HIROSHIMA MATH. J.
16 (1986), 651–663

Oscillation theorems for nonlinear differential systems with general deviating arguments

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(Received December 20, 1985)

1. Introduction

The oscillation theory of nonlinear differential systems with deviating argements has been developed by many authors. Most of them have studied two-dimensional differential systems; see, for example, Kitamura and Kusano [2–4], Shevelo, Varech and Gritsai [8], and Varech and Shevelo [9, 10]. The oscillation results for n-dimensional systems with deviating arguments have been given by Foltynska and Werbowski [1], the present author [5, 6] and Šeda [7].

The purpose of this paper is to obtain oscillation criteria for the nonlinear differential system with general deviating arguments of the form:

$$(S_r)\qquad \begin{aligned} y_i'(t) &= p_i(t)f_i(y_{i+1}(h_{i+1}(t))), \qquad i = 1, 2,\dots, n-1,\\ y_n'(t) &= (-1)^r p_n(t)f_n(y_1(h_1(t))), \qquad r = 1, 2, \end{aligned}$$

where the following conditions are assumed to hold:

(1) a) p_i: $[0, \infty)\to[0, \infty)$, $i=1, 2,\dots, n$, are continuous and not identically zero on any infinite subinterval of $[0, \infty)$, and

$$\int^{\infty} p_i(t)dt = \infty, \quad i = 1, 2,\dots, n-1;$$

b) h_i: $[0, \infty)\to R$ are continuous and $\lim_{t\to\infty} h_i(t)=\infty$, $i=1,\dots, n$;

c) f_i: $R\to R$ are continuous and $uf_i(u)>0$ for $u\neq 0$, $i=1, 2,\dots, n$.

Denote by W the set of all solutions $y(t)=(y_1(t),\dots, y_n(t))$ of the system (S_r) which exist on some ray $[T_y, \infty)\subset[0, \infty)$ and satisfy $\sup\{\sum_{i=1}^{n}|y_i(t)|;\ t\geqq T\}>0$ for all $T\geqq T_y$.

DEFINITION 1. A solution $y\in W$ is called oscillatory if each component has arbitrarily large zeros.

A solution $y\in W$ is called nonoscillatory (resp. weakly nonoscillatory) if each component (resp. at least one component) is eventually of constant sign.

DEFINITION 2. We shall say that the system (S_1) has the property A if for n even every solution $y\in W$ is oscillatory and for n odd it is either oscillatory or

(P_1) y_i $(i=1, 2,\dots, n)$ tend monotonically to zero as $t\to\infty$.

We shall say that the system (S_2) has the property B if for n even every solution $y\in W$ is either oscillatory or (P_1) holds or

(P_2) $|y_i|$ $(i=1, 2,\dots, n)$ tend monotonically to ∞ as $t\to\infty$,

and for n odd it is either oscillatory or (P_2) holds.

We introduce the following notations:

i) Let τ: $[0, \infty)\to R$ be a continuous function such that $\tau(t)\leqq t$ and $\tau(t)\to\infty$ as $t\to\infty$. We define

$$\gamma_\tau(t) = \sup\{s\geqq 0;\ \tau(s)<t\} \quad \text{for all} \quad t>0;$$

ii) Let $i_k\in\{1, 2,\dots, n\}$, $k\in\{1, 2,\dots, n-1\}$, t, $s\in[0, \infty)$. We define:

(2) $I_0 = 1$,

$$I_k(t, s; p_{i_k},\dots, p_{i_1}) = \int_s^t p_{i_k}(x) I_{k-1}(x, s; p_{i_{k-1}},\dots, p_{i_1})dx.$$

It is easy to prove that the following identities hold:

$$(3) \quad I_k(t, s; p_{i_k},\dots, p_{i_1}) = \int_s^t p_{i_1}(x) I_{k-1}(t, x; p_{i_k},\dots, p_{i_2})dx,$$

$$(4) \quad I_k(t, s; p_{i_k},\dots, p_{i_1}) = (-1)^k I_k(s, t; p_{i_1},\dots, p_{i_k}).$$

To obtain main results we need the following lemmas:

LEMMA 1. *Suppose that the conditions* (1a)–(1c) *are satisfied. Let* $y=(y_1,\dots, y_n)\in W$ *be a nonoscillatory solution of* (S_r) *on the interval* $[a, \infty)$, $a\geqq 0$.

I) *Then there exist an integer* $l\in\{1, 2,\dots, n\}$, *with* $n+r+l$ *odd or* $l=n$, *and* $t_0\geqq a$ *such that for* $t\geqq t_0$

$$(5_l) \qquad y_i(t)y_1(t)>0, \qquad i=1, 2,\dots, l,$$

$$(6_l) \qquad (-1)^{l+i}y_i(t)y_1(t)>0, \qquad i=l, l+1,\dots, n.$$

II) *In addition let* $\lim_{t\to\infty}|y_l(t)|=L_l$, $0\leqq L_l\leqq\infty$. *Then*

$$(7) \qquad l>1,\ L_l>0 \Rightarrow \lim_{t\to\infty}|y_i(t)| = \infty, \quad i=1, 2,\dots, l-1,$$

$$l<n,\ L_l<\infty \Rightarrow \lim_{t\to\infty} y_i(t) = 0, \quad i=l+1,\dots, n.$$

