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On the (non)compactness of the radial sobolev spaces

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1. Introduction and basic compactness results

We say that a normed space V is embedded in a normed space H, and write $V \hookrightarrow H$, if V is a linear subspace of H and the injection mapping $x \to x$ from V to H is continuous. If, in addition, the injection mapping is a compact operator from V to H, we say that V is compactly embedded in H and write $V \Subset H$. From the well known Sobolev lemma, we know that

$$H^1(\boldsymbol{R}^n) \hookrightarrow L^q(\boldsymbol{R}^n) \tag{1}$$

holds for $2 \leq q \leq 2n/(n-2)$ if $n > 2$. Moreover, if Ω is a bounded piece-wise smooth domain in $\boldsymbol{R}^n$, then $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ if $2 \leq q < 2n/(n-2)$. However, this fails to hold if Ω is not bounded. To see, for example, that we cannot replace the embedding in (1) by a compact embedding, let $\phi_m(x) = \phi(x + me)$ $(m = 1, 2, \dots)$, where ϕ is a non-zero element of $C_0^\infty(\boldsymbol{R}^n)$ and e is a unit vector in $\boldsymbol{R}^n$. The sequence $\{\phi_m\}$ is clearly bounded in $H^1(\boldsymbol{R}^n)$ but does not have a subsequence converging in L^q.

When solving differential equations, we sometimes construct approximate solutions in a suitable framework of function spaces and discuss their convergence. For non-linear equations specially, the best way to prove convergence in some space is to prove that the sequence of approximate solutions is bounded in some space which is compactly embedded in the original space. There is a lot of information available concerning compact and non-compact embeddings in the theory of Sobolev spaces (cf. [1], [2], [3], [4]). In this paper we restrict our attention to the spaces

$$H_r^m(\boldsymbol{R}^n) := \{u \in H^m(\boldsymbol{R}^n) : u = u(|x|)\};$$

i.e., the class of all radial functions $u(|x|)$ whose derivatives of order up to m belong to $L^2(\boldsymbol{R}^n)$ $(m = 0, 1, 2, \dots)$; $n \geq 3$. We denote by $\|\cdot\|$ the L^2-norm in $\boldsymbol{R}^n$ and we set for $m \geq 0$

$$\|u\|_m^2 = \|u\|^2 + \||\xi|^m \hat{u}\|^2,$$

where $\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix \cdot \xi} u(x) dx$ is the Fourier transform of u. The norm $\|\cdot\|_m$ is equivalent to the standard norm of $H^m(\boldsymbol{R}^n)$; we employ this norm in $H_r^m(\boldsymbol{R}^n)$. We have:

THEOREM 1.1. *If* $2<q<2n/(n-2)$, *then* $H_r^1(\boldsymbol{R}^n)$ *is compactly embedded in* $L^q(\boldsymbol{R}^n)$ $(n\geq 3)$.

This theorem, due to W. A. Strauss, is a consequence of the following two lemmas (cf. [5]).

LEMMA 1. (*Radial lemma*) *Let* $n\geq 2$ *and let* $u\in H_r^1(\boldsymbol{R}^n)$; *then there exists* $U(x)$ *continuous for* $x\neq 0$ *such that* $u(x)=U(x)$ *a.e.* x *and for any* $\rho>0$

$$|U(x)|\leq C|x|^{(1-n)/2}\|u\|_{H^1}\qquad for\quad |x|>\rho,$$

where c depends only on n and ρ.

LEMMA 2. (*Compactness lemma*) *Let* $\{P_l\}$, $\{Q_l\}$ *be two sequences of continuous functions from* $\boldsymbol{R}$ *to* $\boldsymbol{R}$. *For* $c>0$, *let* $\gamma(c)=\sup\{|t|: t=P_l(s)$ for some l, s such that $|Q_l(s)|\leq c|P_l(s)|\}$.

Assume

i) $\gamma(c)>0$ *for all* $c>0$;

ii) $\{u_l\}$ *is a sequence of measurable functions from* $\boldsymbol{R}^n$ *to* $\boldsymbol{R}$ *such that* $d:=\sup_l\int|Q_l(u_l(x))|dx<\infty$;

iii) $P_l(u_l(x))\to v(x)$ *a.e. on* $\boldsymbol{R}^n$. *Then* $\lim_{l\to\infty}\int_B|P_l(u_l)-v|dx=0$ *for all bounded sets B. If, in addition*

iv) $P_l(s)=O(Q_l(s))$ *as* $s\to 0$, *univormly in* l;

v) $u_l(x)\to 0$ *as* $|x|\to\infty$, *uniformly in* l; *then* $\lim_{l\to\infty}\int|P_l(u_l)-v|dx=0$.

