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A characterization of Prüfer v-multiplication domains in terms of polynomial grade

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Prüfer v-multiplication domains, abbreviated to PVMD's, have among their special cases a variety of notions, including Prüfer domains, Krull domains, GCD domains, etc. Many interesting characterizations of PVMD's are given by several authors (see [2], [3], [5], [7], [8], [11]). The main purpose of this paper is to give a characterization of PVMD's in terms of polynomial grade (cf. Theorem 2 and Remark 3). This characterization makes the situation of PVMD's in the class of P-domains clearer.

Moreover, we shall examine some properties of PVMD's by making use of Theorem 2 and Remark 3. First, we shall give some characterizations of PVMD's in the class of intergrally closed domains (cf. Theorem 5 and Proposition 7). In particular, Theorem 5 is a generalization of Theorem 3.4 of [5]. Next, we shall give a necessary and sufficient condition for an FC domain to be integrally closed (cf. Proposition 11). Finally, in case A is a PVMD, we shall give a characterization of G_2-stableness of $A \subset B$, where B is an overring of A (cf. Proposition 12).

To give our results, we include the following notions and notations.

Throughout this paper, A and K denote an integral domain and its quotient field respectively. Moreover, we denote by X an indeterminate. For a fractional ideal I of A, we put $I_v = A:_K(A:_K I)$. We say that I is a *v-ideal* if $I = I_v$, and a v-ideal I is *of finite type* if there is a finitely generated fractional ideal J of A such that $I = J_v$. An integral domain A is called a *Prüfer v-multiplication domain* (*PVMD*), if the set of all v-ideals of A of finite type forms a group under the v-multiplication $I \cdot J = (IJ)_v$, [3]. Let I be an ideal of A. We denote by $\mathrm{gr}(I)$ and $\mathrm{Gr}(I)$ the classical grade of I and the polynomial grade of I respectively, [6]. The following subsets of $\mathrm{Spec}(A)$ are needed for this paper.

$$\mathfrak{P}(A) = \{P \in \mathrm{Spec}(A) \mid P \text{ is minimal over } a: b \text{ for some } a, b \in A\}.$$

$$\mathfrak{G}(A) = \{P \in \mathrm{Spec}(A) \mid \mathrm{Gr}(P) \leqq 1\}.$$

If A_P is a valuation ring for each $P \in \mathfrak{P}(A)$, A is called a *P-domain*, [5]. It is known that a PVMD is a P-domain, ([5], Corollary 1.4). Since $A = \cap \{A_P \mid P \in \mathfrak{P}(A)\}$ by Theorem E of [9] and $\mathfrak{P}(A) \subset \mathfrak{G}(A)$, we have $A = \cap \{A_P \mid P \in \mathfrak{G}(A)\}$.

Let I be an ideal of $A[X]$. We denote by $c(I)$ the ideal of A generated by all coefficients of all polynomials in I and we call it the *content* of I. Let $U=\{f(X)\in A[X] \mid A:{}_K c(f)=A\}$. Then U is a multiplicatively closed subset of $A[X]$ and $A[X]_U$ is a subring of $K(X)$.

We begin with the following lemma which can be proved easily.

LEMMA 1. *Let $Q\in \mathrm{Spec}(A[X])$ with $Q\cap U=\emptyset$. Then we have $c(Q)A[X]\cap U=\emptyset$.*

THEOREM 2. *For A, the following statements are equivalent.*

(1) *$A[X]_U$ is a Prüfer domain.*

(2) *A_P is a valuation ring for each $P\in \mathfrak{G}(A)$.*

PROOF. (1)⇒(2). Let $P\in \mathfrak{G}(A)$. Then we have $PA[X]\cap U=\emptyset$ by Lemma 3.1 of [10]. Then $(A[X]_U)_{PA[X]_U}$ is a valuation ring by the assumption. Therefore, $A_P=(A[X]_U)_{PA[X]_U}\cap K$ is a valuation ring.

(2)⇒(1). Let $P\in \mathrm{Spec}(A[X]_U)$ and put $Q=P\cap A[X]$. Then we have $P=QA[X]_U$ and $Q\cap U=\emptyset$. Therefore, $c(Q)A[X]\cap U=\emptyset$ by Lemma 1. Since U is a multiplicatively closed subset of $A[X]$, there exists $Q_1\in \mathrm{Spec}(A[X])$ with the property that $Q_1\cap U=\emptyset$ and $c(Q)A[X]\subset Q_1$. Put $P_1=Q_1\cap A$. Then $c(Q)\subset P_1$ and $\mathrm{Gr}(P_1)\leqq 1$. Therefore, A_{P_1} is a valuation ring by the assumption. Since $Q\subset P_1A[X]$, we have easily that $(A[X]_U)_P$ is a valuation ring. That is, $A[X]_U$ is a Prüfer domain.