PROOF. a) Let $r=1$. From Lemma 1 of [5] we get the assertions of Lemma 1 in the case I). b) Let $r=2$. Without loss of generality we may suppose that $y_1(t)>0$, $y_1(h_1(t))>0$ for $t\geqq t_1\geqq a$. Because of (1a), (1c), the n-th equation of (S_2) implies that $y_n(t)$ is nondecreasing on $[t_1, \infty)$. Then either $y_n(t)>0$ or $y_n(t)<0$ for $t\geqq t_2\geqq t_1$. i) If $y_n(t)>0$ for $t\geqq t_2$, it is easy to prove that $y_i(t)>0$ for $t\geqq t_3\geqq t_2$, $i=1,\dots, n-1$. ii) Let $y_n(t)<0$ for $t\geqq t_2$. Then in view of the

$(n-1)$-st equation of (S_2) we get $y'_{n-1}(t)y_1(t) \leqq 0$ for $t \geqq t_2$. Then by the case a) with n replaced by $n-1$, there exist an integer $l \in \{1, 2, \ldots, n-1\}$ with $n+l$ odd and a $t_0 \geqq t_2$ such that (5_l), (6_l) hold.

The assertions in the case II) follow from (5_l), (6_l).

LEMMA 2. ([5, Lemma 1]) *Let $y \in W$ be a weakly nonoscillatory solution of (S_r) on $[a, \infty)$. Then there exists a $T \geqq a$ such that y is nonoscillatory on $[T, \infty)$.*

Furthermore we shall consider the system $(\bar{S}_r)$ or the form

$$(\bar{S}_r) \qquad \begin{aligned} y'_i(t) &= p_i(t)y_{i+1}(t), \quad i = 1, 2, \ldots, n-2, \\ y'_{n-1}(t) &= p_{n-1}(t)f_{n-1}(y_n(h_n(t))), \\ y'_n(t) &= (-1)^r p_n(t)f_n(y_1(h_1(t))), \quad r = 1, 2, \end{aligned}$$

where the conditions (1a)–(1c) hold and
(1d) $f_{n-1}(u), f_n(u)$ are nondecreasing functions of u.

LEMMA 3. ([5, Lemma 4]) *Suppose that* (1a)–(1d) *are satisfied. Let $y=(y_1, \ldots, y_n) \in W$ be a solution of $(\bar{S}_r)$ on $[t_0, \infty)$. Then the following relations hold*:

$$(8) \qquad \begin{aligned} y_i(t) = {} & \textstyle\sum_{j=0}^{m} (-1)^j y_{i+j}(s) I_j(s, t; p_{i+j-1}, \ldots, p_i) \\ & + (-1)^{m+1} \int_s^t y_{i+m+1}(x) p_{i+m}(x) I_m(x, t; p_{i+m-1}, \ldots, p_i) dx \\ & \textit{for} \quad m = 0, 1, \ldots, n-i-2, \; i = 1, 2, \ldots, n-2, \; t, s \in [t_0, \infty); \end{aligned}$$

$$(9) \qquad \begin{aligned} y_i(s) = {} & \textstyle\sum_{j=0}^{n-i-1} (-1)^j y_{i+j}(t) I_j(t, s; p_{i+j-1}, \ldots, p_i) \\ & + (-1)^{n-i} \int_s^t p_{n-1}(x) I_{n-i-1}(x, s; p_{n-2}, \ldots, p_i) f_{n-1}(y_n(h_n(x))) dx, \\ & \textit{for} \quad i = 1, 2, \ldots, n-1, \quad t, s \in [t_0, \infty). \end{aligned}$$

LEMMA 4. *Suppose that* (1a)–(1d) *are satisfied Let* $y=(y_1, \ldots, y_n) \in W$ be a *nonoscillatory solution of $(\bar{S}_r)$ on $[a, \infty)$ with $y_1(t) > 0$ on $[a, \infty)$. Then there exist a $t_0 \geqq a$ and an integer $l \in \{1, 2, \ldots, n\}$ with $n+r+l$ odd or $l=n$ such that* (5_l)–(7) *hold. Moreover*

$$(10_l) \qquad y_i(t) \geqq (-1)^{n+l} \int_{t_0}^t p_{n-1}(s) \bar{I}_{n-i-1}(s, t_0) f_{n-1}(y_n(h_n(s))) ds$$

$$\textit{for} \quad l = 2, 3, \ldots, n-1, \quad i = 1, 2, \ldots, l-1, \quad t \geqq t_0;$$

$$(10_n) \qquad y_i(t) \geqq \int_{t_0}^t p_{n-1}(s) I_{n-i-1}(t, s; p_i, \ldots, p_{n-2}) f_{n-1}(y_n(h_n(s))) ds$$

for $i = 1, 2, \ldots, n-1, \quad t \geqq t_0,$

where

$$\bar{I}_{n-i-1}(s, t_0) = \begin{cases} I_{n-i-1}(s, t;\, p_{n-2}, \ldots, p_l, p_i, \ldots, p_{l-1}), & 2 \leqq l \leqq n-2 \\ I_{n-i-1}(s, t_0;\, p_i, \ldots, p_{n-2}), & l = n-1. \end{cases}$$

