HIROSHIMA MATH. J.
16 (1986), 625–629

Asymptotic expansion for the distribution of the discriminant function in the first order autoregressive processes

Hirofumi WAKAKI and Ho-Se OH

(Received January 20, 1986)

1. Introduction

This paper is concerned with the problem of classifying a series of T observations coming from one of two first order autoregressive Gaussian processes. Let Π_j $(j=1, 2)$ denote the first order autoregressive Gaussian process which satisfies the stochastic equation

$$y_t = \alpha_j y_{t-1} + u_t \qquad (t = \cdots -1, 0, 1, \ldots) \tag{1.1}$$

where $\alpha_j(|\alpha_j|<1)$ is known, and u_t's are independent identically distributed as $N(0, \sigma_j^2)$ with known variance σ_j^2. Suppose that $y=(y_1,\ldots, y_T)'$ is a series of T observations coming from Π_1 or Π_2. It is natural to consider a classification method based on the density of y. The density of y is given by

$$f(y; \alpha_j, \sigma_j^2) = (2\pi\sigma_j^2)^{-T/2}(1-\alpha_j^2)^{1/2} \exp\left(-\frac{1}{2} y' \Sigma_j^{-1} y\right) \tag{1.2}$$

when y comes from Π_j, where

$$\Sigma_j^{-1} = \frac{1}{\sigma_j^2}\begin{pmatrix} 1 & -\alpha_j & & & \\ -\alpha_j & 1+\alpha_j^2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & 1+\alpha_j^2 & -\alpha_j \\ & & & -\alpha_j & 1 \end{pmatrix}.$$

Since all the parameters are known, an optimum classification rule is based on the statistic (cf. Anderson [1])

$$V = \log\frac{f(y; \alpha_1, \sigma_1^2)}{f(y; \alpha_2, \sigma_2^2)} = Z - \frac{1}{2}\left\{T \log\frac{\sigma_1^2}{\sigma_2^2} - \log\frac{1-\alpha_1^2}{1-\alpha_2^2}\right\}, \tag{1.3}$$

where

$$Z = \frac{1}{2} y'(\Sigma_2^{-1} - \Sigma_1^{-1})y. \tag{1.4}$$

The rule is to classify $\boldsymbol{y}$ as coming from Π_1 if $V>k$ and from Π_2 if $V\leq k$, where k is a constant. If the prior probabilities q_j of Π_j are known, the rule with $k=\log(q_2/q_1)$ is the Bayes rule.

We study the asymptotic ($T\to\infty$) distribution of V or equivalently of Z. Kligiené [3] and Krzyśko [4] showed in a more general case that the limiting distribution of this function is normal. In this paper we give an approximation for the distribution function of V based on an asymptotic expansion of the distribution of Z up to the term of order T^{-1}.

2. The main result

First we define some coefficients used to describe an asymptotic expansion of the distribution function of Z when $\boldsymbol{y}$ comes from Π_j.

Let

$$(2.1)\qquad c_1=\sigma_j^2\{\sigma_1^{-2}-\sigma_2^{-2}\},\quad c_2=\sigma_j^2\{\alpha_1\sigma_1^{-2}-\alpha_2\sigma_2^{-2}\},$$
$$c_3=\sigma_j^2\{(1+\alpha_1^2)\sigma_1^{-2}-(1+\alpha_2^2)\sigma_2^{-2}\},$$

$$(2.2)\qquad \gamma_0^2=a^{-3}\left\{-\frac{1}{2}(a-2)c_3^2-4\alpha_jc_2c_3-(a^2+2a-4)c_2^2\right\},$$

$$\gamma_1=a^{-1}\{c_3-\alpha_jc_2-c_1\},$$

$$\gamma_2=a^{-5}\left\{-\frac{1}{6}(a^2-6a+6)c_3^3-3(a-2)\alpha_jc_3^2c_2\right.$$
$$\left.-(a^2-12a+12)c_3c_2^2-\frac{1}{3}(a^3+2a^2+12a-24)\alpha_jc_2^3\right\},$$

$$\gamma_3=a^{-4}\left\{\frac{1}{2}(a^2-a-1)c_3^2+2(a+1)\alpha_jc_3c_2\right.$$
$$\left.+\frac{1}{2}(a^3+2a^2-4)c_2^2+\frac{1}{2}a^3c_1^2-2a\alpha_jc_1c_2-(a^2-a)c_1c_3\right\},$$