As a limiting case of Theorem 1.1, we are interested in whether $H_r^1(\boldsymbol{R}^n)\Subset L^2(\boldsymbol{R}^n)$; $H_r^1\Subset L^q(\boldsymbol{R}^n)$ $(q=2n/(n-2))$. As we show in the next section, the answer is negative.

We obtain the following corollaries of Theorem 1.1.

COROLLARY 1 OF THEOREM 1.1.

1) $L^{2-\varepsilon}(\boldsymbol{R}^n)\cap H_r^1(\boldsymbol{R}^n)\Subset L^2(\boldsymbol{R}^n)$ *if* $0<\varepsilon\leq 1$,

2) $L^{q^*+\varepsilon}(\boldsymbol{R}^n)\cap H_r^1(\boldsymbol{R}^n)\Subset L^{q^*}(\boldsymbol{R}^n)$ *if* $\varepsilon>0$, $q^*=2n/(n-2)$

PROOF. Let $1\leq p<p'<q\leq\infty$. By Hölder's inequality,

$$\|u\|_{L^{p'}}\leq\|u\|_{L^p}^{\lambda}\|u\|_{L^p}^{\mu}$$

where $\lambda=p(p'-q)/p'(p-q)$, $\mu=q(p-p')/p'(p-q)$. It follows that a sequence which converges in one of L^p, L^q and is bounded in the other one, converges in $L^{p'}$. To get the first result, select $p=2-\varepsilon$, $p'=2$, $q\in(2, q^*)$; for the second result select $p\in(2, q^*)$, $p'=q^*$, $q=q^*+\varepsilon$ and use Theorem 1.1 to conclude that a sequence bounded in $L^{2-\varepsilon} \cap H_r^1$ or in $L^{q^*+\varepsilon} \cap H_r^1$ has a convergent subsequence in L^2 or in L^{q^*}, respectively. Q. E. D.

Corollary 2 of Theorem 1.1. *We have*

1) $H_r^m(\boldsymbol{R}^n) \Subset L^p(\boldsymbol{R}^n)$ *if* $2 \leq m < n/2,\ 2 < p < 2n/(n-2m)$;
2) $H_r^m(\boldsymbol{R}^n) \Subset L^p(\boldsymbol{R}^n)$ *if* $m \geq n/2,\ 2 < p < \infty$.

Proof. 1) H_r^m is embedded in L^p if $m<n/2, 2\leq p\leq 2n/(n-2m)$ by Sobolev's lemma. If $\{u_k\}$ is a bounded sequence in H_r^m, $m<n/2$, then it is also bounded in L^{p^*}, $p^*=2n/(n-2m)$. Since $H_r^m \hookrightarrow H_r^1 \Subset L^q$ if $2<q<2n/(n-2)$, $\{u_k\}$ has a subsequence converging in L^p. If $2<p<p^*$, we can choose q so that $2<q\leq p$ and conclude that the subsequence (which is bounded in L^{p^*} and converges in L^q) converges in L^p.

The proof of 2) is similar, since H_r^m is embedded in L^p, $2\leq p<\infty$, if $m\geq n/2$. Q. E. D.

2. Non-compactness results

Theorem 2.1. *$H_r^m(\boldsymbol{R}^n)$ is not compactly embedded in $L^2(\boldsymbol{R}^n)$ $(m=1, 2, \ldots)$.*

Proof. Let $u(x)=\beta^{n/4}e^{-\beta|x|^2}$, where $\beta>0$. Then $|\xi|^\alpha \hat{u}(\xi)=(4\beta)^{-n/4}\cdot |\xi|^\alpha e^{-|\xi|^2/4\beta}$; *hence*

$$\| |\xi|^\alpha \hat{u} \|^2 = (4\beta)^{-n/2}\omega_n \int_0^\infty r^{n-1+2\alpha} e^{-r^2/2\beta} dr$$
$$= 2^{\alpha+1-n/2}\beta^\alpha \omega_n \Gamma\left(\alpha + \frac{u}{2}\right),$$

where ω_n is the area of the unit sphere in $\boldsymbol{R}^n$ and Γ is the gamma function. Thus (with $\alpha=0$, m),

$$\|u\|^2 = 2^{1-n/2}\omega_n\Gamma(n/2) \text{ is } > 0, \quad \text{independent of} \quad \beta;$$
$$\|u\|_m^2 = \|u\|^2 + \| |\xi|^m \hat{u} \|^2 \leq 2^{m+1-n/2}\omega_n \Gamma(m+n/2)(1+\beta^m).$$

Select now a sequence $\{\beta_k\}$ in $(0, \infty)$ with $\lim_{k\to\infty} \beta_k=0$ and let $u_k=u$ with $\beta=\beta_k$. The estimates above show that $\{u_k\}$ is bounded in $H_r^m(\boldsymbol{R}^n)$ and $\|u_k\|=c_n>0$ for all k. On the other hand, $\lim_{k\to\infty} u_k(x)=0$ for all $x\in \boldsymbol{R}^n$, thus $\{u_k\}$ cannot have a subsequence converging in $L^2(\boldsymbol{R}^n)$. Q. E. D.