PROOF. Suppose that $l \in \{2, 3, \ldots, n-1\}$, $i = 1, 2, \ldots, l-1$. Putting $m = l-i-1$, $s = t_0$, $x = u$ in (8) and then using (4) and (5_l), we have

$$(11) \qquad y_i(t) \geqq \int_{t_0}^{t} y_l(u) p_{l-1}(u) I_{l-i-1}(t, u;\, p_i, \ldots, p_{l-2})\,du, \quad t \geqq t_0.$$

On the other hand, we put $i = l$, $s = u$ in (9) and then use (6) to get

$$(12) \qquad y_l(u) \geqq (-1)^{n+l} \int_{u}^{t} p_{n-1}(x) I_{n-l-1}(x, u;\, p_{n-2}, \ldots, p_l) \cdot f_{n-1}(u_n(h_n(x)))\,dx \quad \text{for} \quad t \geqq u.$$

Substituting (12) in (11), we obtain

$$(13) \qquad y_i(t) \geqq (-1)^{n+l} \int_{t_0}^{t} \left(\int_{u}^{t} p_{n-1}(x) I_{n-l-1}(x, u;\, p_{n-2}, \ldots, p_l) \cdot f_{n-1}(y_n(h_n(x)))\,dx \right) p_{l-1}(u) I_{l-i-1}(t, u;\, p_i, \ldots, p_{l-2})\,du$$
$$\geqq (-1)^{n+l} \int_{t_0}^{t} p_{n-1}(x) H_l(x, t_0) f_{n-1}(y_n(h_n(x)))\,dx$$

for $t \geqq t_1 = \gamma_{h_n}(t_0)$, where

$$(14) \qquad H_l(x, t_0) = \int_{t_0}^{x} I_{n-l-1}(x, u;\, p_{n-2}, \ldots, p_l) p_{l-1}(u) \cdot I_{l-i-1}(x, u;\, p_i, \ldots, p_{l-2})\,du, \quad \text{for} \quad x \geqq t_1.$$

i) Let $2 \leqq l \leqq n-2$. In view of (3), H_l can be written in the form

$$(15) \qquad H_l(x, t_0) = I_{n-l}(x, t_0;\, p_{n-2}, \ldots, p_l, p_{l-1} I_{l-i-1}(x, \cdot\,;\, p_i, \ldots, p_{l-2})).$$

Using the following relation for $t_0 \leqq s \leqq x$:

$$\int_{t_0}^{s} P_{l-1}(u) I_{l-i-1}(x, u;\, p_i, \ldots, p_{l-2})\,du$$
$$\geqq \int_{t_0}^{s} p_{l-1}(u) I_{l-i-1}(s, u;\, p_i, \ldots, p_{l-2})\,du = I_{l-i}(s, t_0;\, p_i, \ldots, p_{l-1})$$

and then using (2), (3) $n-l$ times in (14), we obtain

(16) $H_l(x, t_0) \geqq I_{n-i-1}(x, t_0; p_{n-2},..., p_l, p_i,..., p_{l-1})$.

ii) Let $l=n-1$. Then with regard to $I_0=1$, (14) implies

$$H_{n-1}(x, t_0) = \int_{t_0}^{x} p_{n-2}(u) I_{n-i-2}(x, u; p_i,..., p_{n-3}) du$$
$$= I_{n-i-1}(x, t_0; p_i,..., p_{n-2}).$$

If we put (16) and the last equality in (13) we get (10_l).

Let $l=n$. Putting $t=t_0$, $s=t$ in (9) and then using (4), (5_n), we obtain (10_n). The proof of Lemma 4 is complete.

2. Main results

DEFINITION 3. System $(\bar{S}_r)$ is called (α, β)-superlinear, if there exist positive numbers α, β such that $\alpha\beta > 1$ and

$$\frac{|f_i(u)|}{|u|^{\gamma_i}} \geqq \frac{|f_i(v)|}{|v|^{\gamma_i}} \quad \text{for } |u| > |v|,\ uv > 0,$$

$i = n-1, n$, $\gamma_{n-1} = \alpha$, $\gamma_n = \beta$.

Let $m \in \{1, 2,..., n\}$. We denote

$$(17) \quad J_{n-2}^{m}(t, T) = \begin{cases} I_{n-2}(t, T; p_{n-2},..., p_1) & \text{for } 1 \leqq m \leqq 2, \\ I_{n-2}(t, T; p_{n-1},..., p_m, p_2,..., p_{m-1}) & \text{for } 2 \leqq m \leqq n-1, \\ I_{n-2}(t, T; p_2,..., p_{n-1}) & \text{for } 2 \leqq m = n; \end{cases}$$

$$P(t) = \int_t^{\infty} p_n(s) ds.$$

THEOREM 1. *Let the system $(\bar{S}_r)$ be (α, β)-suplerlinear. Suppose that there exist continuous increasing functions g, h: $[0, \infty) \to R$ such that*

(18) $g(t) \leqq h_1(t)$, $\lim_{t\to\infty} g(t) = \infty$

(19) $h(t) \geqq \max\{h_n(t), g^{-1}(t)\}$, *where g^{-1} indicates the inverse function of g.*

If

$$(20) \quad \int_{\gamma_h(T)}^{\infty} p_{n-1}(t) J_{n-2}^{l}(t, T) f_{n-1}(LP(h(t))) dt = \infty \quad \textit{for} \quad l = 1, 2,$$

$$\int_{\gamma_h(T)}^{\infty} p_1(t) J_{n-2}^{l}(t, T) f_{n-1}(LP(h(t))) dt = \infty \quad \textit{for} \quad l = 3, 4,..., n$$

for every constant $L > 0$, then the system $(\bar{S}_1)$ has the property A and the system $(\bar{S}_2)$ has the property B.

