(A,\ \xi)\to 0$ as $A\to\infty$. Also by Lemma 3.2 (ii) and the assumption (3.1) we get

$$\begin{aligned}M_1(A,\ \xi) &= (-i) \sum_{j=1}^n \xi_j\gamma(A)^{-1/2}E[\gamma(A)^{1/2}\alpha_j - x(At_j)] \\ &= (-i)\xi\cdot\alpha.\end{aligned}$$

To calculate $M_2(A,\ \xi)$ we use Lemma 3.2 (iii) and (v);

$$\begin{aligned}M_2(A,\ \xi) &= (1/2) \sum_{j=1}^n \xi_j^2\gamma(A)^{-1} \int_0^\infty \int_W (I_j+J_j)dxPx \\ &\quad + (1/2) \sum\sum_{j,k=1,\,j\neq k}^n \xi_j\xi_k\gamma(A)^{-1} \int_0^\infty \int_W (I_jI_k+J_jJ_k)dPdx \\ &= (1/2) \sum_{j=1}^n \xi_j^2\gamma(A)^{-1}E|x(At_j) - \gamma(A)^{1/2}\alpha_j|\end{aligned}$$

$$+ (1/2) \sum\sum_{j,k=1,j\neq k}^{n} \xi_j\xi_k \gamma(A)^{-1}(1/2)\{E|x(At_j) - \gamma(A)^{1/2}\alpha_j|$$
$$+ E|x(At_k) - \gamma(A)^{1/2}\alpha_k| - E|x(At_j) - x(At_k) - (\alpha_j - \alpha_k)\gamma(A)^{1/2}|\}$$
$$\to (1/2) \sum_{j=1}^{n} \xi_j^2 \rho(t_j, 0) + (1/2) \sum\sum_{j,k=1,j\neq k}^{n} \xi_j\xi_k (1/2)$$
$$\{\rho(t_j, 0) + \rho(t_k, 0) + \rho(t_j, t_k)\} \, (A \to \infty).$$

Thus we obtain (3.5) and the proof of Theorem 3.1 is completed.

§4. Weak convergence

In this section, in addition to the previous conditions (3.1) and (3.2), we will assume that $x(t)$ has stationary increments and that

$$\gamma(A) = A^h, \quad \text{for some} \quad h > 0, \quad \text{and} \tag{4.1}$$

$$E|t^{-h}x(t)|^{2m+2} \quad \text{is bounded in} \quad t > 0, \tag{4.2}$$

where m is the smallest integer larger than $1/h$.

We note that $x(t)$ has a continuous modification so that we may assume that $x(t)$ is a continuous process. Then $y_0(t)$ also is a continuous process. Moreover it should be noted that $h \leqq 1$ follows automatically from the above assumption.

Under the above assumptions we have

THEOREM 4.1. *The process $A^{-h/2}y_0(At)$ converges to a fractional Brownian motion $y^\infty(t)$ as $A \to \infty$ in the sense of weak convergence of probability distributions on the continuous path space $C([0, 1], \mathbf{R})$, where $y^\infty(t)$ is a centered Gaussian process of which covariance function is given by*

$$E[y^\infty(t)y^\infty(s)] = (1/2)\rho(1, 0)(|t|^h + |s|^h - |t-s|^h), \; 0 \leqq s, \, t \leqq 1. \tag{4.3}$$

In order to prove the theorem it suffices to show that the family of probability distributions on $C([0, 1], \mathbf{R})$ induced by $A^{-h/2}y_0(At)$ is tight.