$$\gamma_4=a^{-7}\left\{-\frac{1}{8}(a^3-12a^2+30a-20)c_3^4-4(a^2-5a+5)\alpha_jc_3^2c_2\right.$$
$$-\frac{3}{2}(a^3-22a^2+60a-40)c_3^2c_2^2-4(3a^2-20a+20)\alpha_jc_3c_2^3$$
$$\left.+\frac{1}{4}(a^5+2a^4+80a^2-240a+160)c_2^4\right\},$$

where $a=1-\alpha_j^2$.

We state here the main theorem and its corollary, which will be used for the evaluation of the probabilities of misclassification.

THEOREM. *Let Z be the random variable defined by* (1.4), *and put*

$$\tilde{Z} = \frac{1}{\rho\sqrt{T}}(Z-\mu), \tag{2.3}$$

where $\mu=(2a)^{-1}T(2\alpha_j c_2 - c_3)+\gamma_1$, $\rho=(\gamma_0^2+2\gamma_3 T^{-1})^{1/2}$. *Then the distribution function of* $\tilde{Z}$ *can be expanded as*

$$\mathrm{P}(Z \le x) = \Phi(x) - \phi(x)\left\{\Gamma_1(x)\frac{1}{\sqrt{T}} + \Gamma_2(x)\frac{1}{T}\right\} + 0(T^{-3/2}) \tag{2.4}$$

if y *comes from* Π_j, *where* $\Phi(x)$ *and* $\phi(x)$ *are the cdf and the pdf of* $N(0, 1)$ *respectively, and*

$$\Gamma_1(x) = \rho^{-3}\gamma_2(x^2-1), \tag{2.5}$$

$$\Gamma_2(x) = \frac{1}{2}\rho^{-6}\{\gamma_2^2 x^5 + 2(\rho^2\gamma_4 - 5\gamma_2^2)x^3 + 5(3\gamma_2^2 - 2\rho^2\gamma_4)x\}.$$

COROLLARY. *The distribution function of V when* y *comes from* Π_j *can be approximated as*

$$\mathrm{P}(V \le v) \cong \Phi(x_v) - \phi(x_v)\left\{\Gamma_1(x_v)\frac{1}{\sqrt{T}} + \Gamma_2(x_v)\frac{1}{T}\right\} \tag{2.6}$$

for large T, where

$$x_v = \frac{1}{\rho\sqrt{T}}\left\{v + \frac{1}{2}\left(T\log\frac{\sigma_1^2}{\sigma_2^2} - \log\frac{1-\alpha_1^2}{1-\alpha_2^2}\right) - \mu\right\}. \tag{2.7}$$

Let $\mathrm{P}(i|j)$ be the probability of misclassifying y into Π_i when it comes in fact from Π_j $(i \ne j)$. Then

$$\mathrm{P}(1|2) = 1 - \mathrm{P}(V \le k|\Pi_2),$$
$$\mathrm{P}(2|1) = \mathrm{P}(V \le k|\Pi_1).$$

Therefore, the corollary can be used for the evaluation of P(1|2) and P(2|1).

3. Derivation of the asymptotic expansion

To obtain the asymptotic expansion of $\tilde{Z}$ we consider the characteristic function of $Z_1=(1/\sqrt{T})Z$,

$$\psi_1(t) = E[\exp\{(it/\sqrt{T})Z\}] \tag{3.1}$$

when y comes from Π_j. By (1.2) we can write $\psi_1(t)$ as

$$\psi_1(t) = (1-\alpha_j^2)^{1/2}|Q|^{-1/2} \tag{3.2}$$

where

$$Q = \begin{pmatrix} r & q & & & 0 \\ q & p & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & p & q \\ 0 & & & q & r \end{pmatrix}$$

with $r=1+c_1itT^{-1/2}$, $q=-\alpha_j-c_2itT^{-1/2}$, $p=1+\alpha_j^2+c_3itT^{-1/2}$ and c_i's are given by (2.1). Following arguments similar to Ochi [5], we expand $\psi_1(t)$ in a power series with respect to $(1/\sqrt{T})$. We use the formula (cf. Anderson [2], Ochi [5])