PROOF. Suppose that $(\bar{S}_r)$ has a weakly nonoscillatory solution $y=(y_1,\dots,y_n)\in W$. Then, by Lemma 2, y is nonoscillatory. Without loss of generality we may suppose that $y_1(t)>0$, $y_1(h_1(t))>0$ for $t\geqq t_1>0$. Then the n-th equation of $(\bar{S}_r)$ implies $(-1)^r y_n'(t)\geqq 0$ for $t\geqq t_1$ and it is not identically zero on any infinite interval of $[t_1, \infty)$. Then, by Lemma 4, there exist a $t_2\geqq t_1$ and an integer $l\in\{1, 2,\dots, n\}$ with $n+r+l$ odd or $l=n$ such that (5_l)–(8), (11) hold for $t\geqq t_2$.

A) Consider the system $(\bar{S}_1)$, i.e. $r=1$ and $n+l$ is even. Integrating the n-th equation of $(\bar{S}_1)$ from t $(\geqq t_2)$ to ∞, we have

$$(21)\qquad y_n(t)\geqq y_n(t)-y_n(\infty)=\int_t^\infty p_n(s)f_n(y_1(h_1(s)))ds,\quad t\geqq t_2.$$

I) Let $l\geqq 2$. Then, y_1 is an increasing function and therefore there exist $C>0$ and $t_3\geqq t_2$ such that $y_1(h_1(t))\geqq C$ for $t\geqq t_3$. Using the last inequality, (21) implies

$$(22)\qquad y_n(t)\geqq f_n(C)\int_t^\infty p_n(s)ds = LP(t)$$

where $L=f_n(C)$. Because the system $(\bar{S}_1)$ is (α, β)-superlinear, in view of (22) and $y_1(h_1(t))\geqq C$, we have

$$(23)\qquad f_{n-1}(y_n(h(t)))\geqq \frac{f_{n-1}(LP(h(t)))}{(LP(h(t)))^\alpha}(y_n(h(t)))^\alpha,\quad t\geqq t_3,$$

$$(24)\qquad f_n(y_1(h_1(t)))\geqq M(y_1(h_1(t)))^\beta,\quad t\geqq t_3,\quad M=C^{-\beta}L.$$

If we put (24) in (21), then using (18), (19) and the monotonicity of y_1, we get

$$y_n(t)\geqq M\int_t^\infty p_n(s)(y_1(g(s)))^\beta ds\geqq M(y_1(g(t)))^\beta P(t),\quad t\geqq t_3,$$

or

$$(25)\qquad y_n(h(t))\geqq My_1(g(h(t)))^\beta P(h(t))\geqq M(y_1(t))^\beta P(h(t)),$$

for $t\geqq\gamma_h(t_3)=T_1$.

i) Let $2<l\leqq n$ ($n+l$ is odd). Putting $i=2$, $t_0=T_1$ in (10_l), (10_n), and using (19), the monotonicity of h, y_n, f_{n-1}, (4), (17) and (23), we obtain

$$(26_l)\qquad y_2(t)\geqq\int_{T_1}^t p_{n-1}(s)\bar{I}_{n-3}(s, T_1)f_{n-1}(y_n(h_n(s)))ds$$
$$\geqq (y_n(h(t)))^\alpha\frac{f_{n-1}(LP(h(t)))}{(LP(h(t)))^\alpha}J_{n-2}^l(t, T_1),\quad t\geqq T_1,$$
$$l=3, 4,\dots, n-2,$$

$$(26_n)\qquad y_2(t) \geqq f_{n-1}(y_n(h(t))) \int_{T_1}^{t} p_{n-1}(s) I_{n-3}(t, s; p_2, \ldots, p_{n-2}) ds$$

$$\geqq (y_n(h(t)))^{\alpha} \frac{f_{n-1}(LP(h(t)))}{(LP(h(t)))^{\alpha}} J_{n-2}^{n}(t, T_1), \quad t \geqq T_1,$$

respectively. Combining (25) with (26), we get

$$(27)\qquad y_2(t) \geqq C^{-\gamma}(y_1(t))^{\gamma} f_{n-1}(LP(h(t))) J_{n-2}^{l}(t, T_1), \quad t \geqq T_1, \quad \gamma = \alpha\beta.$$

Multiplying (27) by $p_1(t)(y_1(t))^{-\gamma}$ and then using the first equation of $(\bar{S}_1)$, we get

$$(28)\qquad y_1'(t)(y_1(t))^{-\gamma} \geqq C^{-\gamma} p_1(t) f_{n-1}(LP(h(t))) J_{n-2}^{l}(t, T_1), \quad t \geqq T_1.$$

