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On the existence of limits along lines of Beppo Levi functions

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1. Introduction

Fefferman [2] proved that if u is a continuously differentiable function on R^n with gradient in $L^p(R^n)$ and $1<p<n$, then

$$\lim_{x_n\to\infty} u(x', x_n) = \text{const.} \tag{1}$$

holds for almost every $x' \in R^{n-1}$. The author improved his result in [8] by showing that the set of $(x', 0)$ for which (1) does not hold is of Bessel capacity of index $(1, p)$ zero (see Meyers [6] for the definition of Bessel capacities).

In this paper we deal with Beppo Levi functions of general order (cf. [1]) and discuss the existence of radial and perpendicular limits. For this purpose we establish an integral representation of Beppo Levi functions as a generalization of [7; Theorem 4.1], and apply the technique of [5] to study the behavior at infinity of potential type functions.

2. Integral representation of Beppo Levi functions

Let R^n denote the n-dimensional euclidean space. For a multi-index $\lambda=(\lambda_1,\dots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \cdots + \lambda_n, \quad \lambda! = \lambda_1!\cdots\lambda_n!, \quad x^\lambda = x_1^{\lambda_1}\cdots x_n^{\lambda_n}$$
$$D^\lambda = (\partial/\partial x_1)^{\lambda_1}\cdots(\partial/\partial x_n)^{\lambda_n},$$

where $x=(x_1,\dots, x_n)$ is a point of R^n. Following Deny-Lions [1], we use the notation $BL_m(L^p(R^n))$ to denote the space of all functions $u \in L^p_{loc}(R^n)$ such that $D^\lambda u \in L^p(R^n)$ for any λ with $|\lambda|=m$, where $1<p<\infty$, m is a positive integer and the derivatives are taken in the sense of distributions. A function $u \in BL_m(L^p(R^n))$ is called a Beppo Levi function of order m attached to the space $L^p(R^n)$, or briefly an (m, p)-BL function on R^n, if u is (m, p)-quasi continuous in the sense of [7].

Let k_m denote the Riesz kernel of order $2m$, which is defined by

$$k_m(x) = \begin{cases} |x|^{2m-n} & \text{if } \ 2m < n \text{ or if } 2m > n \text{ and } n \text{ is odd,} \\ -|x|^{2m-n}\log|x| & \text{if } \ 2m \geqq n \text{ and } n \text{ is even.} \end{cases}$$

For a nonnegative integer ℓ and a multi-index λ, we set

$$k_{m,\lambda}(x) = D^{\lambda} k_m(x)$$

and

$$K_{m,\lambda,\ell}(x, y) = \begin{cases} k_{m,\lambda}(x-y) - \sum_{|\mu| \leq \ell} (\mu!)^{-1} x^{\mu} (D^{\mu} k_{m,\lambda})(-y) & \text{if } |y| \geqq 1, \\ k_{m,\lambda}(x-y) & \text{if } |y| < 1. \end{cases}$$

We first state some properties of functions $K_{m,\lambda,\ell}$, which can be proved by elementary calculus (cf. [3; Lemma 4.2], [10; Lemma 4]).

LEMMA 1. (i) *The function* $K_{m,\lambda,\ell}(\cdot, y)$ *is polyharmonic of order* m *in* $R^n - \{y\}$, *that is,*

$$\Delta^m K_{m,\lambda,\ell}(\cdot, y) = 0 \qquad on \quad R^n - \{y\}.$$

(ii) *If* $2m - n - |\lambda| - \ell \leqq 0$, *then*

$$|K_{m,\lambda,\ell}(x, y)| \leqq \text{const.}\, |x|^{\ell+1} |y|^{2m-n-|\lambda|-\ell-1}$$

whenever $|y| \geqq 2|x| \geqq 1$.

(iii) *If* $|\lambda| = m$ *and* $m - n - \ell \leqq 0$, *then*

$$|K_{m,\lambda,\ell}(x, y)| \leqq \text{const.}\, |x|^{\ell} \times \begin{cases} \log(4|x|/|y|) & \text{in case} \quad \ell = m - n, \\ |y|^{m-n-\ell} & \text{in case} \quad \ell > m - n, \end{cases}$$

whenever $1 \leqq |y| \leqq 2|x|$ *and* $|x - y| \geqq |x|/2$.

Let a_λ be constants so chosen that

$$\phi(x) = \sum_{|\lambda|=m} a_\lambda \int k_{m,\lambda}(x-y) D^{\lambda} \phi(y) dy \qquad \text{for any} \quad \phi \in C_0^{\infty}(R^n) \tag{2}$$

(see Wallin [12; p. 71]).

THEOREM 1. *If* $u \in BL_m(L^p(R^n))$ *and* $mp \geqq n$, *then*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y) D^{\lambda} u(y) dy + P_\ell(x)$$

holds for almost every $x \in R^n$, *where* ℓ *is a nonnegative integer such that* $\ell \leqq m - n/p < \ell + 1$ *and* P_ℓ *is a polynomial of degree at most* $m-1$. *If* u *is an* (m, p)*-BL function on* R^n, *then the equality holds for* $x \in R^n$ *except those in a set whose Bessel capacity of index* (m, p) *is zero.*

REMARK. Kurokawa [4] has obtained an integral representation of Beppo Levi functions different from ours, and applied it to the discussion of weighted

L^p estimates of Beppo Levi functions.

To prove Theorem 1, we prepare the following lemma.

LEMMA 2. *Let $mp \geqq n$ and ℓ be the integer such that $\ell \leqq m - n/p < \ell + 1$. If $f \in L^p(R^n)$ and $|\lambda| = m$, then*

$$\int_{\{y;|y| \geqq 2|x|\}} |K_{m,\lambda,\ell}(x, y)f(y)|dy \leqq \text{const. } |x|^{m-n/p} \|f\|_p$$

whenever $|x| > 1$.

PROOF. By Lemma 1 (ii) we have for $x \in R^n - B(0, 1)$,

$$\int_{\{y;|y| \geqq 2|x|\}} |K_{m,\lambda,\ell}(x, y)f(y)|dy$$
$$\leqq \text{const. } |x|^{\ell+1} \int_{\{y;|y| \geqq 2|x|\}} |y|^{m-n-\ell-1} |f(y)|dy.$$

By our assumptions, $p'(m-n-\ell-1)+n<0$, so that the required assertion follows from Hölder's inequality, where $1/p + 1/p' = 1$.