REMARK 3 (cf. [7], Theorem & [2], Theorem 3.6). It is known that the following statements are all equivalent to (1) of Theorem 2.

(3) *$A[X]_U$ is a Bezout domain.*

(4) *A is integrally closed and each prime ideal of $A[X]_U$ is the extension of a prime ideal of A.*

(5) *A is a PVMD.*

Since $\mathfrak{P}(A)\subset \mathfrak{G}(A)$, Theorem 2 and Remark 3 imply that a PVMD is a P-domain. Moreover, we have the following two characterizations of PVMD's.

COROLLARY 4. *The following statements are equivalent.*

(1) *A is a PVMD.*

(2) *$A[X]$ is a PVMD.*

(3) *$A[X]_P$ is a valuation ring for each prime ideal P of $A[X]$ with* $\mathrm{gr}(P)\leqq 1$.

PROOF. (2)⇔(3). This equivalence follows easily from Proposition 3.4 of [10].

(2)⇒(1). Assume that $A[X]$ is a PVMD and let $P\in \mathfrak{G}(A)$. Then $PA[X]\in \mathfrak{G}(A[X])$. By Theorem 2 and Remark 3, $A[X]_{PA[X]}$ is a valuation ring. Therefore, $A_P=A[X]_{PA[X]}\cap K$ is a valuation ring. This implies that A is a PVMD.

(1)⇒(2). Assume that A is a PVMD and let $Q \in \mathfrak{G}(A[X])$. If $Q \cap A = (0)$ and $Q \neq (0)$, then we have $QK[X] = f(X)K[X]$ for some irreducible polynomial $f(X) \in K[X]$. Therefore, $A[X]_Q = K[X]_{f(X)K[X]}$ is a valuation ring.

Next, assume that $Q \cap A = P \neq (0)$. Then we have $\mathrm{Gr}(P) = 1$. Moreover, since $Q \cap A \neq (0)$, $Q \cap U = \emptyset$ by Lemma 3.1 of [10]. Therefore, $A[X]_Q = (A[X]_U)_{QA[X]_U}$ is a valuation ring by Theorem 2 and Remark 3. That is, $A[X]$ is a PVMD.

THEOREM 5 (cf. [5], Theorem 3.4). *Let A be integrally closed. Then the following statements are equivalent.*

(1) *A is a PVMD.*

(2) *Let $P \in \mathfrak{P}(A[X])$ and $P \neq (0)$. If $P \cap U = \emptyset$, then $P \cap A \neq (0)$.*

PROOF. (1)⇒(2). Assume that A is a PVMD. Let $P \in \mathfrak{P}(A[X])$ and $P \neq (0)$. Suppose that $P \cap U = \emptyset$. Then $PA[X]_U$ is a prime ideal of $A[X]_U$. Therefore, $PA[X]_U$ is the extension of a prime ideal of A by Remark 3. That is, we have $P \cap A \neq (0)$.

(2)⇒(1). Let $Q \in \mathfrak{G}(A)$ and $Q \neq (0)$. Then we have $QA[X] \cap U = \emptyset$ by Lemma 3.1 of [10]. Let P be a prime ideal of $A[X]$ contained in $QA[X]$. Suppose that $(P \cap A)A[X] \neq P$ and take $f(X) \in P - (P \cap A)A[X]$. Then there exists $P_1 \in \mathfrak{P}(A[X])$ such that $f(X) \in P_1 \subset P$. Since $P_1 \subset QA[X]$, $P_1 \cap U = \emptyset$. Therefore, we have $P_1 \cap A \neq (0)$ by the assumption. Thus, $P_1 = (P_1 \cap A)A[X]$ and $P_1 \cap A \in \mathfrak{P}(A)$ by Corollary 8 of [1]. Since $P_1 \subset P$, $f(X) \in P_1 \subset (P \cap A)A[X]$. This is a contradiction. Hence, we have $P = (P \cap A)A[X]$. Since A is integrally closed, A_Q is a valuation ring by Theorem (19.15) of [3]. Therefore, A is a PVMD by Theorem 2 and Remark 3.

Given an extension of integral domains $A \subset B$ and $P \in \mathrm{Spec}(A)$, we say the extension satisfies INC at P if distinct comparable prime ideals of B do not contract to P, [8]. If $W \subset \mathrm{Spec}(A)$, we say that the extension satisfies INC on W if it satisfies INC at each $P \in W$, [8]. If $A \subset B$ satisfies INC on $\mathrm{Spec}(A)$, then as usual we say $A \subset B$ satisfies INC. Given an extension of integral domains $A \subset B$, we say that an element u in B is *super-primitive over A*, if u is the root of a polynomial $f(X) \in A[X]$ with $A :_K c(f) = A$. The following proposition is a characterization of super-primitive elements.