LEMMA 4.2. *Under the assumption* (4.2) *we have some $C>0$ such that*

$$E(A^{-h/2}y_0(At))^{2m} \leqq Ct^{mh}, \quad 0 \leqq t \leqq \min(1, 1/A).$$

PROOF. Let $\gamma(A) = A^h$. Using a Poisson property we can easily check that

$$P[\gamma(A)^{-1/2}y_0(At) \leqq \alpha]$$
$$= P[u(At, A^{h/2}\alpha) - v(At, A^{h/2}\alpha) + w(At, A^{h/2}\alpha) \geqq 1]$$
$$= e(1) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a(n+k)b(k) + e(0) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a(n+1+k)b(k),$$

where

$$e(1)=\psi_t^A(\alpha)=P[x(At)\leqq A^{h/2}\alpha],\quad e(0)=1-e(1),$$

$$a(k)=P[\textstyle\sum_{l=1}^{\infty}\chi(x_l(At)\leqq A^{h/2}\alpha)=k]$$

(4.5)
$$=(1/k!)\left\{\int_0^\infty \psi_t^A(\alpha-x)dx\right\}^k \exp\left\{-\int_0^\infty \psi_t^A(\alpha-x)dx\right\},$$

$$b(k)=P[\textstyle\sum_{l=-\infty}^{-1}\chi(x_l(At)>A^{h/2}\alpha)=k]$$

$$=(1/k!)\left\{\int_0^\infty (1-\psi_t^A(\alpha+x))dx\right\}^k \exp\left\{-\int_0^\infty (1-\psi_t^A(\alpha+x))dx\right\}.$$

$$E(A^{-h/2}y_0(At))^{2m}=\int_{-\infty}^{\infty}\alpha^{2m}dP[\gamma(A)^{-1/2}y_0(At)\leqq\alpha]$$

$$=\int_{-\infty}^{\infty}\alpha^{2m}\textstyle\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}b(k)\{e(1)da(n+k)/d\alpha+e(0)da(n+1+k)/d\alpha\}d\alpha$$

$$+\int_{-\infty}^{\infty}\alpha^{2m}\textstyle\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\{a(n+k)e(1)+a(n+1+k)e(0)\}db(k)/d\alpha d\alpha$$

$$+\int_{-\infty}^{\infty}\alpha^{2m}\textstyle\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\{a(n+1+k)b(k)-a(n+k)b(k)\}de(0)$$

$$=I_1+I_2+I_3.$$

By (4.4) we see that

$$da(k)/d\alpha=\{a(k-1)-a(k)\}\psi_t^A(\alpha)\qquad(k\geqq 1),$$

$$da(0)/d\alpha=-a(0)\psi_t^A(\alpha),$$

(4.5)

$$db(k)/d\alpha=-\{b(k-1)-b(k)\}\{1-\psi_t^A(\alpha)\}\qquad(k\geqq 1),$$

$$db(0)/d\alpha=b(0)\{1-\psi_t^A(\alpha)\}.$$

In order to estimate I_1, I_2, I_3, we use the following inequalities which are easily checked.

LEMMA 4.3.

(i) $\sum_{k=0}^{\infty}a(k)b(k+1)\leqq\int_0^\infty\{1-\psi_t^A(\alpha+x)\}dx,$

(ii) $\sum_{k=0}^{\infty}a(k+1)b(k)\leqq\int_0^\infty\psi_t^A(\alpha-x)dx,$

(iii) $\int_{-\infty}^{0}\alpha^{2m}\psi_t^A(\alpha)d\alpha=\int_0^\infty\alpha^{2m}\{1-\psi_t^A(\alpha)\}d\alpha$

$$\leqq(2m+1)^{-1}E|A^{-h/2}x(At)|^{2m+1},$$

(iv) $\int_0^\infty d\alpha \int_0^\infty dx\alpha^{2m}\{1-\psi_t^A(\alpha+x)\} = \int_{-\infty}^0 d\alpha \int_0^\infty dx\alpha^{2m}\psi_t^A(\alpha-x)$

$\leqq (2m+1)^{-1}(2m+2)^{-1}E(A^{-h/2}x(At))^{2m+2}$.