From this lemma, we can easily derive the following two facts.

COROLLARY 1. *Under the assumptions in Lemma 2 we have*

$$\lim_{|x| \to \infty} |x|^{(n-mp)/p} \int_{\{y;|y| \geqq 2|x|\}} K_{m,\lambda,\ell}(x, y)f(y)dy = 0.$$

COROLLARY 2. *Under the assumptions in Lemma 2, if $\phi \in C_0^\infty(R^n)$, then*

$$\iint |K_{m,\lambda,\ell}(x, y)f(y)\phi(x)|dydx < \infty.$$

PROOF OF THEOREM 1. In view of the facts in [12; p. 71], for $\phi \in C_0^\infty(R^n)$ we find

$$\int K_{m,\lambda,\ell}(x, y)\Delta^m \phi(x)dx = c_m(-1)^{|\lambda|} D^\lambda \phi(y), \tag{3}$$

where c_m is a constant independent of λ, ℓ and ϕ. Hence, $\Delta^m = c_m \sum_{|\lambda|=m} a_\lambda D^{2\lambda}$ by (2) and (3). In view of Corollary 2 to Lemma 2 we can apply Fubini's theorem to obtain

$$\int \left\{ \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y) D^\lambda u(y)dy \right\} \Delta^m \phi(x)dx$$
$$= c_m(-1)^m \sum_{|\lambda|=m} a_\lambda \int D^\lambda \phi(y) D^\lambda u(y)dy$$
$$= c_m \int u(y) \sum_{|\lambda|=m} a_\lambda D^{2\lambda} \phi(y)dy = \int u(y) \Delta^m \phi(y)dy$$

for $\phi \in C_0^\infty(R^n)$, which implies

$$\Delta^m\Big(u - \textstyle\sum_{|\lambda|=m} a_\lambda \displaystyle\int K_{m,\lambda,\ell}(\cdot, y) D^\lambda u(y) dy\Big) = 0.$$

Now Theorem 1 follows from Lemma 4.1 in [7] and the following result.

LEMMA 3. *If $f \in L^p(R^n)$, then $\int K_{m,\lambda,\ell}(x, y) f(y) dy$ are (m, p)-BL functions on R^n whenever m, p, λ and ℓ are given as in Lemma 2.*

PROOF. Set $v(x) = \int K_{m,\lambda,\ell}(x, y) f(y) dy$. For $r > 0$, we write $v_r(x) = \int_{B(0,r)} K_{m,\lambda,\ell}(x, y) f(y) dy$, where $B(0, r) = \{y \in R^n; |y| < r\}$. Then it is easy to see that $v - v_r$ is continuous on $B(0, r)$. Since $v_r(x) = \int_{B(0,r)} (D^\lambda k_m)(x-y) f(y) dy +$ a polynomial, Lemma 3.3 (iii) in [7] implies that v_r is (m, p)-quasi continuous on R^n. Thus v is (m, p)-quasi continuous on R^n.

On the other hand, for $\phi \in C_0^\infty(R^n)$ and a multi-index μ with length m, we have

$$\begin{aligned}\int v(x) D^\mu \phi(x) dx &= \int \Big\{\int K_{m,\lambda,\ell}(x, y) D^\mu \phi(x) dx\Big\} f(y) dy \\ &= \int \Big\{\int k_{m,\lambda}(x-y) D^\mu \phi(x) dx\Big\} f(y) dy \\ &\leqq \text{const.}\ \|\phi\|_{p'} \|f\|_p\end{aligned}$$

on account of Lemma 3.3 (ii) in [7], where $1/p + 1/p' = 1$. This implies that $v \in BL_m(L^p(R^n))$, and hence v is an (m, p)-BL function on R^n.

3. Radial limits

We denote by $B(a, r)$ the open ball with center at a and radius r, and by S the boundary $\partial B(0, 1)$. By using the integral representations in [7] and the same method as in the proof of Corollary 4.7 in [5], we can establish the following result.

THEOREM 2. *If $mp < n$ and u is an (m, p)-BL function on R^n, then there exist a polynomial P of degree at most $m-1$ and a set $E \subset S$ such that $B_{m,p}(E) = 0$ and*

$$\lim_{r\to\infty} r^{(n-mp)/p}\{u(r\Theta) - P(r\Theta)\} = 0 \qquad \text{for every} \quad \Theta \in S - E,$$

where $B_{m,p}$ denotes the Bessel capacity of index (m, p) (see [6]).

Our main aim in this section is to extend Theorem 2 to general cases. For that purpose we need some lemmas.

LEMMA 4. *Suppose there exists a nonnegative integer ℓ such that $\ell < m - n/p < \ell + 1$. If $|\lambda| = m$ and $f \in L^p(R^n)$, then*

$$\lim_{|x|\to\infty} |x|^{(n-mp)/p} \int_{\{y;|x-y|\geqq|x|/2\}} K_{m,\lambda,\ell}(x, y) f(y) dy = 0.$$

PROOF. By Corollary 1 to Lemma 2 we have only to prove

$$\lim_{|x|\to\infty} |x|^{(n-mp)/p} \int_{E(x)} K_{m,\lambda,\ell}(x, y) f(y) dy = 0,$$

where $E(x) = \{y;\ |x-y| \geqq |x|/2,\ |y| < 2|x|\}$. If $|x| > 1$ and $y \in E(x)$, then $|K_{m,\lambda,\ell}(x, y)| \leqq$ const. $|x|^{\ell}|y|^{m-n-\ell} \log(4|x|/|y|)$ on account of Lemma 1 (iii). Hence we obtain for $\eta > 1$,

$$\begin{aligned}
&|x|^{(n-mp)/p} \int_{E(x)} |K_{m,\lambda,\ell}(x, y) f(y)| dy \\
&\leqq \text{const. } |x|^{(n-mp)/p+\ell} \int_{B(0,2|x|)} |y|^{m-n-\ell} |f(y)| \log(4|x|/|y|) dy \\
&\leqq \text{const. } |x|^{(n-mp)/p+\ell} \int_{B(0,\eta)} |y|^{m-n-\ell} |f(y)| \log(4|x|/|y|) dy \\
&\quad + \text{const. } \left\{ \int_{R^n - B(0,\eta)} |f(y)|^p dy \right\}^{1/p},
\end{aligned}$$

so that

$$\begin{aligned}
&\lim_{|x|\to\infty} |x|^{(n-mp)/p} \int_{E(x)} |K_{m,\lambda,\ell}(x, y) f(y)| dy \\
&\leqq \text{const. } \left\{ \int_{R^n - B(0,\eta)} |f(y)|^p dy \right\}^{1/p}
\end{aligned}$$

Letting $\eta \to \infty$, we derive the desired equality.