Integrating (28) from $T_2 = \gamma_h(T_1)$ to τ, and then letting $\tau \to \infty$, we have

$$\int_{T_2}^{\infty} p_1(t) J_{n-2}^{l}(t, T_1) f_{n-1}(LP(h(t))) dt \leqq \frac{C^{\gamma} y_1(T_2)}{\gamma - 1} < \infty,$$

which contradicts (20_l) for $l \geqq 3$.

ii) Let $l = 2 = n$. If we put the second equation in the first equation of $(\bar{S}_1)$ and then use (19), (23) and (25), we get

$$y_1'(t)(y_1(t))^{-\gamma} \geqq C^{-\gamma} p_1(t) f_1(LP(h(t))).$$

Integrating the last inequality from T_1 to τ and letting $\tau \to \infty$, we get a contradiction to (20_l) for $l = 2 = n$.

iii) Let $l = 2 < n$. Putting $i = 2$ in (9) and using (6), (19), the monotonicity of h, y_n, f_{n-1}, we obtain

$$(29)\qquad y_2(s) \geqq \int_{s}^{t} p_{n-1}(x) f_{n-1}(y_n(h(x))) I_{n-3}(x, s; p_{n-2}, \ldots, p_2) dx.$$

Combining (25) with (23) and using the monotonicity of y_1, from (29) we get

$$(30)\qquad y_2(s) \geqq C^{-\gamma}(y_1(s))^{\gamma} \int_{s}^{t} p_{n-1}(x) f_{n-1}(LP(h(x))) I_{n-3}(x, s; p_{n-2}, \ldots, p_2) dx.$$

Multiplying (30) by $p_1(s)(y_1(s))^{-\gamma}$ and using the first equation of $(\bar{S}_1)$, we have

$$y_1'(s)(y_1(s))^{-\gamma} \geqq C^{-\gamma} p_1(s) \int_{s}^{t} p_{n-1}(x) I_{n-3}(x, s; p_{n-2}, \ldots, p_2) \cdot$$

$$f_{n-1}(LP(h(x))) dx.$$

Integrating the last inequality from T_1 to t, we get

$$\frac{C^{\gamma}(y_1(T_1))^{1-\gamma}}{\gamma - 1} \int_{T_1}^{t} p_1(s) \int_{s}^{t} p_{n-1}(x) I_{n-3}(x, s; p_{n-2}, \ldots, p_2) \cdot$$

$$f_{n-1}(LP(h(x))) dx ds \geqq \int_{T_1}^{t} p_{n-1}(LP(h(x))) J_{n-2}^{2}(x, T_1) dx,$$

which contradicts (20_2) as $t \to \infty$.

II) Let $l=1$ (n is odd). Then $y_1(t) \downarrow K$ as $t \uparrow \infty$, where $K \geqq 0$. Assume that $K>0$. If we put $i=1$, $s=T_1$ in (9), and use (6), (19) and the monotonicity of h, y_n, f_{n-1}, we obtain

$$y_1(T_1) \geqq \int_{T_2}^{t} p_{n-1}(x) f_{n-1}(y_n(h(x))) I_{n-2}(x, T_1; p_{n-2}, \dots, p_1) dx$$

for $t \geqq T_2 = \gamma_n(T_1)$. Further using (22), we have

$$y_1(T_1) \geqq \int_{T_2}^{t} p_{n-1}(x) J_{n-2}^{1}(x, T_1) f_{n-1}(LP(h(x))) dx,$$

which contradicts (20_1) as $t \to \infty$. Therefore $K=0$, $\lim_{t\to\infty} y_1(t)=0$. Then by (7) $\lim_{t\to\infty} y_i(t)=0$ for $i=1, 2, \dots, n$.

B) Consider the system $(\bar{S}_2)$, i.e. $r=2$ and $n+l$ is odd.

I) By virtue of Lemma 3, (5_n) holds. Then the n-th equation of $(\bar{S}_2)$ implies, in view of $y_1(h_1(t))>0$, (1a) and (1c), that $y_n(t)$ is a nondecreasing function and therefore $\lim_{t\to\infty} y_n(t) = L_n \leqq \infty$. Then it follows from (7) that $\lim_{t\to\infty} y_i(t) = \infty$ for $i=1, 2, \dots, n-1$. We shall prove that $L_n = \infty$. Suppose that $L_n < \infty$. In view of the monotonicity of y_n and y_1, there exist $T_2 \geqq t_0$, $K_1 > 0$ and $C>0$ such that

$$K_1 \leqq y_n(h_n(t)) \leqq L_n, \tag{31}$$

$$C \leqq y_n(g(t)) \qquad \text{for} \quad t \geqq T_2. \tag{32}$$

Integrating the n-th equation of $(\bar{S}_2)$ and using (18), (31) and the monotonicity of f_n, y_1, we get

$$L_n \geqq \int_t^{\infty} p_n(s) f_n(y_1(h_1(s))) ds \geqq f_n(y_1(g(t))) P(t), \quad t \geqq T_2. \tag{33}$$

In view of (32), the inequality (33) implies

$$L_n \geqq f_n(C) P(h(t)) = LP(h(t)), \quad L = f_n(C), \quad t \geqq T_3 = \gamma_h(T_2). \tag{34}$$