Then we have

$$
\begin{aligned}
I_1 &= \int_{-\infty}^\infty \alpha^{2m} \textstyle\sum_{n=0}^\infty \sum_{k=0}^\infty b(k)\{a(n+1+k)-a(n+k)\}\psi_t^A(\alpha)^2 d\alpha \\
&\quad + \int_{-\infty}^\infty \alpha^{2m} \textstyle\sum_{n=0}^\infty \sum_{k=0}^\infty b(k)\{a(n+k)-a(n+1+k)\}\psi_t^A(\alpha)\{1-\psi_t^A(\alpha)\}d\alpha \\
&\leqq \int_{-\infty}^\infty \alpha^{2m}\{\textstyle\sum_{k=0}^\infty a(k)b(k+1)\}\psi_t^A(\alpha)^2 d\alpha \\
&\quad + \int_{-\infty}^\infty \alpha^{2m}\{\textstyle\sum_{k=0}^\infty a(k)b(k)\}\psi_t^A(\alpha)\{1-\psi_t^A(\alpha)\}d\alpha.
\end{aligned}
$$

The first term is estimated by Lemma 4.3 (i), (iii), (iv):

$$
\begin{aligned}
&\int_{-\infty}^\infty \alpha^{2m}\{\textstyle\sum_{k=0}^\infty a(k)b(k+1)\}\psi_t^A(\alpha)^2 d\alpha \\
&\leqq \int_0^\infty d\alpha \int_0^\infty dx\alpha^{2m}\{1-\psi_t^A(\alpha+x)\} + \int_{-\infty}^0 \alpha^{2m}\psi_t^A(\alpha)d\alpha \\
&\leqq \text{const}\,\{E(A^{-h/2}x(At))^{2m+2} + E|A^{-h/2}x(At)|^{2m+1}\}.
\end{aligned}
$$

The second term is estimated by Lemma 4.3 (iii):

$$
\begin{aligned}
&\int_{-\infty}^\infty \alpha^{2m}\{\textstyle\sum_{k=0}^\infty a(k)b(k)\}\psi_t^A(\alpha)\{1-\psi_t^A(\alpha)\}d\alpha \\
&\leqq \int_0^\infty \alpha^{2m}\{1-\psi_t^A(\alpha)\}d\alpha + \int_{-\infty}^0 \alpha^{2m}\psi_t^A(\alpha)d\alpha \\
&\leqq \text{const}\, E|A^{-h/2}x(At)|^{2m+1}.
\end{aligned}
$$

Similarly by Lemma 4.3 (ii) $\sim$ (iv),

$$
\begin{aligned}
I_2 &= \int_{-\infty}^\infty \alpha^{2m} \textstyle\sum_{n=0}^\infty \sum_{k=0}^\infty a(n+k)\{b(k)-b(k-1)\}\psi_t^A(\alpha)\{1-\psi_t^A(\alpha)\}d\alpha \\
&\quad + \int_{-\infty}^\infty \alpha^{2m} \textstyle\sum_{n=0}^\infty \sum_{k=0}^\infty a(n+1+k)\{b(k)-b(k-1)\}\{1-\psi_t^A(\alpha)\}^2 d\alpha \\
&\leqq \int_{-\infty}^\infty \alpha^{2m}\{\textstyle\sum_{k=0}^\infty a(k)b(k)\}\psi_t^A(\alpha)\{1-\psi_t^A(\alpha)\}d\alpha \\
&\quad + \int_{-\infty}^\infty \alpha^{2m}\{\textstyle\sum_{k=0}^\infty a(k+1)b(k)\}\{1-\psi_t^A(\alpha)\}^2 d\alpha \\
&\leqq \text{const}\,\{E|A^{-h/2}x(At)|^{2m+1} + E(A^{-h/2}x(At))^{2m+2}\}.
\end{aligned}
$$

Finally we get easily

$$I_3 = \int_{-\infty}^{\infty} \alpha^{2m}\{\textstyle\sum_{k=1}^{\infty} a(k)b(k)\}\,d\psi_t^A(\alpha) \leqq E(A^{-h/2}x(At))^{2m}.$$