Remark 1. In case $m=1$, other examples of sequences which are bounded in H_r^1, bounded away from 0 in L^2 and converging pointwise to 0 are given by

Example 1: $f_k(x)=k^{(1-n)/2}\phi(|x|-k)$, $k=1, 2, \ldots$. where $\phi\in C_0^\infty(\boldsymbol{R})$, $\phi\not\equiv 0$, $\operatorname{supp}\phi\subseteq [0, 1]$, EXAMPLE 2: $u_k(x)=e^{-|x|^2/2k}k^{-(n/4)}$, $k=1, 2,\dots$.

COROLLARY OF THEOREM 2.1. *Let m_1, m_2 be integers, $0\leq m_2<m_1$. Then $H_r^{m_1}(\boldsymbol{R}^n)$ is not compactly embedded in $H_r^{m_2}(\boldsymbol{R}^n)$.*

THEOREM 2.2. *Let $1\leq m<n/2$. Then $H_r^m(\boldsymbol{R}^n)$ is not compactly embedded in $L^p(\boldsymbol{R}^n)$; $p=2n/(n-2m)$.*

PROOF. Let $\phi\in C_0^\infty(\boldsymbol{R})$, $\phi\not\equiv 0$, $\operatorname{supp}\phi\subseteq[0, 1]$. We set $\psi(x)=\phi(|x|^2-1)$ for $x\in\boldsymbol{R}^n$. Then $\psi\in C_0^\infty(\boldsymbol{R}^n)$; hence $\psi\in H_r^1(\boldsymbol{R}^n)$. Let $f_k(x)=\alpha_k\phi(\beta_k|x|^2-1)$ for $k=1, 2,\dots$, where $\alpha_k=\beta_k^{(n-2m)/4}=\beta_k^{n/2p}$, $\beta_k>0$ for $k=0, 1,\dots$ and $\lim_{k\to\infty}\beta_k=\infty$. Then

$$\hat{f}_k(\xi) = \alpha_k\beta_k^{-n/2}\hat{\psi}(\xi/\sqrt{\beta_k}),\quad \xi\in\boldsymbol{R}^n;\ k = 1, 2,\dots ;$$

hence

$$\|f_k\|^2 = \|\hat{f}_k\|^2 = \alpha_k^2\beta_k^{-n/2}\|\hat{\psi}\|^2 = \beta_k^{-m}\|\hat{\psi}\|^2 \longrightarrow 0$$

as $k\to\infty$, and

$$\||\xi|^m\hat{f}_k\|^2 = \alpha_k^2\beta_k^{m-n/2}\||\xi|^m\hat{\psi}\|^2 = \||\xi|^m\hat{\psi}\|^2,$$

which is bounded uniformly in k. Hence $\{f_k\}$ is bounded in H_r^m; since $\lim_{k\to\infty}f_k=0$ in L^2, the only possible limit for a convergent subsequence in L^p is 0. However,

$$\|f_k\|_{L^p}^p= \alpha_k^p\beta_k^{-n/2}\int|\psi(x)|^p dx = \int|\psi(x)|^p dx > 0,$$

since $\alpha_k^p\beta_k^{-n/2}=\beta_k^0=1$. Thus no subsequence of $\{f_k\}$ can converge to 0 in L^p.

Q. E. D.

REMARK 2. In case $m=1$, we have other examples:

EXAMPLE 3: Letting ϕ be the same as in Remark 1, we put

$$f_k(x) = a_k\phi(2^k|x|-1)\quad (k = 1, 2,\dots).$$

Then we can determine the sequence $\{a_k\}$ so that

$$(1)\ \ \overline{\lim}\,\|f_k\|_1 < \infty,\quad (2)\ \ \lim\|f_k\| = 0,\quad (3)\ \ \underline{\lim}\,\|f_k\|_{L^p} > 0.$$

EXAMPLE 4: Let $\alpha_k>0$, $\lim_{k\to\infty}\alpha_k=\infty$. Then

$$u_k(x) = \alpha_k\exp[-\alpha_k^{4|x|^2/(n-2)}]\qquad (k = 1, 2,\dots)$$

satisfies above conditions.

REMARK 3. Although we have considered only the spaces $H_r^m(\boldsymbol{R}^2)$, we will obtain analogous results for the general Sobolev spaces $W_r^{m,p}(\boldsymbol{R}^n)$.

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