HIROSHIMA MATH. J.
16 (1986), 613–616

When does LCM-stability ensure flatness at primes of depth one?

Susumu ODA and Ken-ichi YOSHIDA

(Received January 13, 1986)

Let R be a Noetherian integral domain and let M be an R-module. We say that M is LCM-stable over R if $(aR \cap bR)M = aM \cap bM$ for any elements $a, b \in R$ (cf. [1], [5]). F. Richman [4] proved that when A is an overring of R, that is, A is an intermediate ring between R and the field of quotients $K(R)$ of R, A is flat over R if and only if A is LCM-stable over R. The obstruction ideal $\mathscr{F}_R(A)$ (cf. [3]) has only depth one prime divisors. So if A is flat over R at primes of depth one, A is flat over R. Therefore the following question will arise:

When is the LCM-stable R-module M flat over R at each prime of depth one?

It is known that there is a module which is flat over a Noetherian normal domain R at each prime of depth one but is not LCM-stable over R. Our objective is to prove the following result which shows that the LCM-stable module over a Noetherian integral domain is not necessarily flat at primes of depth one:

Let R be a Noetherian integral domain and let M be a torsion-free, finite R-module. Assume that M is LCM-stable over R. Then M is reflexive if and only if $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for each $\mathfrak{p} \in Dp_1(R)$ $(:=\{\mathfrak{p} \in Spec\ R \mid \text{depth } R_{\mathfrak{p}}=1\})$, i.e., M is flat over R at primes of depth one.

The following notation is fixed throughout this paper:

R denotes a (commutative) Noetherian integral doamin,
K the field of quotients of R,
$\bar{R}$ the integral closure of R in K and
M a non-zero torsion-free finite R-module.

We start with the following definition.

1. DEFINITION. Regard M as an R-submodule of $M_K := M \otimes_R K$. Define $\mathscr{R}(M)$ by

$$\mathscr{R}(M) := \{\alpha \in K \mid \alpha M \subseteq M\}.$$

2. ROPOSITION. *$\mathscr{R}(M)$ is an integral domain which contains R and is integral over R.*

PROOF. It is obvious that $\mathscr{R}(M)$ is an integral domain which contains R.

Let $\{m_1,\dots, m_n\}$ be a set of generators of M. For any $\alpha \in \mathscr{R}(M)$, $\alpha m_i = \sum a_{ij} m_j$ $(a_{ij} \in R)$. Thus $det(\alpha\delta_{ij} - a_{ij}) = 0$, where δ_{ij} is Kronecker symbol, since M is torsion-free. This yields an integral dependence of α over R. Q. E. D.

3. We call $\mathscr{R}(M)$ a full coefficient ring of an R-module M. R is said to be *full* on M if $\mathscr{R}(M) = R$.

4. DEFINITION. An R-module N is called *LCM-stable* over R if $(aR \cap bR)M = aM \cap bM$ for any elements a, $b \in R$.

5. PROPOSITION. *If M is LCM-stable over R, R is full on M.*

PROOF. For $\alpha \in \mathscr{R}(M)$, put $I_\alpha = \{a \in R | \alpha a \in R\}$, which is a non-zero ideal of R. Then we have that $\alpha \in R$ if and only if $I_\alpha = R$. Suppose that $I_\alpha \neq R$ and put $\alpha = b/a$ $(a, b \in R)$. It is easy to see that $I_\alpha = (a/b)R \cap R$. By the LCM-stability of M, $(aR \cap bR)M = aM \cap bM$. This yields $(R \cap (a/b)R)M = (a/b)M \cap M$. Hence since $\alpha = b/a \in \mathscr{R}(M)$, we have $(b/a)M \subseteq M$ and hence $M \subseteq (a/b)M$. So $I_\alpha M = M$. Since M is a non-zero torsion-free finite R-module, we have $I_\alpha = 0$, which is absurd. Hence $I_\alpha = R$ and consequently $\alpha \in R$. Q. E. D.

6. Let N be an R-module and $N^*: = Hom_R(N, R)$ an R-dual of N. If N is torsion-free over R, a canonical R-homomorphism $N \to N^{**}$ is injective. N is called reflexive if this canonical homomorphism is bijective.

7. REMARK. Let N, N_1, N_2 be R-modules. Then it is easy to see that: (i) $N_1 \oplus N_2$ is LCM-stable (resp. reflexive) over R if and only if both N_1 and N_2 are LCM-stable (resp. reflexive) over R.

(ii) N is LCM-stable (resp. reflexive) over R if and only if so is $N_\mathfrak{p}$ for any $\mathfrak{p} \in Spec\ R$.

The next result will be required in the proof of Theorem 9 below.