From the above arguments we have that if $0 \leqq t \leqq \min(1, 1/A)$

$$\begin{aligned}
&E(A^{-h/2}y_0(At))^{2m}\\
&\quad\leqq \text{const}\,\{E(A^{-h/2}x(At))^{2m} + E(A^{-h/2}x(At))^{2m+2}\}\\
&\quad\leqq \text{const}\,\{E((At)^{-h}x(At))^{2m}(At)^{2mh}A^{-mh}\\
&\qquad + E((At)^{-h}x(At))^{2m+2}(At)^{2(m+1)h}A^{-(m+1)h}\}\\
&\quad\leqq \text{const}\; t^{mh}.
\end{aligned}$$

From Chebyshev's inequality and Lemma 2.3, we obtain

COROLLARY 4.4. *Under the assumption* (4.2), *if* $x(t)$ *has stationary increments, for some* $C>0$,

$$P[|A^{-h/2}y_0(At) - A^{-h/2}y_0(As)| \geqq \varepsilon] \leqq C\varepsilon^{-4m}(t-s)^{mh},$$
$$0 \leqq s \leqq t \leqq 1, \quad 0 \leqq t - s \leqq \min(1, 1/A), \quad \varepsilon > 0.$$

On the other hand we will show

LEMMA 4.5. *Under the assumption* (4.2), *if* $x(t)$ *has stationary increments, for some* $C>0$,

$$P[|A^{-h/2}y_0(At) - A^{-h/2}y_0(As)| \geqq \varepsilon] \leqq C\varepsilon^{-4m}(t-s)^{mh},$$
$$0 \leqq s \leqq t \leqq 1, \quad \min(1, 1/A) \leqq t - s \leqq 1, \quad \varepsilon > 0.$$

In order to show Lemma 4.5 we need the following lemma. Let $l(t) = \sum_{k=-\infty}^{-1} \chi[x_k(t)>0]$, $m(t) = \sum_{k=1}^{\infty} \chi[x_k(t) \leqq 0]$, $n(t) = \chi[x_0(t)>0]$ and $L(t) = l(t) + n(t) - m(t)$.

LEMMA 4.6. *Under the assumption* (4.2), *for some* $C>0$,

$$P[|L(At)| \geqq M] \leqq CM^{-4m}(At)^{2mh}, \quad \min(1, 1/A) \leqq t \leqq 1, \quad M > 0.$$

PROOF. We will evaluate the characteristic function of $L(At)$,

$$E[\exp\{i\xi L(At)\}] - g(\xi)f(\xi), \qquad \xi \in \mathbf{R},$$

where

$$g(\xi) = \exp(i\xi)P[x(At)>0] + P[x(At) \leqq 0],$$
$$f(\xi) = \exp[\{\exp(i\xi)-1\}E[x(At)^+] + \{\exp(-i\xi)-1\}E[(-x(At))^+]].$$

Noting that

$$d^{4m}E[\exp\{i\xi L(At)\}]/d\xi^{4m}|_{\xi=0} = (i)^{4m}E[L(At)^{4m}],$$

$$|g^{(n)}(0)| \leqq 1 \qquad (n = 0, 1, 2,\ldots), \quad f(0) = 1 \quad \text{and}$$

$$\max\{|f^{(2n-1)}(0)|, |f^{(2n)}(0)|\} \leqq C(E|x(At)|)^n \ (n = 1, 2,\ldots), \quad \text{for some } C > 0,$$

we easily obtain

$$E[L(At)^{4m}] \leqq \text{const} \max\{1, (E|x(At)|)^{2m}\}. \tag{4.5}$$

By using Chebyshev's inequality, (4.6) and the assumption (4.2), we obtain

$$P[|L(At)| \geqq M] \leqq \text{const}\, M^{-4m}(At)^{2mh},$$
$$\text{for any} \quad M > 0 \quad \text{and} \quad \min(1, 1/A) \leqq t \leqq 1.$$

We proceed to the proof of Lemma 4.5. For each $t \geqq 0$, let $\tilde{y}_{L(t)+k}(t) = y_k(t)$, $k \in \mathbf{Z}$. Note that $\tilde{y}_0(t)$ is the position of the nonpositive nearest particle from zero.