Because the system $(\bar{S}_2)$ is (α, β)-superlinear, in view of (32)–(34) and (19) we have

$$L_n \geqq M(y_1(g(h(t))))^{\beta} P(h(t)) \geqq M(y_1(t))^{\beta} P(h(t)), \quad M = LC^{-\beta} \tag{35}$$

$$f_{n-1}(L_n) \geqq \frac{f_{n-1}(LP(h(t)))}{(LP(h(t)))^{\alpha}} (L_n)^{\alpha}, \quad t \geqq T_3. \tag{36}$$

a) Let $n>2$. From (10_n) for $i=2$, $t_0=T_4$, in view of (31) and (17), we get

$$y_2(t) \geqq f_{n-1}(K_1) J_{n-2}^{n}(t, T_3), \quad t \geqq T_3. \tag{37}$$

Multiplying (37) by $f_{n-1}(L_n)p_1(t)(y_1(t))^{-\gamma}$ and then using (35), (36) and the first equation of $(\bar{S}_2)$, we have

$$(38) \qquad y_1'(t)(y_1(t))^{-\gamma} \geqq \frac{f_{n-1}(K_1)}{f_{n-1}(L_n)} C^{-\alpha} p_1(t) f_{n-1}(LP(h(t))) J_{n-2}^n(t, T_3).$$

b) Let $n=2$. From the first equation of $(\bar{S}_2)$ and in view of (31) we obtain $y_1'(t) \geqq p_1(t) f_1(K_1)$, $t \geqq T_2$. Multiplying the last inequality by $L_n^\alpha(y_1(t))^{-\gamma}$ and then using (35) and (36), we get (38) for $n=2$ $(J_0=1)$. Integrating (38) from T_3 to ∞, we get a contradiction to (20_n). Therefore $L_n=\infty$ and $\lim_{t\to\infty} y_i(t)=\infty$, $i=1, 2,\dots, n$.

II) Let $l \in \{1, 2,\dots, n-1\}$. Then (6) implies that $y_n(t)<0$ for $t \geqq t_2$ and it is an increasing function. Integrating the n-th equation of $(\bar{S}_2)$ from t $(\geqq t_2)$ to ∞, we have

$$- y_n(t) \geqq \int_t^\infty p_n(s) f_n(y_1(h_1(s))) ds, \quad t \geqq t_2.$$

Further proceeding in the same way as in the cases A–I), A–II) of this proof except that $y_n(t)$ is replaced by $-y_n(t)$ (>0), we get a contradiction to (20_l) for $l=1, 2,\dots, n-1$. In the case n is even and $l=1$ we obtain $\lim_{t\to\infty} y_i(t)=0$ for $i=1, 2,\dots, n$.

The proof of Theorem 1 is complete.

Theorem 1 represents a certain generalization of Theorem 5 in [4].

THEOREM 2. *Let the system $(\bar{S}_r)$ be (α, β)-superlinear. Suppose that*

$$(39) \qquad h_n(t) \leqq t, \; g_1(t) \leqq \min\{h_1(t), t\} \quad \text{on} \quad [0, \infty),$$
$$\text{where } g_1'(t) \geqq 0 \quad \text{on} \quad [0, \infty), \lim_{t\to\infty} g_1(t) = \infty.$$

If

$$(40_l) \quad \int_{\gamma g_1(T)}^\infty p_{n-1}(g_1(t)) g_1'(t) J_{n-2}^l(g_1(t), T) \frac{f_{n-1}(LP(g_1(t)))}{(P(g_1(t)))^\alpha} (P(t))^\alpha dt = \infty \quad \text{for } l = 1, 2,$$

$$\int_{\gamma g_1(T)}^\infty p_1(g_1(g_1(t)) g_1'(t) J_{n-2}^l(g_1(t), T) \frac{f_{n-1}(LP(g_1(t)))}{(P(g_1(t)))^\alpha} (P(t))^\alpha dt = \infty \quad \text{for } l = 3, 4,\dots, n,$$

then the system $(\bar{S}_1)$ has the property A, and the system $(\bar{S}_2)$ has the property B.

PROOF. Let $y=(y_1,\dots, y_n) \in W$ be a weakly nonoscillatory solution of $(\bar{S}_r)$. Then by Lemma 2 it is nonoscillatory. Let $y_1(t)>0$, $y_1(h_1(t))>0$ for $t \geqq t_1 > 0$.

Proceeding in the same way as in the proof of Theorem 1, we see that (5_l)–(8), (11) hold for $t \geqq t_2 \geqq t_1$. Let $T_2 \geqq t_2$ be so large that $g_1(t) \geqq t_2$, $h_n(t) \geqq t_2$ for $t \geqq T_2$.

A) Consider the system $(\bar{S}_1)$, i.e. $r=1$ and $n+l$ is even. From the n-th equation of $(\bar{S}_1)$ we get (21).