PROPOSITION 6 (cf. [8], Corollary 2.2). *Let $A \subset B$ be an extension of integral domains and assume that $u \in B$ is algebraic over A. Then u is super-primitive over A if and only if $A \subset A[u]$ satisfies INC on $\mathfrak{G}(A)$.*

PROOF. Let $I = \mathrm{Ker}(A[X] \to A[u])$, where the homomorphism is the evaluation map.

First, assume that u is super-primitive over A. Then there exists $f(X) \in I$

such that $A:{}_K c(f)=A$. Hence, $c(I)\not\subset P$ for each $P\in\mathfrak{G}(A)$ by Theorem 8 of Chapter 5 of [6]. Then $A\subset A[u]$ satisfies INC on $\mathfrak{G}(A)$ by Proposition 2.0 of [8].

Conversely, assume that u is not super-primitive over A. Then we have $\mathrm{Gr}(c(I))=1$ by Theorem 11 of Chapter 5 of [6]. Since $c(I)\neq A$, there exists $P\in\mathfrak{G}(A)$ with $c(I)\subset P$ by Theorem 16 of Chapter 5 of [6]. Therefore, $A\subset A[u]$ does not satisfy INC at P by Proposition 2.0 of [8]. That is, $A\subset A[u]$ does not satisfy INC on $\mathfrak{G}(A)$.

Therefore, we have easily the following proposition by Proposition 2.5 of [8] and Proposition 6.

PROPOSITION 7 (cf. [8], Corollary 2.2 & Proposition 2.5). *Let Ω be the algebraic closure of K and assume that A is integrally closed. Then the following statements are equivalent.*

(1) *A is a PVMD.*

(2) *$A\subset A[u]$ satisfies INC on $\mathfrak{G}(A)$ for each $u\in K$.*

(3) *$A\subset A[u]$ satisfies INC on $\mathfrak{G}(A)$ for each $u\in\Omega$.*

(4) *For each $u\in K$, u is super-primitive over A.*

(5) *For each $u\in\Omega$, u is super-primitive over A.*

Here, we shall give two conditions which imply that a P-domain is a PVMD.

PROPOSITION 8. *Let $\mathfrak{P}(A)$ be compact as a subspace of* Spec(A) *in the Zariski topology. Then A is a PVMD if and only if A is a P-domain.*

PROOF. By Lemma 3.1 of [8] and Theorem E of [9], $\mathfrak{P}(A)$ is compact if and only if given any ideal I of A with $\mathrm{Gr}(I)=1$, there exists $P\in\mathfrak{P}(A)$ such that $I\subset P$. Therefore, this proposition follows easily from Theorem 2 and Remark 3.

A partially ordered set is said to form a *tree* in case no two unrelated elements have a common upper bound.

PROPOSITION 9. *A is a PVMD if and only if it is a P-domain and $\mathfrak{G}(A)$ forms a tree.*

PROOF. By virtue of Theorem 2 and Remark 3, it is sufficient to prove the 'if' part. Assume that A is not a PVMD. Then, by Theorem 2 and Remark 3, there exists $P\in\mathfrak{G}(A)$ and exist two elements a, b in A such that $a: b\subset P$ and $b: a\subset P$. Moreover, there exist $Q_1, Q_2\in\mathfrak{P}(A)$ such that $a: b\subset Q_1\subset P$ and $b: a\subset Q_2\subset P$. Since $Q_1, Q_2\in\mathfrak{P}(A)$, both A_{Q_1} and A_{Q_2} are valuation rings. Therefore, $b: a\not\subset Q_1$ and $a: b\not\subset Q_2$. That is, $Q_1\not\subset Q_2$ and $Q_2\not\subset Q_1$. This is a contradiction.

Recall that an integral domain A is said to be an *FC domain*, in case $Aa\cap Ab$

is finitely generated for each a, $b \in A$.