$$|Q| = \frac{1}{x_1-x_2}[\{(r^2-q^2)x_1+(p-2r)q^2\}x_1^{T-2} - \{(r^2-q^2)x_2+(p-2r)q^2\}x_2^{T-2}], \tag{3.3}$$

where $x_1=\frac{1}{2}(p+\sqrt{p^2-4q^2})$ and $x_2=\frac{1}{2}(p-\sqrt{p^2-4q^2})$. Using the definition of p and q, we can see $x_1=1+0(T^{-1/2})$ and $x_2=\alpha_j^2+0(T^{-1/2})$. Noting that x_2^{T-2} has higher order convergence to zero than any fixed power of T, we have

$$\log\psi_1(t) = \frac{1}{2}\log(1-\alpha_j^2) + \frac{1}{4}\log(p^2-4q^2) - \frac{1}{2}\log\{(r^2-q^2)x_1+(p-2r)q^2\} + \left(1-\frac{T}{2}\right)\log x_1 + 0(T^{-3/2}). \tag{3.4}$$

Using the definition of p, q and r and expanding each term in the right hand side of (3.4), we obtain

$$\log\psi_1(t) = \frac{1}{2}a^{-1}(2\alpha_jc_2-c_3)it\sqrt{T} + \frac{1}{2}\gamma_0^2(it)^2 + \{\gamma_1(it)+\gamma_2(it)^3\}T^{-1/2} + \{\gamma_3(it)^2+\gamma_4(it)^4\}T^{-1} + 0(T^{-3/2}). \tag{3.5}$$

Therefore the characteristic function of $\tilde{Z}$ can be written as

$$\begin{aligned}\psi(t) &= \exp\{-it\mu/(\rho\sqrt{T})\}\psi_1(t/\rho) \\ &= \exp\left(-\frac{1}{2}t^2\right)[1+g_1(it)^3T^{-1/2} + \{g_2(it)^4+g_3(it)^6\}T^{-1}] + 0(T^{-3/2}),\end{aligned} \tag{3.6}$$

where $g_1=\gamma_2\rho^{-3}$, $g_2=\gamma_4\rho^{-4}$ and $g_3=\frac{1}{2}\gamma_2^2\rho^{-6}$. Now we invert the characteristic function (3.6) term by term. We note

$$\int_{-\infty}^{x}\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\exp(-itu)\exp\left(-\frac{1}{2}t^2\right)(it)^j dt\right\}du = (-1)^j\phi^{(j-1)}(x) = -h_{j-1}(x)\phi(x), \tag{3.7}$$

where $h_j(x)$'s are Hermite polynomials; $h_2(x)=x^2-1$, $h_3(x)=x^3-5x$, $h_5(x)=x^5-10x^3+15x$. Using (3.7), we obtain the expression (2.4) with $\Gamma_1(x)=g_1h_2(x)$ and $\Gamma_2(x)=g_2h_3(x)+g_3h_5(x)$. It is easy to see that the expressions $\Gamma_1(x)$ and $\Gamma_2(x)$ are the same as the ones given by (2.5), which proves (2.4).

Acknowledgement

We would like to thank Professor Y. Fujikoshi, Hiroshima University, for his guidance and valuable advice.

References

[1] T. W. Anderson, An Introduction to Multivariate Statistical Analysis, Wiley, New York, 1958.

[2] T. W. Anderson, The Statistical Analysis of Time Series, Wiley, New York, 1971.

[3] N. Kligienė, The probability of the error in the Bayes classifying procedure of the random sequences, Statistical Problem of Control **19**, Institute of Mathematics and Cybernetics of the Academy of Sciences of Lithuanian SSR, Vilnius, 1977, 59–79 (in Russian).

[4] M. Krzyśko, Asymptotic distribution of the discriminant function, Statistics & Probability Letters **1**, 1983, 243–250.

[5] Y. Ochi, Asymptotic expansion for the distribution of an estimator in the first-order autoregressive process, Journal of Time Series Analysis **4**, 1983, 56–67.

Department of Mathematics,
Faculty of Science,
Hiroshima University
and
Department of Industrial Engineering,
Kyuing Nam Junior College of Technology