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Nonoscillation of nonlinear first order differential equations with forcing term

Hiroshi ONOSE

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1. Introduction

We consider the following first order differential equations

$$x'(t) + p(t)f(x(t)) = q(t), \quad t \geqq a, \tag{1}$$

and

$$x'(t) + p(t)f(x(t)) = 0, \quad t \geqq a, \tag{2}$$

where $p \in C[[a, \infty), R]$, $q \in C[[a, \infty), R]$ and $f \in C[R, R]$, $R=(-\infty, \infty)$. A solution $x(t)$ of (1) is called *oscillatory* if $x(t)$ has zeros for arbitrarily large t; otherwise, a solution $x(t)$ is said to be *nonoscillatory*. Equation (1) is nonoscillatory if every solution of (1) is nonoscillatory. The oscillatory properties of the first order functional differential equation

$$x'(t) + p(t)f(x(t-\tau)) = q(t) \tag{3}$$

are investigated by some authors (Cf. [3] and [4]). But there is scarce litrature on the ordinary case of (1) (Cf. [5]). In this paper, we mainly propose a theorem for nonoscillation of (1).

2. The unforced case

THEOREM 1. *Suppose that* $f(x)=0$ *for* $x=0$, $f(x)\neq 0$ *for* $x\neq 0$ *and* $|p(t)|>0$ *on* $[a, \infty)$. *Then every solution of* (2) *has at most one zero.*

PROOF. Assume that $x(t)$ is a solution of (2) which has two consecutive zeros t_1, t_2 with the property

$$x(t_1) = x(t_2) = 0 \qquad \text{for} \quad a \leqq t_1 < t_2.$$

Let $|x(t)|>0$ for $t_1<t<t_2$. By Rolle's theorem, we can take a τ such that $x'(\tau)=0$, $t_1<\tau<t_2$. From (2) we obtain

$$0 = x'(\tau) = -p(\tau)f(x(\tau)).$$

By this and $|p(t)|>0$ for $t\in[a, \infty)$ we have $x(\tau)=0$. This is a contradiction.
Q. E. D.

EXAMPLE 1. Consider the ordinary differential equation

$$x'(t) - 9(\cos t)(\sin t)^7 x(t)^{1/9} = 0, \quad t \geqq \tau. \tag{4}$$

The conditions of Theorem 1 are violated. In fact a solution $x(t)=(\sin t)^9$ of (4) is oscillatory.

EXAMPLE 2. Consider the differential equation

$$x'(t) - x(t)^{1/3} = 0, \quad t \geqq 0. \tag{5}$$

A solution $x(t)=(2t/3)^{3/2}$ of (5) has one zero at $t=0$, which is seen by Theorem 1.

EXAMPLE 3. Consider the differential equation

$$x'(t) - t(1+x(t)^2) = 0, \quad t \geqq -5. \tag{6}$$

A function $x(t)=\tan(t^2/2-\pi/6)$ is a solution of (6) and has zeros more than two. Since $f(x)\equiv 1+x^2$, (6) is not examined by Theorem 1.

3. The forced case

THEOREM 2. *If there exist* $\alpha\in(0, 1]$ *and* $K>0$ *such that*

$$|f(u)| \leqq K|u|^{\alpha} \qquad \textit{for all} \quad u \in R, \tag{7}$$

$$\int_a^{\infty} |p(t)|dt < \infty, \quad \left|\int_a^{\infty} q(t)dt\right| < \infty \tag{8}$$

and

$$\liminf_{t\to\infty}\left|\int_t^{\infty} q(s)ds\right| \Big/ \left(\int_t^{\infty} |p(s)|ds\right) = \beta, \ \beta\in(0, \infty], \tag{9}$$

then every local solution of (1) *is extendable to* $+\infty$ *and every nontrivial solution of* (1) *is bounded and nonoscillatory.*