LEMMA 5. *Suppose $\ell = m - n/p$ is a nonnegative integer. If $|\lambda| = m$ and $f \in L^p(R^n)$, then*

$$\lim_{|x|\to\infty} |x|^{(n-mp)/p} (\log|x|)^{-1/p'} \int_{\{y;|x-y|\geqq|x|/2\}} K_{m,\lambda,\ell}(x, y) f(y) dy = 0.$$

PROOF. In view of Lemma 1 (iii), we have

$$\begin{aligned}
&|x|^{(n-mp)/p} (\log|x|)^{-1/p'} \int_{E(x)} |K_{m,\lambda,\ell}(x, y) f(y)| dy \\
&\leqq \text{const. } (\log|x|)^{-1/p'} \int_{E(x)-B(0,1)} |y|^{-n/p'} |f(y)| dy
\end{aligned}$$

$$+ \text{const.}\, |x|^{-\ell}(\log|x|)^{-1/p'} \int_{B(0,1)} |x-y|^{m-n}|f(y)| \log(2+|x-y|)\, dy$$

with $E(x)$ defined as above. Here the second term of the right hand side clearly tends to zero as $|x|\to\infty$, and the first term of the right hand side can be evaluated as in the previous proof.

Following [5], we say that a set E in R^n is (m, p)-thin at infinity if

$$\sum_{j=1}^{\infty} B_{m,p}(E'_j) < \infty,$$

where $E'_j=\{x\in R^n;\ 2^j x\in E,\ 1\leqq|x|<2\}$. In case $mp>n$, we find easily that $B_{m,p}(A)\geqq B_{m,p}(\{0\})>0$ whenever A is not empty, so that E is (m, p)-thin at infinity if and only if E is bounded.

LEMMA 6. *Let $mp\leqq n$. If $f\in L^p(R^n)$, $|\lambda|=m$ and ℓ is a nonnegative integer, then there exists a set E, which is (m, p)-thin at infinity, such that*

$$\lim_{|x|\to\infty, x\in R^n-E} |x|^{(n-mp)/p} \int_{\{y;|x-y|<|x|/2\}} K_{m,\lambda,\ell}(x, y) f(y)\, dy=0.$$

PROOF. Since $m<n$, $|K_{m,\lambda,\ell}(x, y)|\leqq \text{const.}\, |x-y|^{m-n}$ whenever $1\leqq|x|/2\leqq |y|\leqq 2|x|$. Now, applying Lemma 4.4 i) in [5], we obtain the required assertion.

LEMMA 7. *Suppose $mp>n$. If f, λ and ℓ are as in Lemma 6, then*

$$\lim_{|x|\to\infty} |x|^{(n-mp)/p} \int_{\{y;|x-y|<|x|/2\}} K_{m,\lambda,\ell}(x, y) f(y)\, dy=0.$$

PROOF. From the definition of $K_{m,\lambda,\ell}$, it follows that

$$|K_{m,\lambda,\ell}(x, y)| \leqq \text{const.} \begin{cases} |x-y|^{m-n} & \text{if } m<n, \\ |x-y|^{m-n} \log(|x|/|x-y|) & \text{if } m\geqq n \end{cases}$$

whenever $|x|\geqq 1$ and $|x-y|<|x|/2$. Now Hölder's inequality yields the desired equality.

For simplicity, define

$$A(r) = \begin{cases} r^{(n-mp)/p} & \text{if } m-n/p \text{ is not a nonnegative integer,} \\ r^{(n-mp)/p}(\log r)^{-1/p'} & \text{if } m-n/p \text{ is a nonnegative integer.} \end{cases}$$

THEOREM 3. *Let u be an (m, p)-BL function on R^n.*

(i) *If $mp>n$, then there exists a polynomial P of degree at most $m-1$ such that* $\lim_{|x|\to\infty} A(|x|)\{u(x)-P(x)\}=0$.

(ii) *If $mp=n$, then there exists a polynomial P of degree at most $m-1$*

and a set E such that E is (m, p)-thin at infinity and

$$\lim_{|x|\to\infty, x\in R^n-E} (\log|x|)^{-1/p'}\{u(x)-P(x)\} = 0.$$

Applying the contractive property of Bessel capacities (cf. [9; Lemma 5]), we can prove the following radial limit theorem; we also refer to Theorem 4.5 and its Corollary 4.7 in [5].

COROLLARY. *If $mp=n$ and u is an (m, p)-BL function on R^n, then there exist a polynomial P of degree at most $m-1$ and a set $E\subset S$ such that $B_{m,p}(E)=0$ and*

$$\lim_{r\to\infty} (\log r)^{-1/p'}\{u(r\Theta)-P(r\Theta)\} = 0$$

for every $\Theta\in S-E$.