I) Let $l \geqq 2$. Proceeding in the same way as in the case A–I) in the proof of Theorem 1, we get (22)–(24). Combining (24) and (21) and using (39) and the monotonicity of y_n, y_1, we have

$$y_n(g_1(t)) \geqq y_n(t) \geqq M \int_t^\infty p_n(s)(y_1(h_1(s)))^\beta ds \geqq M(y_1(g_1(t)))^\beta P(t), \quad t \geqq T_2, \tag{41}$$

and (22) implies

$$y_n(g_1(t)) \geqq LP(t) \qquad \text{for} \quad t \geqq T_2. \tag{42}$$

i) Let $l=n$ or $2<l \leqq n-2$. Putting $i=2$, $t_0=T_2$ in (10_l), (10_n) and using (39), the monotonicity of g_1, y_n, f_{n-1}, (5_l), (17) and the superlinearity of $(\bar{S}_1)$, we obtain

$$(43_l) \qquad y_2(t) \geqq f_{n-1}(y_n(t)) \int_{T_2}^t p_{n-1}(s) \bar{I}_{n-3}(s, T_2) ds \geqq f_{n-1}(y_n(t)) J_{n-2}^l(t, T_2) \geqq \frac{f_{n-1}(LP(t))}{(LP(t))^\alpha} (y_n(t))^\alpha J_{n-2}^l(t, T_2), \quad l = 3, 4, \ldots, n-2,$$

or

$$(43_n) \qquad y_2(t) \geqq f_{n-1}(y_n(t)) \int_{T_2}^t p_{n-1}(s) I_{n-3}(t, s; p_2, \ldots, p_{n-2}) ds \geqq \frac{f_{n-1}(LP(t))}{(LP(t))^\alpha} (y_n(t))^\alpha J_{n-2}^n(t, T_2), \quad t \geqq T_2,$$

respectively.

From (43) and in view of (39) we get

$$y_2(g_1(t)) \geqq \frac{f_{n-1}(LP(g_1(t)))}{(LP((g_1(t)))^\alpha} J_{n-2}^l(g_1(t), T_2)(y_n(t))^\alpha, \tag{44}$$

for $t \geqq T_3 = \gamma_{g_1}(T_2)$. Combining (44) with (41), we have

$$y_2(g_1(t)) \geqq C^{-\gamma} \frac{f_{n-1}(LP(g_1(t)))}{(P(g_1(t)))^\alpha} J_{n-2}^l(g_1(t), T_2) \cdot (y_1(g_1(t)))^\gamma (P(t))^\beta, \quad t \geqq T_3. \tag{45}$$

Multiplying (45) by $p_1(g_1(t))g_1'(t)(y_1(g_1(t)))^{-\gamma}$ and using the first equation of $(\bar{S}_1)$, we get

$$(46)\qquad \frac{y_1'(g_1(t))g_1'(t)}{(y_1(g_1(t)))^\alpha} \geqq C^{-\gamma}p_1(g_1(t))g_1'(t)\frac{f_{n-1}(LP(g_1(t)))}{(P(g_1(t)))^\alpha}\cdot J_{n-2}(g_1(t), T_2)(P_1(t))^\alpha,\ t \geqq T_3\,.$$

Integrating (46) from T_3 to ∞, we obtain a contradiction to (40_l) for $l \geqq 3$.

ii) Let $l=2=n$. If we put the second equation in the first equation of $(\bar{S}_1)$ and use (39), (23) and (41), then we get

$$\begin{aligned} y_1'(g_1(t)) &\geqq p_1(g_1(t))f_1(y_2(g_1(t))) \\ &\geqq C^{-\gamma}p_1(g_1(t))(y_1(g_1(t)))^\gamma \frac{f_1(LP(g_1(t)))(P(t))^\alpha}{(L(g_1(t)))^\alpha},\quad t \geqq T_2\,. \end{aligned}$$

Integrating the last ineqaulity from T_3 to ∞, we have a contradiction to (40_l) for $l=2=n$.

iii) Let $l=2<n$. If we put $i=2$ in (9) and use (7), (39), the monotonicity of y_n, f_{n-1}, we obtain

$$(47)\qquad y_2(s) \geqq \int_s^t p_{n-1}(x)I_{n-3}(x, s; p_{n-2},\dots, p_2)f_{n-1}(y_n(x))dx.$$

Combining (23) with (25) and then using the monotonicity of y_1, g_1, we obtain from (47)

$$(48)\qquad y_2(g_1(s)) \geqq C^{-\gamma}(y_1(g_1(s)))^\gamma \int_{g_1(s)}^t p_{n-1}(x)\cdot I_{n-3}(x, g_1(s); p_{n-2},\dots, p_2)f_{n-1}(LP(x))dx,$$

because $g_1(g_1(s)) \leqq g_1(s)$. Multiplying (48) by $p_1(g_1(s))g_1'(s)(y_1(g_1(s)))^{-\gamma}$ and using the first equation of $(\bar{S}_1)$, we get

$$\frac{y_1'(g_1(s))g_1'(s)}{(y_1(g_1(s)))^\gamma} \geqq C^{-\gamma}p_1(g_1(s))g_1'(s)\int_{g_1(s)}^t p_{n-1}(x)\cdot I_{n-3}(x, g_1(s); p_{n-2},\dots, p_2)\, f_{n-1}(L(P(x)))dx.$$

Integration of the above from T_2 to t yields

$$\begin{aligned} \infty &> \int_{T_2}^t p_1(g_1(s))g_1'(s)\int_{g_1(s)}^t p_{n-1}(x)I_{n-3}(x, g_1(s); p_{n-2},\dots, p_2)\cdot f_{n-1}(LP(x))dx = \int_{g_1(T_2)}^{g_1(t)} p_{n-1}(x)f_{n-1}(LP(x))\int_{\gamma(T_2)}^x p_1(u)\cdot I_{n-3}(x, u; p_{n-2},\dots, p_2)dudx \\ &= \int_{T_2}^t p_{n-1}(g_1(s))g_1'(s)J_{n-2}^l(g_1(s), T_2)f_{n-1}(LP(g_1(s)))ds, \end{aligned}$$

which contradicts (40_2)