For any $\varepsilon > 0$,

$$\begin{aligned} &P[|A^{-h/2}y_0(At)| \geqq 2\varepsilon] \\ &\quad = P[|\tilde{y}_{L(At)}(At)| \geqq 2A^{h/2}\varepsilon, |L(At)| < M] \\ &\qquad + P[|\tilde{y}_{L(At)}(At)| \geqq 2A^{h/2}\varepsilon, |L(At)| \geqq M] \\ &\quad \leqq P[|\tilde{y}_{L(At)}(At) - L(At)| \geqq A^{h/2}\varepsilon, |L(At)| < M] \\ &\qquad + P[|L(At)| \geqq A^{h/2}\varepsilon] + P[|L(At)| \geqq M] \\ &\quad = J_1 + J_2 + J_3. \end{aligned}$$

Note that $X_n = \tilde{y}_{n+1}(At) - \tilde{y}_n(At)$, $n \in \mathbf{Z}$, i.i.d. random variables with exponentaial distributions of mean one. Let $\tilde{X}_n = X_n - 1$. By Lemma 2.3 we can evaluate

$$J_1 \leqq 2P[\max_{1 \leqq k \leqq M} |\tilde{X}_1 + \cdots + \tilde{X}_k| > A^{h/2}\varepsilon].$$

Noting that $EX_1 = 0$, and $E(X_1)^{2p} < \infty$, for any integer $p \geqq 1$, by Lemma 2.4 we get

$$J_1 \leqq \text{const}\, M^p A^{-ph}\varepsilon^{-2p}.$$

By Lemma 4.6 we get

$$J_2 \leqq \text{const}\, t^{2mh}\varepsilon^{-4m}, \quad J_3 \leqq \text{const}\,(At)^{2mh}M^{-4m}.$$

Choosing $M = \varepsilon A^{3h/4}$ and $p = 4m$, we get

$$J_1 + J_2 + J_3 \leqq \text{const}\ t^{mh}\varepsilon^{-4m}.$$

By Lemma 2.3 we complete the proof of Lemma 4.5.

LEMMA 4.7. *Under the assumption* (4.2) *if* $x(t)$ *has stationary increments, then*

$$\{A^{-h/2}y_0(At)\},\ A>0, \quad \textit{is tight in} \quad C([0,1],\mathbf{R}).$$

PROOF. Combining Corollary 4.4 and Lemma 4.5, we obtain

$$P[|A^{-h/2}y_0(At)-A^{-h/2}y_0(As)|\geqq\varepsilon] \leqq \text{const}\ \varepsilon^{-4m}(t-s)^{mh},$$

$$\text{for any} \quad \varepsilon>0,\ A>0 \quad \text{and} \quad 0\leqq s\leqq t\leqq 1.$$

Following the fundamental theorem for tightness in $C([0,1],\mathbf{R})$ (see Billingsley [1], Theorem 12.3), we easily obtain that

$$\{A^{-h/2}y_0(At)\},\ A>0, \quad \text{is tight in} \quad C([0,1],\mathbf{R}).$$

Finally by Lemma 4.7 the proof of Theorem 4.1 is completed.

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Added in Proof

I came to know that Dürr, Goldstein, and Lebowitz [8] recently solved this problem by a different method from our's.

[8] D. Dürr, S. Goldstein, and J. L. Lebowitz, Asymptotics of particle trajectories in infinite one-dimensional systems with collisions, Comm. Pure Appl. **38** (1985), 573–597.

Department of Applied Physics,
Faculty of Science,
Tokyo Institute of Technology