PROOF OF THEOREM 3. Let ℓ be a nonnegative integer such that $\ell\leqq m-n/p<\ell+1$. In view of Theorem 1, we can find a polynomial P of degree at most $m-1$ and a set E_1 such that $B_{m,p}(E_1)=0$ and

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy + P(x) \qquad \text{for} \quad x\in R^n - E_1.$$

We note here that E_1 is (m, p)-thin at infinity; in case $mp>n$, E_1 is empty and the functions defined by the above integrals are all continuous. We write $u=u_1+u_2+P$ outside E_1, where

$$u_1(x) = \sum_{|\lambda|=m} a_\lambda \int_{\{y;|x-y|\geqq|x|/2\}} K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy$$

and

$$u_2(x) = \sum_{|\lambda|=m} a_\lambda \int_{\{y;|x-y|<|x|/2\}} K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy.$$

We infer from Lemmas 4 and 5 that $\lim_{|x|\to\infty} A(|x|)u_1(x)=0$. Moreover, taking Lemmas 6 and 7 into consideration, we can show the existence of E_2 such that E_2 is (m, p)-thin at infinity and

$$\lim_{|x|\to\infty, x\in R^n-E_2} |x|^{(n-mp)/p}\, u_2(x) = 0.$$

Since $E=E_1\cup E_2$ is (m, p)-thin at infinity, our theorem is proved.

4. Perpendicular limits

By the integral representations of Beppo Levi functions in [7] and the proof of Proposition 1 in [8], we can prove the following result.

THEOREM 4. *If $mp<n$ and u is an (m, p)-BL function on R^n, then there exist a polynomial P of degree at most $m-1$ and a set $E\subset R^{n-1}\times\{0\}$ such that $B_{m,p}(E)=0$ and*

(4) $\lim_{t\to\infty}\{u(x', t)-P(x', t)\} = 0$ *for every* $(x', 0)\in R^{n-1}\times\{0\}-E$.

REMARK. Unlike the conclusion of Theorem 2, (4) can not be replaced by

$$\lim_{t\to\infty} A(t)\{u(x', t)-P(x', t)\} = 0$$

(see Example 6 in Section 5).

In view of Theorem 3, only the case $mp=n$ remains to be discussed for the existence of perpendicular limits of (m, p)-BL functions.

THEOREM 5. *If $mp=n$ and u is an (m, p)-BL function on R^n, then there exist a polynomial P of degree at most $m-1$ and a set E such that*

$$\lim_{x_n\to\infty, x\in R^n-E}(\log|x|)^{-1/p'}\{u(x)-P(x)\} = 0$$

and

$$\textstyle\sum_{j=1}^{\infty} B_{m,p}(E^{(j)}) < \infty,$$

where x_n denotes the n-th coordinate of x, $1/p+1/p'=1$ and

$$E^{(j)} = \{x\in E; j\leqq x_n<j+1\}.$$

PROOF. By Theorem 1 we can find a set E_1 and a polynomial P of degree at most $m-1$ such that $B_{m,p}(E_1)=0$ and

$$u(x) = \textstyle\sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,0}(x, y)D^\lambda u(y)dy + P(x)$$

for any $x\in R^n-E_1$. Write $u=u_1+u_2+u_3+P$ on R^n-E_1, where

$$u_j(x) = \textstyle\sum_{|\lambda|=m} a_\lambda \int_{D(j)} K_{m,\lambda,0}(x, y)D^\lambda u(y)dy, \qquad j = 1, 2, 3,$$

with $D(1)=\{y\in R^n; |x-y|\geqq|x|/2\}$, $D(2)=\{y\in R^n; 1\leqq|x-y|<|x|/2\}$ and $D(3)=B(x, 1)$. It follows from Lemma 5 and Hölder's inequality that

$$\lim_{|x|\to\infty}(\log|x|)^{-1/p'}\{u_1(x)+u_2(x)\} = 0.$$

By the definition of functions $K_{m,\lambda,0}$ we can find a nonnegative function f in $L^p(R^n)$ such that

$$|u_3(x)| \leqq \int_{B(x,1)} g_m(x-y)f(y)dy,$$

where g_m denotes the Bessel kernel of order m (see [6], [11]). Take a sequence $\{t_j\}$ of positive numbers such that $\lim_{j\to\infty} t_j=\infty$ and

$$\sum_{j=1}^{\infty} t_j \int_{\{y=(y',y_n);\, j-1<y_n<j+2\}} f(y)^p dy<\infty.$$

Define

$$E^{(j)} = \left\{x = (x', x_n);\, j \leqq x_n < j+1, \int_{B(x,1)} g_m(x-y)f(y)dy > t_j^{-1/p}\right\}$$

for each positive integer j. By the definition of Bessel capacities we have

$$B_{m,p}(E^{(j)}) \leqq t_j \int_{\{y;\, j-1<y_n<j+2\}} f(y)^p dy,$$

from which it follows that $\sum_{j=1}^{\infty} B_{m,p}(E^{(j)})<\infty$. Clearly,

$$\lim_{x_n\to\infty, x\in R^n-E_2} \int_{B(x,1)} g_m(x-y)f(y)dy = 0,$$

where $E_2=\bigcup_{j=1}^{\infty} E^{(j)}$. Therefore $E=E_1\cup E_2$ has the required properties in our theorem, and the proof is complete.

For $E\subset R^n$, denote by E^* the projection of E to the hyperplane $R^{n-1}\times\{0\}$. If $\sum_{j=1}^{\infty} B_{m,p}(E^{(j)})<\infty$ with the above notation, then $B_{m,p}(\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} E^{(j)*})=0$ on account of the contractive property of Bessel capacities (cf. [9; Lemma 5]). Thus Theorem 5 has the following corollary.

COROLLARY. *If $mp=n$ and u is an (m, p)-BL function on R^n, then there exist a polynomial P of degree at most $m-1$ and a set $E\subset R^{n-1}\times\{0\}$ such that $B_{m,p}(E)=0$ and*

$$\lim_{t\to\infty} (\log t)^{-1/p'}\{u(x', t) - P(x', t)\} = 0$$

for every $(x', 0)\in R^{n-1}\times\{0\}-E$.