PROOF. I) Extendability

The case of $\alpha=1$: Let $x(t)$, $t\in[t_0, T)\to R$, $t_0\geqq a$, be a local solution of (1) with $T(<\infty)$ the right-end boundary point of its maximal interval of existence. Integration of (1) from t_0 to t, $t_0\leqq t<T$, yields

$$|x(t)| \leqq |x(t_0)| + \int_{t_0}^{t} K|p(s)|\,|x(s)|ds + K_1, \tag{10}$$

where $K_1 = \left|\int_{t_0}^{\infty} q(t)dt\right|$. An application of Gronwall's inequality in (10) implies

$$|x(t)| \leqq (|x(t_0)| + K_1)\exp\left\{\int_{t_0}^{t} K|p(s)|ds\right\}. \tag{11}$$

This inequality implies that the solution $x(t)$ of (1) is extendable to the point $t=T$. This is a contradiction. That is, all local solutions of equation (1) are extendable to $+\infty$. From (8) and (11), we obtain

$$|x(t)| \leqq (|x(t_0)| + K_1)\exp\left\{\int_{t_0}^{\infty} K|p(s)|ds\right\} < K_2 < \infty, \tag{12}$$

for $t \geqq t_0$, where $K_2 > 1$ is a constant.

The case of $0<\alpha<1$: By the same argument as the case of $\alpha=1$, we obtain

$$\begin{aligned} |x(t)| &\leqq |x(t_0)| + \int_{t_0}^{t} K|p(s)|\,|x(s)|^{\alpha}ds + \left|\int_{t_0}^{\infty} q(s)ds\right| \\ &\leqq A + B\int_{t_0}^{t} |p(s)|\,|x(s)|^{\alpha}ds, \end{aligned} \tag{13}$$

where $A = \max\left\{\left(|x(t_0)| + \left|\int_{t_0}^{\infty} q(s)ds\right|\right), 1\right\}$ and $B = \max\{K, 1\}$.

By applying of Lemma of Dhongade and Deo [1] as $v>1$ in it, we have

$$|x(t)| \leqq AB[1 + F^{-1}(B^{\alpha}\int_{t_0}^{t} |p(s)|ds)] \quad \text{for} \quad t \geqq t_0, \tag{14}$$

where $F(t) \equiv \int_{t_0}^{t} (1+s)^{-\alpha}ds$. This inequality implies that the solution $x(t)$ of (1) is extendable to the point T. This is also a contradiction. Thus all local solutions of equation (1) are extendable to $+\infty$. From (8) and (14) we obtain

$$|x(t)| \leqq AB\left[1 + F^{-1}\left(B^{\alpha}\int_{t_0}^{\infty} |p(s)|ds\right)\right] < K_3 < +\infty \tag{15}$$

for $t \geqq t_0$, where $K_3 > 1$ is a constant.

(II) Nonoscillation

The case of $0<\alpha\leqq 1$: Suppose that $x(t)$ is an oscillatory solution of (1). Let $\{t_n\}_{n=1}^{\infty}$ be the zeros of $x(t)$. From (1), we obtain

$$x(t) = -\int_{t_n}^{t} p(s)f(x(s))ds + \int_{t_n}^{t} q(s)ds. \tag{16}$$

By using (7), (12) and (15), we have

$$\int_{t_n}^{t} |p(s)f(x(s))|ds \leqq K_4 K\int_{t_n}^{t} |p(s)|ds,$$

where $K_4 = \max\{K_2, K_3^\alpha\}$, and the last integral converges as $t \to \infty$. Equation (16) implies

$$x(\infty) = -\int_{t_n}^{\infty} p(s)f(x(s))ds + \int_{t_n}^{\infty} q(s)ds,$$

where $x(\infty) = \lim_{t\to\infty} x(t)$. Since $x(t)$ is oscillatory, we must have $x(\infty)=0$. This means that

$$\int_{t_n}^{\infty} q(s)ds = \int_{t_n}^{\infty} p(s)f(x(s))ds, \qquad \text{for} \quad n = 1, 2, \dots . \tag{17}$$

Moreover, for a γ, $0<\gamma<\beta$, there exists t_1 such that

$$K|x(t)|^{\alpha} < \gamma \qquad \text{for} \quad t \geqq t_* \geqq t_0 . \tag{18}$$

From (17) and (18), we obtain

$$\left|\int_{t_n}^{\infty} q(s)ds\right| \leqq \int_{t_n}^{\infty} K|p(s)|\,|x(s)|^{\alpha}ds < \gamma \int_{t_n}^{\infty} |p(s)|ds$$

for sufficiently large n, say $t_n \geqq t_*$. This means that

$$\liminf_{n\to\infty}\left(\left|\int_{t_n}^{\infty} q(s)ds\right|\right)\Big/\left(\int_{t_n}^{\infty} |p(s)|ds\right) \leqq \gamma,$$

which is a contradiction to (9). Q.E.D.