LEMMA 10. *Let A be integrally closed and take a, $b \in A-\{0\}$. Assume that $a\colon b$ is finitely generated and put $I=(a\colon b)+(b\colon a)$. Then we have $A\colon {}_K I=A$.*

PROOF. Since $a\colon b$ is finitely generated, there exist $a_1, a_2,\dots, a_n \in A$ such that $a\colon b=(a_1, a_2,\dots, a_n)$. Moreover, for $1 \leqq i \leqq n$, there exists $b_i \in A$ such that $a_i b=ab_i$. Then we have $b\colon a=(b_1, b_2,\dots, b_n)$. Assume that $x \in A\colon {}_K I$. Put $xa_i=\alpha_i$ and $xb_i=\beta_i$ for $1 \leqq i \leqq n$. Then $\alpha_i \in A$ and $\beta_i \in A$. Moreover, we have $\alpha_i \in a\colon b$ for $1 \leqq i \leqq n$. Therefore, for $1 \leqq i \leqq n$, there exist $\lambda_{ij} \in A$ $(1 \leqq j \leqq n)$ such that $\alpha_i=\sum_{j=1}^n \lambda_{ij} a_j$. Since $xa_i=\sum_{j=1}^n \lambda_{ij} a_j$ for $1 \leqq i \leqq n$, x integral over A. On the other hand, A is integrally closed. Thus, $x \in A$. This implies that $A\colon {}_K I=A$.

The following proposition contains the result of Theorem 2 of [11].

PROPOSITION 11. *Let A be an FC domain. Then the following statements are equivalent.*

(1) *A is integrally closed.*

(2) *$A\colon {}_K((a\colon b)+(b\colon a))=A$ for each a, $b \in A-\{0\}$.*

(3) *A is a PVMD.*

PROOF. The implication (3)⇒(1) is obvious. Moreover, the implication (1)⇒(2) follows easily from Lemma 10.

(2)⇒(3). Let $P \in \mathfrak{G}(A)$ and assume that A_P is not a valuation ring. Then there exists $u \in K$ such that u, $u^{-1} \notin A_P$. Put $u=a/b \in A-\{0\}$. Since u, $u^{-1} \notin A_P$, we have $a\colon b \subset P$ and $b\colon a \subset P$. Moreover, $A\colon {}_K((a\colon b)+(b\colon a))=A$ by the assumption. On the other hand, since A is an FC domain, $(a\colon b)+(b\colon a)$ is finitely generated. Therefore, we have $\mathrm{Gr}(P) \geqq \mathrm{Gr}((a\colon b)+(b\colon a)) \geqq 2$. This is a contradiction. Thus, A_P is a valuation ring for each $P \in \mathfrak{G}(A)$. That is, A is a PVMD by Theorem 2 and Remark 3.

Let $A \subset B$ be an extension of integral domains. We say that $A \subset B$ is G_2-*stable* if for each finitely generated ideal I of A with $\mathrm{Gr}(I) \geqq 2$, $\mathrm{Gr}(IB) \geqq 2$, [10]. It is obvious that if $A \subset B$ is flat, then $A \subset B$ is G_2-stable. But the converse is false as is seen in $\boldsymbol{Z}[\sqrt{5}] \subset \boldsymbol{Z}[1+\sqrt{5}/2]$, where $\boldsymbol{Z}$ is the ring of integers. As for *overrings*, we have the following

PROPOSITION 12 (cf. [5], Proposition 5.1). *Let A be a PVMD and B an overring of A. Then $A \subset B$ is G_2-stable if and only if $B=\cap\{A_P \mid P \in Y\}$ for some $Y \subset \mathfrak{G}(A)$. Moreover, in this case, B is also a PVMD.*

PROOF. Assume that $A \subset B$ is G_2-stable and let $Q \in \mathfrak{G}(B)$. Put $P=Q \cap A$. Since $A \subset B$ is G_2-stable, we have $\mathrm{Gr}(P)=1$. Therefore, A_P is a valuation ring by Theorem 2 and Remark 3. Since $A_P \subset B_Q \subset K$, B_Q is a valuation ring. Hence, B is a PVMD by Theorem 2 and Remark 3. Moreover, we have $B_Q = A_P$ by Theorem 65 of [4]. Put $Y = \{Q \cap A \mid Q \in \mathfrak{G}(B)\}$. Then we have $Y \subset \mathfrak{G}(A)$ and $B = \cap \{B_Q \mid Q \in \mathfrak{G}(B)\} = \cap \{A_P \mid P \in Y\}$.

Conversely, assume that $B = \cap \{A_P \mid P \in Y\}$ for some $Y \subset \mathfrak{G}(A)$. Let w be the $*$-operation induced by the valuation ring A_P for $P \in Y$. Suppose that I is a finitely generated ideal of A with $\mathrm{Gr}(I) \geqq 2$. Then we have $(IB)_w = \cap \{IA_P \mid P \in Y\} = \cap \{A_P \mid P \in Y\} = B$. Hence, we have $(IB)_v = B$ by Theorem (34.1) of [3]. That is, $B:_K(B:_K IB) = B$. Therefore, $B:_K IB = B:_K(B:_K(B:_K IB)) = B$. Then $\mathrm{Gr}(IB) \geqq 2$. This implies that $A \subset B$ is G_2-stable.

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