5. Best possibility with respect to the order at infinity

We shall give below examples which show the best possibility of our theorems with respect to the order at infinity. The functions appearing in the following examples will be of potential type; so we prepare

PROPOSITION. *Let $m>0$, $p>1$ and $\alpha\leqq m-n/p<\alpha+1$. Let $K(x, y)$ be a Borel function on $R^n\times R^n$ for which there is $M>0$ such that*

$$|K(x, y)| \leqq M|x|^{\alpha+1}|y|^{m-n-\alpha-1} \quad \textit{when} \quad |y| \geqq 2|x| > 1,$$

$$|K(x, y)| \leqq M|x|^{\alpha}|y|^{m-n-\alpha} \qquad \textit{when} \quad 1 \leqq |y| < 2|x| \quad \textit{and} \quad |x-y| > |x|/2$$

and

$$|K(x, y)| \leqq M|x-y|^{m-n} \qquad \textit{when} \quad |x-y| \leqq |x|/2.$$

For $f \in L^p(R^n)$, *we define* $Kf(x) = \int K(x, y)f(y)dy$. *Then there exists a set* $E \subset R^n$ *which is* (m, p)*-thin at infinity such that*

$$\lim_{|x| \to \infty, x \in R^n - E} A(|x|)Kf(x) = 0.$$

If $mp \leqq n$, *then there exists a set* $F \subset R^n$ *such that* $\sum_{j=1}^{\infty} B_{m,p}(F^{(j)}) < \infty$ *and*

$$\lim_{x_n \to \infty, x \in R^n - F} B(|x|)Kf(x) = 0,$$

where $F^{(j)} = \{x \in F; j \leqq x_n < j+1\}$, $B(r) = 1$ *in case* $mp < n$ *and* $B(r) = A(r)$ *in case* $mp = n$.

The proof of the proposition is similar to those of Theorems 3 and 5.

If $|\lambda| = m$ and $\ell \leqq m - n/p < \ell + 1$, then $K_{m,\lambda,\ell}$ satisfies all the conditions on K with $\alpha = \ell + \varepsilon$ for some $\varepsilon \geqq 0$.

We shall give other examples of K. Let m be a positive integer and set

$$R_m(x) = \begin{cases} |x|^{m-n} \log |x| & \text{if} \quad m - n \text{ is a nonnegative even integer,} \\ |x|^{m-n} & \text{otherwise.} \end{cases}$$

Letting ℓ be a nonnegative integer, we define

$$R_{m,\ell}(x, y) = \begin{cases} R_m(x-y) - \sum_{|\lambda| \leqq \ell} (\lambda!)^{-1} x^{\lambda} (D^{\lambda} R_m)(-y) & \text{if} \quad y \in R^n - B(0, 1), \\ R_m(x-y) & \text{if} \quad y \in B(0, 1). \end{cases}$$

If $mp \geqq n$ and $\ell \leqq m - n/p < \ell + 1$, then $R_{m,\ell}$ satisfies all the conditions on K in the proposition and, in the same manner as in the proof of Lemma 3, $R_{m,\ell} f$ is shown to be an (m, p)-*BL* function on R^n whenever $f \in L^p(R^n)$, on account of Lemmas 3.3 and 4.3 in [7].

EXAMPLE 1. Let $mp = n$ and h be a nondecreasing function on R^1 such that $\lim_{t \to \infty} h(t) = \infty$. Then we can find an (m, p)-*BL* function u on R^n such that $\lim_{|x| \to \infty, x \in R^n - E} (\log |x|)^{-1/p'} u(x) = 0$ for some E which is (m, p)-thin at infinity but $\lim_{|x| \to \infty, x \in A} h(|x|)(\log |x|)^{-1/p'} u(x) = \infty$ for some A which is not (m, p)-thin at infinity.

This example shows that Theorem 3 (ii) is best possible as to the order at infinity.

For the construction of such u, take a sequence $\{k_j\}$ of positive integers such that $2k_j < k_{j+1}$, $h(2^{2k_j}) > 0$ and $\sum_{j=1}^{\infty} h(2^{2k_j})^{-1} < \infty$. Now we define

$$E_j = \{y = (y', y_n) \in R^n;\ |y| < \sqrt{2}\, y_n,\ 2^{k_j} < |y| < 2^{2k_j}\},$$

$$f(y) = \begin{cases} h(2^{2k_j})^{-1/p} |y|^{-m} (\log |y|)^{-1/p} & \text{when } -y \in E_j, \\ 0 & \text{when } -y \in R^n - \cup_{j=1}^{\infty} E_j \end{cases}$$

and

$$u(x) = -R_{m,0} f(x) = -\int \{|x-y|^{m-n} - |y|^{m-n}\} f(y) dy.$$

Then we see easily that $f \in L^p(R^n)$, and hence, by the consideration given after the Proposition, u is an (m, p)-BL function on R^n. If $2^{2k_j} \leqq |x| < 2^{2k_j+1}$ and $|x| < \sqrt{2}\, x_n$, then $|x-y| > |y|$ whenever $-y \in \cup_{j=1}^{\infty} E_j$. Consequently, if j is large enough, then

$$\begin{aligned} u(x) &\geqq \int_{\{y;\, -y \in E_j\}} \{|y|^{m-n} - |x-y|^{m-n}\} f(y) dy \\ &\geqq h(2^{2k_j})^{-1/p} \left\{ \int_{E_j} |y|^{-n} (\log |y|)^{-1/p} dy \right. \\ &\qquad \left. - c_1 |x|^{m-n} \int_{E_j} |y|^{-m} (\log |y|)^{-1/p} dy \right\} \\ &\geqq c_2 h(2^{2k_j})^{-1/p} k_j^{1/p'}, \end{aligned}$$

where c_1 and c_2 are positive constants independent of j. Thus, setting $A = \cup_{j=1}^{\infty} \{x;\ |x| < \sqrt{2}\, x_n,\ 2^{2k_j} < |x| < 2^{2k_j+1}\}$, we obtain

$$\lim_{|x| \to \infty, x \in A} h(|x|) (\log |x|)^{-1/p'} u(x) = \infty.$$

Since A is not (m, p)-thin at infinity, u satisfies the last condition in Example 1. Thus, in view of the Proposition, u is a required function.

EXAMPLE 2. Let $0 < m - n/p < 1$ and h be as in Example 1. Then, in the same manner as above, we can construct an (m, p)-BL function u on R^n such that $\lim_{|x| \to \infty} |x|^{(n-mp)/p} u(x) = 0$ but

$$\limsup_{t \to \infty} h(t) t^{(n-mp)/p} u(0, t) = \infty.$$

This example together with the following three examples will show the best possibility of Theorem 3 (i) as to the order at infinity.