Example 4. Consider the equation

$$x'(t) - (1/t^3)x(t) = -(1/t^2) - (1/t^4), \qquad t \geqq 10. \tag{19}$$

By using Theorem 2, we see that every nontrivial solution of (19) is nonoscillatory. In fact, $x(t) = 1/t$ is such a nonoscillatory solution of (19).

Example 5. Consider the equation

$$x'(t) + (1/t)x(t) = -t^{-1}\sin t, \qquad t \geqq \pi. \tag{20}$$

Since the conditions of Theorem 2 are violated, equation (20) may have an oscillatory solution. In fact, $x(t) = t^{-1}(\cos t - 1)$ is such an oscillatory solution of (20).

Example 6. Consider the equation

$$x'(t) + (t^{-1}\sin t)x(t) = t^{-2}(\sin t - 1), \qquad t \geqq \pi. \tag{21}$$

This equation has an oscillating forcing term, but $x(t) = 1/t$ is a nonoscillatory solution of (21).

THEOREM 3. *Suppose that* $uf(u)>0$ *for* $u\neq 0$, $p(t)>0$ *for* $t\geqq a$, $\limsup_{t\to\infty} Q_\alpha(t)=+\infty$ *and* $\liminf_{t\to\infty} Q_\alpha(t)=-\infty$, *where* $Q_\alpha(t)\equiv\int_\alpha^t q(s)ds$ *for any fixed constant* $\alpha\geqq a$. *Then every solution* $x(t)$ *of* (1) *is oscillatory.*

PROOF. Suppose that $x(t)>0$ for sufficiently large t, say $t\geqq T$. From (1), we obtain

$$x'(t) = q(t) - p(t)f(x(t)) \leqq q(t) \qquad \text{for} \quad t \geqq T.$$

By integrating this we have

$$0 < x(t) \leqq x(T) + Q_T(t) \qquad \text{for} \quad t \geqq T.$$

This is a contradiction, since $\liminf_{t\to\infty} Q_T(t)=-\infty$. If we suppose that $x(t)<0$, then we obtain

$$0 > x(t) \geqq x(\tau) + Q_\tau(t) \qquad \text{for} \quad t \geqq \tau.$$

This also leads to a contradiction, since $\limsup_{t\to\infty} Q_\tau(t)=+\infty$. Q. E. D.

EXAMPLE 7. Consider the equation

$$x'(t) + tx(t)^{1/3} = t\sin t + 3(\sin^2 t)\cos t \qquad \text{for} \quad t \geqq \tau. \tag{22}$$

Since the function

$$Q_\alpha(t)=\int_\alpha^t q(s)ds=\int_\alpha^t (u\sin u+3(\sin^2 u)\cos u)du$$
$$= -t\cos t + \sin t + \sin^3 t - \sin^3\alpha - \sin\alpha + \alpha\cos\alpha$$

satisfies the condition of Theorem 3, every solution of (22) is oscillatory. In fact $x(t)=\sin^3 t$ is such an oscillatory solution.

4. Kartsatos's conjecture

Consider the differential equation

$$x^{(n)} + \sum_{i=0}^{n-1} P_{n-i}(t, x, x', \dots, x^{(n-1)})x^{(i)} = 0. \tag{23}$$

Recently, Kartsatos [2] gave an interesting nonoscillatory result on equation (23) as follows.