II) Let $l=1$ (n is odd). Then $y_1(t)\downarrow K$ as $t\uparrow\infty$, where $K\geqq 0$. Assume that $K>0$. If we put $i=1$, $s=T_1$ in (9) and use (7), (39), the monotonicity of y_n, f_{n-1}, and (22), then we obtain

$$y_1(T_1)\geqq\int_{T_1}^{t}p_{n-1}(x)f_{n-1}(y_n(x))I_{n-2}(x,T_1;P_{n-2},\dots,p_1)dx$$
$$\geqq\int_{g_1^{-1}(T_1)}^{g_1^{-1}(t)}p_{n-1}(g_1(s))g_1'(s)J_{n-2}^1(g_1(t),T_1)\cdot\frac{f_{n-1}(LP(g_1(s)))}{(P(g_1(s)))^\alpha}(P_1(s))^\alpha ds.$$

Since $g_1^{-1}(t)\to\infty$ as $t\to\infty$, the last inequality gives a contradiction to (40_1). Therefore $K=0$, i.e. $\lim_{t\to\infty}y_1(t)=0$. Then it follows from (7) that $\lim_{t\to\infty}y_i(t)=0$ for $i=1,2,\dots,n$.

B) Consider the system $(\bar S_2)$, i.e. $r=2$ and $n+l$ is odd.

I) By virtue of Lemma 3, (5_n) holds. Exactly as in the case B–I) of the proof of Theorem 1 we get $\lim_{t\to\infty}y_i(t)=\infty$ for $i=1,2,\dots,n-1$. We shall prove that $\lim_{t\to\infty}y_n(t)=L_n=\infty$. Suppose that $0<L_n<\infty$. Proceeding as in the case B–I), we get (31)–(33), in which we replace $g(t)$ by $g_1(t)$. Combining (33) with (32) gives

$$L_n\geqq LP(g_1(t)),\quad L=f_n(C),\quad t\geqq T_3. \tag{49}$$

Because the system $(\bar S_2)$ is (α,β)-superlinear, in view of (32) and (49) we have

$$L_n\geqq M(y_1(g_1(t)))^\beta P(t),\quad M=LC^{-\beta} \tag{50}$$

$$f_{n-1}(L_n)\geqq\frac{f_{n-1}(LP(g_1(t)))}{(LP(g_1(t)))^\alpha}(L_n)^\alpha,\quad t\geqq T_3. \tag{51}$$

a) Let $n>2$. From (10_n) for $i=2$, $t_0=T_3$, in view of (31) and (17) we obtain (37). From (37) we get

$$y_2(g_1(t))\geqq f_{n-1}(K_1)J_{n-2}^n(g_1(t),T_3),\quad t\geqq T_4=\gamma_{g_1}(T_3).$$

Multiplying the last inequality by $f_{n-1}(L_n)$ and using (51) and (50), we have

$$f_{n-1}(L_n)y_2(g_1(t))\geqq\frac{f_{n-1}(LP(g_1(t)))}{(P(g_1(t)))^\alpha}C^{-\gamma}(y_1(g_1(t)))^\gamma\cdot(P(t))^\alpha f_{n-1}(K_1)J_{n-2}^n(g_1(t),T_3),\quad t\geqq T_4. \tag{52}$$

If we use the first equation of $(\bar S_2)$, (52) implies

(53) $$\frac{y_1'(g_1(t))g_1'(t)}{(y_1(g_1(t)))^\gamma} \geqq \frac{f_{n-1}(K_1)}{f_{n-1}(L_n)} C^{-\gamma} p_1(g_1(t))g_1'(t) \cdot \frac{f_{n-1}(LP(g_1(t)))}{(P(g_1(t)))^\alpha} J_{n-2}^n(g_1(t), T_3)(P(t))^\alpha \text{ for } t \geqq T_4 .$$

b) Let $n=2$. From the first equation of $(\bar{S}_2)$, in view of (31) we obtain $y_1'(g_1(t)) \geqq p_1(g_1(t))f_1(K_1)$ for $t \geqq T_2$. Multiplying the last inequality by $f_1(L_1)g_1'(t)(y_1(g_1(t)))^{-\gamma}$ and using (50), (51), we get (53) for $n=2$ ($J_0=1$). Integrating (53) from T_4 to ∞, we have a contradiction to (40_n). Therefore $L_n=\infty$, i.e. $\lim_{t\to\infty} y_i(t)=\infty$ for $i=1, 2,\dots, n$

II) Let $l \in \{1, 2,\dots, n-1\}$. If we proceed as in the cases A–I), A–II) of this proof by replacing $y_n(t)$ by $-y_n(t)$, we obtain a contradiction to (40_l) for $l=1, 2,\dots, n-1$. In the case where n is even and $l=1$ we have $\lim_{t\to\infty} y_i(t)=0$ for $i=1, 2,\dots, n$. The proof of Theorem 2 is complete.

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