EXAMPLE 3. Let h be a nondecreasing function on R^1 such that $\lim_{r \to \infty} h(r) = \infty$. Suppose $m - n < \ell \leqq m - n/p < \ell + 1$ for some positive even integer ℓ. Then we can find a function $u \in BL_m(L^p(R^n))$ satisfying $\lim_{|x| \to \infty} A(|x|) u(x) = 0$ and

$$\limsup_{r\to\infty} h(r)A(r)(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} u(x)dS(x) = \infty,$$

where σ_n denotes the area of the boundary $\partial B(0, 1)$ of $B(0, 1)$.

For this purpose, find a nonnegative function $f \in L^p(R^n)$ for which there exists a sequence $\{r_j\}$ of positive numbers tending to ∞ such that

$$\lim_{j\to\infty} h(r_j)A(r_j)r_j^{\ell}\int_{B(0,r_j)} |y|^{m-n-\ell}f(y)dy = \infty$$

and

$$\lim_{j\to\infty}\left\{\int_{B(0,r_j)}|y|^{m-n-\ell}f(y)\,dy\right\}\left\{r_j\int |y|^{m-n-\ell+1}(r_j+|y|)^{-2}f(y)\,dy\right\}^{-1} = \infty;$$

see (A) and (C) in Appendix. Applying (3.2) in [7], we establish

$$\begin{aligned}\sum_{|\lambda|=2j}(\lambda!)^{-1}(D^{\lambda}R_m)(-y)(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} x^{\lambda}dS(x)\\ = c_j r^{2j}(\Delta^j R_m)(-y) = r^{2j}|y|^{m-n-2j}\{d'_j + d''_j \log|y|\}\end{aligned}$$

for $j=1,\dots, \ell^* = \ell/2$, where c_j, d'_j and d''_j are constants such that $d'_{\ell^*} \neq 0$ and $d''_{\ell^*}=0$. Consider $v(x)=R_{m,\ell}f(x)$. Then $v \in BL_m(L^p(R^n))$ as remarked after the Proposition. Further we obtain

$$\begin{aligned}(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} v(x)dS(x) = \int\Big\{(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} R_m(x-y)\,dS(x)\\ - \sum_{j=0}^{\ell^*} r^{2j}|y|^{m-n-2j}[d'_j + d''_j\log|y|]\Big\}f(y)dy.\end{aligned}$$

If $m-n$ is not a nonnegative even integer, then

$$(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} R_m(x-y)dS(x) \leqq M_1 r^{m-n}$$

for $y \in B(0, 2r)$ and $d''_j=0$ for all $j=0,\dots, \ell^*$, where M_1 is a positive constant. If $m-n$ is a nonnegative even integer, then $|x|^{m-n}$ is a polynomial of degree less than ℓ, so that,

$$\begin{aligned}&(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} v(x)dS(x)\\ &= \int\Big\{(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} |x-y|^{m-n}\log(|x-y|/r)dS(x)\\ &\quad - \sum_{j=0}^{\ell^*} r^{2j}|y|^{m-n-2j}[d'_j + d''_j\log(|y|/r)]\Big\} f(y)dy.\end{aligned}$$

Since $\int_{\partial B(0,1)} |x-y|^{m-n}|\log|x-y||dS(x)$ is continuous on R^n, there exists $M_2>0$ such that

$$(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} |x-y|^{m-n}\log(|x-y|/r)dS(x) \geqq -M_2 r^{m-n}$$

whenever $y\in B(0, 2r)$. Hence, noting that $|R_{m,\ell}(x, y)|\leqq M_3|x|^{\ell+1}|y|^{m-n-\ell-1}$ whenever $|y|\geqq 2|x|>1$, we establish

$$\begin{aligned}
&(-d'_{\ell*})^{-1}h(r)A(r)(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} v(x)dS(x)\\
&\geqq h(r)A(r)r^{\ell}\Big\{\int_{B(0,2r)}|y|^{m-n-\ell}f(y)\,dy - M_4 r^{m-n-\ell}\int_{B(0,2r)} f(y)dy\\
&\quad - M_5\textstyle\sum_{j=0}^{\ell*-1} r^{2j-\ell}\displaystyle\int_{B(0,2r)}|y|^{m-n-2j}(|y|/r)^{-1}f(y)dy\\
&\quad - M_6 r\int_{R^n-B(0,2r)}|y|^{m-n-\ell-1}f(y)dy\Big\}\\
&\geqq h(r)A(r)r^{\ell}\Big\{\int_{B(0,r)}|y|^{m-n-\ell}f(y)\,dy - M_7 r^{-1}\int_{B(0,r)}|y|^{m-n-\ell+1}f(y)dy\\
&\quad - M_8 r\int_{R^n-B(0,r)}|y|^{m-n-\ell-1}f(y)\,dy\Big\},
\end{aligned}$$

where $M_3\sim M_8$ are positive constants. By the construction of f, the right hand side is not bounded above. Thus $u=(-d'_{\ell*})^{-1}v$ satisfies the last condition in Example 3, and thus it is a required function in view of the Proposition.

If ℓ is odd, then we need consider the weighted mean value of u over the surface $\partial B(0, r)$.

EXAMPLE 4. Let h be as above. Suppose $\ell\leqq m-n/p<\ell+1$ for some positive odd integer ℓ. Then we can find a function $u\in BL_m(L^p(R^n))$ satisfying $\lim_{|x|\to\infty} A(|x|)u(x)=0$ and

$$\limsup_{r\to\infty} h(r)A(r)(\tau_n r^n)^{-1}\int_{\partial B(0,r)} u(x)x_n dS(x) = \infty,$$

where $\tau_n=\int_{\partial B(0,1)}|x_n|dS(x)$ and $x=(x_1,\dots, x_n)$.