THEOREM 4 ([2], Theorem). *Let* $P_{n-i}: R_+\times R^n\to R$, *where* $R=(-\infty,\infty)$ *and* $R_+=[0,\infty)$ *be such that the functions* $P_{n-i}(t, u_1, u_2,\dots, u_n)u_{i+1}$ *are continuous on* $R_+\times R^n$. *Moreover, let*

$$|P_{n-i}(t, u_1, u_2,\dots, u_n)| \leqq F_{n-i}(t),$$

where $F_{n-i}: R_+ \to R_+$ *are continuous and such that*

$$\int_0^\infty t^{n-1} e^t F_{n-i}(t) dt < \infty.$$

Then every local solution of (23) *is extendable to* $+\infty$ *and every nontrivial solution is nonoscillatory.*

In the paper, he proposes a conjecture that his result may remain true without the factor e^t in the integral condition of Theorem 4. In this section, we note that the conjecture is true in the case of $n=1$. Now, we consider the following first order differential equation which is a special case of $n=1$ in (23).

$$x'(t) + p(t, x(t))|x(t)|^\alpha \operatorname{sgn} x(t) = 0, \quad t \in [0, \infty), \tag{24}$$

where α is a constant with $0<\alpha\leqq 1$.

THEOREM 5. *Let* $p: R_+ \times R \to R$ *be such that the function* $p(t, u)u$ *is continuous on* $R_+ \times R$. *Moreover, let*

$$|p(t, u)| \leqq F(t)$$

where $F: R_+ \to R_+$ *is continuous and such that*

$$\int_0^\infty F(t) dt < \infty.$$

For the case $\alpha=1$, *every local solution of* (24) *is extendable to* $+\infty$ *and every nontrivial solution is bounded and has no zeros in its interval of existence. For the case of* $0<\alpha<1$, *every local solution of* (24) *is extendable to* $+\infty$ *and every nontrivial solution is bounded.*

PROOF. *The case* $\alpha=1$. Let $x(t)$, $t \in [t_0, T) \to R$ be a local solution of (24) with $T(<\infty)$ the right-end boundary point of its maximal interval of existence. Integration of (24) from t_0 to t, $t_0 \leqq t < T$, yields

$$|x(t)| \leqq |x(t_0)| + \int_{t_0}^t |p(s, x(s))|\,|x(s)| ds. \tag{25}$$

An application of Gronwall's inequality in (25) implies

$$|x(t)| \leqq |x(t_0)| \exp\left\{\int_{t_0}^t F(s) ds\right\}. \tag{26}$$

This appraisal implies that the solution $x(t)$ of (24) is extendable to the point $t=T$. This is a contradiction. Hence, all local solutions of the equation (24) are extendable to $+\infty$ and bounded on $t \in [t_0, \infty)$. Now, we assume that $x(t)$ is a solution of (24), and there exists $t_1 \in [t_0, \infty)$ such that $x(t_1)=0$. Taking this

t_1 instead of t_0 in (26), we have

$$|x(t)| \leqq 0 \qquad \text{for all} \quad t \in [t_1, \infty).$$

This is a contradiction.

The case of $0<\alpha<1$: Let $x(t)$, $t \in [t_0, T) \to R$ be a local solution of (24). By the same argument as in Theorem 2, we have

$$|x(t)| \leqq A_0\left[1+G^{-1}\left(\int_{t_0}^{t} F(s)ds\right)\right] < K_3 < \infty$$

where $G(t) \equiv \int_{t_0}^{t} (1+s)^{-\alpha} ds$, K_3 and A_0 are positive constants. This shows that all local solutions of (24) are extendable to $+\infty$ and bounded on $t \in [t_0, \infty)$.

Q. E. D.

EXAMPLE 8. Consider the equation

$$x'(t) - ((3\sin 2t)/2)x(t)^{1/3} = 0, \quad t \geqq 0. \tag{27}$$

The conditions of Theorem 5 are violated. In fact, $x(t) = \sin^3 t$ is an oscillatory solution of (27).

EXAMPLE 9. Consider the equation

$$x'(t) - x(t) = 0, \quad t \geqq 0. \tag{28}$$

The conditions of Theorem 5 are violated. In fact, the unbounded function e^t is a solution of (28).

REMARK. Theorem 2 is valid for the case of $q(t) \not\equiv 0$ in (1) and Therorem 5 is valid for the case of $q(t) \equiv 0$ in (1).

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Department of Mathematics,
College of General Education,
Ibaraki University