For this, we first note by (3.2) in [7] that for $j=1,\dots,\ \ell^*=(\ell-1)/2$,

$$\begin{aligned}
&\textstyle\sum_{|\lambda|=2j+1}(\lambda!)^{-1}(D^{\lambda}R_m)(-y)(\tau_n r^n)^{-1}\displaystyle\int_{\partial B(0,r)} x^{\lambda}x_n dS(x)\\
&\quad = c_j r^{2j+1}((\partial/\partial x_n)\Delta^j R_m)(-y)
\end{aligned}$$

$$= r^{2j+1} y_n |y|^{m-n-2j-2}\{d'_j + d''_j \log |y|\}$$

with constants c_j, d'_j and d''_j such that $d'_{\ell *} \neq 0$ and $d''_{\ell *}=0$. Find a nonnegative function $f \in L^p(R^n)$ for which there exists a sequence $\{r_j\}$ of positive numbers tending to ∞ such that

$$\lim_{j\to\infty} h(r_j)A(r_j)r_j^{\ell} \int_{B(0,r_j)} y_n |y|^{m-n-\ell-1} f(y)dy = \infty$$

and

$$\lim_{j\to\infty}\left\{\int_{B(0,r_j)} y_n|y|^{m-n-\ell-1}f(y)dy\right\}\left\{r_j\int |y|^{m-n-\ell+1}(r_j+|y|)^{-2}f(y)dy\right\}^{-1} = \infty;$$

see Appendix (D). Setting $v(x)=R_{m,\ell}f(x)$, we find, as in the above arguments for Example 3,

$$(-d'_{\ell *})^{-1}(\tau_n r^n)^{-1}\int_{\partial B(0,r)} v(x)x_n\,dS(x) \geqq r^{\ell}\int_{B(0,r)} y_n|y|^{m-n-\ell-1}f(y)dy$$

$$- M_1 r^{\ell-1}\int_{B(0,r)} |y|^{m-n-\ell+1}f(y)dy - M_2 r^{\ell+1}\int_{R^n-B(0,r)} |y|^{m-n-\ell-1}f(y)dy$$

with positive constants M_1 and M_2. Thus we see that $u=(-d'_{\ell *})^{-1}v$ has the properties required in Example 4.

EXAMPLE 5. Suppose $m-n$ is a positive even integer and $m-n/p<m-n+1$, that is, $n/p'<1$. If h is as above, then we can find $u \in BL_m(L^p(R^n))$ such that $\lim_{|x|\to\infty} A(|x|)u(x)=0$ and

$$\limsup_{r\to\infty} h(r)A(r)(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} u(x)dS(x) = \infty.$$

For the construction of such u, find a nonnegative function $f \in L^p(R^n)$ for which there is a sequence $\{r_j\}$ of positive numbers such that $\lim_{j\to\infty} r_j=\infty$,

$$\lim_{j\to\infty} h(r_j)r_j^{-n/p'}\int_{B(0,r_j)} f(y)\log(r_j/|y|)dy = \infty$$

and

$$\lim_{j\to\infty}\left\{\int_{B(0,r_j)} f(y)\log(r_j/|y|)dy\right\}\left\{r_j\int(r_j+|y|)^{-1}f(y)dy\right\}^{-1} = \infty;$$

see Appendix (B). Letting $\ell=m-n$, we consider $v(x)=R_{m,\ell}f(x)$. Since ℓ is even, we obtain

$$(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} v(x)dS(x)$$

$$=\int\left\{(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} |x-y|^{\ell}\log(|x-y|/r)\,dS(x)\right.$$

$$- \sum_{j=0}^{\ell^*} r^{2j}|y|^{\ell-2j}[d'_j + d''_j \log(|y|/r)]\Big\} f(y)\,dy,$$

where d'_j and d''_j are constants such that $d''_{\ell^*} \neq 0$ and $\ell^* = \ell/2$. Thus, in the same manner as above, we derive

$$(d''_{\ell^*})^{-1}(\sigma_n r^{n-1})^{-1}\int_{\partial B(0,r)} v(x)\,dS(x) \geqq r^{\ell}\int_{B(0,r)} f(y)\log(r/|y|)\,dy$$

$$- M_1 r^{\ell}\int_{B(0,r)} f(y)\,dy - M_2 r^{\ell+1}\int_{R^n - B(0,r)} |y|^{-1} f(y)\,dy$$

for some positive constants M_1 and M_2. As before, $u = (d''_{\ell^*})^{-1}v$ is seen to satisfy the required assertions in Example 5.

We next consider the best possibility as to the order at infinity of our results concerning perpendicular limits.

EXAMPLE 6. Let $mp<n$ and h be a nondecreasing function on R^1 such that $\lim_{t\to\infty} h(t) = \infty$. Then there exists a nonnegative function $f \in L^p(R^n)$ such that $\limsup_{t\to\infty} h(t)\int |(x', t) - y|^{m-n} f(y)dy = \infty$ for any $x' \in R^{n-1}$.

In view of Lemma 3.3 in [7], the potential $R_m f(x) \equiv \int |x-y|^{m-n} f(y)dy$ is an (m, p)-BL function on R^n. Further, in the same way as in the proof of Proposition 1 in [8], we can find a set $E \subset R^{n-1} \times \{0\}$ such that $B_{m,p}(E) = 0$ and $\lim_{t\to\infty} R_m f(x', t) = 0$ for any $x' \in R^{n-1}$ with $(x', 0) \notin E$. Hence Theorem 4 is best possible as to the order at infinity.

For the construction of such f, take $\phi \in C_0^\infty(R^n)$ such that $\phi \geqq 0$ on R^n, $\phi = 1$ on $B(0, 1/2)$ and $\phi = 0$ outside $B(0, 1)$, and find a sequence $\{r_j\}$ of positive numbers such that $r_j + 1 < r_{j+1} - 1$ and $\sum_{j=1}^{\infty} h(r_j)^{-1} < \infty$. Now define $f(y) = \sum_{j=1}^{\infty} h(r_j)^{-1/p}\phi(y - r_j e)$ with $e = (0, \ldots, 0, 1) \in R^n$. Then we see easily that $f \in L^p(R^n)$. Further, setting $x^{(j)} = (x', r_j) \in R^n$, we have

$$h(r_j) R_m f(x^{(j)}) \geqq h(r_j)^{1/p'} R_m \phi((x', 0)) \longrightarrow \infty \qquad \text{as} \quad j \longrightarrow \infty.$$

EXAMPLE 7. Let $0 \leqq m - n/p < 1$ and h be as above. Then the functions u obtained in Examples 1 and 2 may be taken to satisfy

$$\limsup_{t\to\infty} h(t)A(t)u(x', t) = \infty \qquad \text{for any} \quad x' \in R^{n-1}.$$

This shows that Theorem 5 is best possible as to the order at infinity.

Appendix

Let h be a positive nondecreasing function on R^1 such that $\lim_{r\to\infty} h(r) = \infty$.

(A) Let a be a positive number such that $n/p'-1<a<n/p'$. Then we shall find a nonnegative function $f \in L^p(R^n)$ satisfying

$$(A_1) \qquad \lim_{j\to\infty} h(r_j) r_j^{a-n/p'} \int_{B(0,r_j)} |y|^{-a} f(y)dy = \infty;$$

$$(A_2) \qquad \lim_{j\to\infty} \left\{\int_{B(0,r_j)} |y|^{-a} f(y)dy\right\}\left\{r_j \int |y|^{-a+1}(r_j+|y|)^{-2} f(y)dy\right\}^{-1} = \infty$$

for some sequence $\{r_j\}$ of positive numbers which tends to ∞.

For this purpose, take sequences $\{s_j\}$ and $\{\varepsilon_j\}$ of positive numbers such that $\lim_{j\to\infty} \varepsilon_j = 0$, $\varepsilon_{j+1} < \varepsilon_j < 1/2$, $s_j < \varepsilon_j s_{j+1}$ and $\sum_{j=1}^{\infty} h(s_j)^{-1} < \infty$. We now define

$$f(y) = \begin{cases} h(s_j)^{-1/p}|y|^{-n/p} & \text{if} \quad s_j < |y| < 2s_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exist positive constants c_1, c_2, c_3 and c_4 such that for $2s_k < r < s_{k+1}$ we have

$$(A_3) \qquad \int f(y)^p dy = c_1 \sum_{j=1}^{\infty} h(s_j)^{-1} < \infty;$$

$$(A_4) \qquad \int_{B(0,r)} |y|^{-a} f(y)dy = c_2 \sum_{j=1}^{k} h(s_j)^{-1/p} s_j^{-a+n/p'};$$

$$(A_5) \qquad r^{-1} \int_{B(0,r)} |y|^{-a+1} f(y)dy = c_3 \sum_{j=1}^{k} h(s_j)^{-1/p} s_j^{n/p'-a}(s_j/r);$$

$$(A_6) \qquad \int_{R^n-B(0,r)} |y|^{-a-1} f(y)dy = c_4 \sum_{j=k+1}^{\infty} h(s_j)^{-1/p} s_j^{-a+n/p'-1} < \infty.$$

From (A_3) it follows that $f \in L^p(R^n)$. By (A_4) we have

$$(A_7) \qquad h(r) r^{a-n/p'} \int_{B(0,r)} |y|^{-a} f(y)dy \geqq c_2 h(s_k)^{1/p'} (r/s_k)^{a-n/p'}.$$

Since $s_j > \varepsilon_k^{k-j} s_k$ for $j > k$, we derive from (A_6) that

$$(A_8) \qquad r \int_{R^n-B(0,r)} |y|^{-a-1} f(y)dy$$
$$\leqq c_4 h(s_k)^{-1/p} s_k^{-a+n/p'} (r/s_k) \varepsilon_k^{a-n/p'+1} (1-\varepsilon_k^{a-n/p'+1})^{-1}.$$

We can choose a sequence $\{M_j\}$ of positive numbers such that

$$2 < M_j < s_{j+1}/s_j, \quad \lim_{j\to\infty} M_j = \infty$$
$$\lim_{j\to\infty} h(s_j)^{1/p'} M_j^{a-n/p'} = \infty \quad \text{and} \quad \lim_{j\to\infty} M_j \varepsilon_j^{a-n/p'+1} = 0.$$

Then, by setting $r_k = M_k s_k$, (A_7) implies (A_1), and (A_5) together with (A_8) implies (A_2).

(B) Suppose $n/p' < 1$. Then, in the same manner, we can construct a nonnegative function $f \in L^p(R^n)$ for which there exists a sequence $\{r_j\}$ of positive numbers tending to ∞ such that

$$(\mathrm{B}_1) \qquad \lim_{j\to\infty} h(r_j) r_j^{-n/p'} \int_{B(0,r_j)} f(y) \log(r_j/|y|)dy = \infty;$$

$$(\mathrm{B}_2) \qquad \lim_{j\to\infty} \left\{\int_{B(0,r_j)} f(y)\log(r_j/|y|)dy\right\}\left\{r_j \int (r_j+|y|)^{-1} f(y)dy\right\}^{-1} = \infty.$$

(C) As in the discussion after Example 1 we can find a nonnegative function $f \in L^p(R^n)$ and a sequence $\{r_j\}$ of positive numbers tending to ∞ such that

$$(\mathrm{C}_1) \qquad \lim_{j\to\infty} h(r_j)(\log r_j)^{-1/p'} \int_{B(0,r_j)} |y|^{-n/p'} f(y)dy = \infty;$$

$$(\mathrm{C}_2) \qquad \lim_{j\to\infty} \left\{\int_{B(0,r_j)} |y|^{-n/p'} f(y)dy\right\} \times \left\{r_j \int |y|^{-n/p'+1}(r_j+|y|)^{-2} f(y)dy\right\}^{-1} = \infty.$$

(D) Let a be a positive number such that $n/p' - 1 < a \leqq n/p'$. Then we can construct a nonnegative function $f \in L^p(R^n)$ which has a sequence $\{r_j\}$ of positive numbers tending to ∞ such that

$$(\mathrm{D}_1) \qquad \lim_{j\to\infty} h(r_j)\tilde{A}(r_j) \int_{B(0,r_j)} y_n |y|^{-a-1} f(y)dy = \infty;$$

$$(\mathrm{D}_2) \qquad \lim_{j\to\infty} \left\{\int_{B(0,r_j)} y_n |y|^{-a-1} f(y)dy\right\} \times \left\{r_j \int |y|^{-a+1}(r_j+|y|)^{-2} f(y)dy\right\}^{-1} = \infty,$$

where $\tilde{A}(r) = r^{a-n/p'}$ if $a < n/p'$ and $\tilde{A}(r) = (\log r)^{-1/p'}$ if $a = n/p'$.

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