8. PROPOSITION ([6]). *Assume that $\bar{R}$ is a finite R-module. Then for $\mathfrak{p} \in Dp_1(R)$, either $\mathfrak{p} \in Ass_\mathfrak{p}(\bar{R}/R)$ or $R_\mathfrak{p}$ is a discrete valuation ring.*

9. THEOREM. *Assume that $\bar{R}$ is a finite R-module and that M is LCM-stable over R. Then the following statements are equivalent:*

(i) *M is reflexive,*

(ii) *$M_\mathfrak{p}$ is reflexive for any $\mathfrak{p} \in Dp_1(R)$,*

(iii) *$M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for any $\mathfrak{p} \in Dp_1(R)$.*

PROOF. (i)→(iii): Take $\mathfrak{p} \in Dp_1(R)$. If $R_\mathfrak{p}$ is a discrete valuation ring, $M_\mathfrak{p}$ is flat (free) over $R_\mathfrak{p}$ because M is a torsion-free finite R-module. We assume that $R_\mathfrak{p}$ is not a discrete valuation ring. By Proposition 8, $\mathfrak{p} \in Ass_R(\bar{R}/R)$. We may assume that R is a local ring with the maximal ideal $\mathfrak{m} \in Dp_1(R)$. Let

$A=\{\alpha\in K|I_\alpha=R$ or $I_\alpha=\mathfrak{m}\}$. Then A is an overring of R and integral over R (cf. the proof of Proposition 3). It is easy to see that the conductor $\mathscr{C}(A/R)=\mathfrak{m}$. For $f\in M^*=Hom_R(M, R)$, if $f(M)\not\subseteq\mathfrak{m}$ then $f(M)=R$ and hence R is a direct summand of M. Let $M=M'\oplus R\oplus\cdots\oplus R$, where M' does not contain R as a direct summand. We shall show that $M'=0$. Suppose the contrary. We may assume that M does not contain R as a direct summand. Since M is LCM-stable, we have $\mathscr{R}(M)=R$ by Proposition 5. If we suppose that $\phi\in M^{**}=Hom_R(M^*, R)$ is such that $\phi(M^*)\not\subseteq\mathfrak{m}$, then $\phi(M^*)=R$. So M^* contains R as a direct summand and hence $M^{**}=M$ contains R as a direct summand, which is absurd. So for any $\phi\in M^{**}$, we have $\phi(M^*)\subseteq\mathfrak{m}$. Since $\mathscr{C}(A/R)=\mathfrak{m}$, for any $\alpha\in A$, $\alpha\phi(M^*)\subseteq R$. This implies that $\mathscr{R}(M^{**})\supseteq A$. But since $\mathscr{R}(M^{**})=\mathscr{R}(M)=R$, we have $A=R$, that is, $\mathfrak{m}=\mathscr{C}(A/R)=R$, a contradiction.

(iii)→(i): Since $M\subseteq M\otimes_R K$ and $M\otimes_R K$ is a K-vector space, we have $M\subseteq M^{**}\subseteq M\otimes_R K$. Suppose that $M\subsetneq M^{**}$. For any $\mathfrak{p}\in Ass_R(M^{**}/M)$, we have *depth* $M_\mathfrak{p}=1$. [Indeed, suppose *depth* $M_\mathfrak{p}>1$. Then there exist $a, b\in\mathfrak{p}$ such that a, b is an $M_\mathfrak{p}$-sequence. so $aM_\mathfrak{p}\cap bM_\mathfrak{p}=abM_\mathfrak{p}$. Since $\mathfrak{p}\in Ass_R(M^{**}/M)$, there exists $m\in M^{**}$ with $Ann_\mathfrak{p}\overline{m}=\mathfrak{p}R_\mathfrak{p}$, where $\overline{m}$ denotes the residue class of m in M^{**}/M. Since $a, b\in\mathfrak{p}$, both am and bm belong to M. Hence $abm\in aM_\mathfrak{p}\cap bM_\mathfrak{p}=abM_\mathfrak{p}$. Consequently, $m\in M_\mathfrak{p}$, which contradicts the choice of m.] Since M is LCM-stable, *depth* $M_\mathfrak{p}=1$ implies *depth* $R_\mathfrak{p}=1$. [Indeed, suppose *depth* $R_\mathfrak{p}>1$. There exist $a, b\in\mathfrak{p}$ such that a, b is an $R_\mathfrak{p}$-sequence. So $aR_\mathfrak{p}\cap bR_\mathfrak{p}=abR_\mathfrak{p}$. Thus $(aR_\mathfrak{p}\cap bR_\mathfrak{p})M_\mathfrak{p}=abM_\mathfrak{p}$. As $M_\mathfrak{p}$ is LCM-stable over $R_\mathfrak{p}$, $abM_\mathfrak{p}=aM_\mathfrak{p}\cap bM_\mathfrak{p}$. But since *depth* $M_\mathfrak{p}=1$, the homothety:

$$M_\mathfrak{p}/aM_\mathfrak{p}\xrightarrow{\cdot b\cdot}M_\mathfrak{p}/aM_\mathfrak{p}$$

is not injective, which implies that $aM_\mathfrak{p}\cap bM_\mathfrak{p}\supsetneq abM_\mathfrak{p}$, which is absurd.] Since $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$, $M_\mathfrak{p}^{**}=(M^{**})_\mathfrak{p}=M_\mathfrak{p}$. Hence $\mathfrak{p}\notin Ass_R(M^{**}/M)$, which contradicts the choice of $\mathfrak{p}$.

(i)⇔(ii) is obvious. Q.E.D.

10. PROPOSITION. *Assume that $\overline{R}$ is a finite R-module. If both M and $M^*=Hom_R(M, R)$ are LCM-stable over R, then M is reflexive over R.*

PROOF. By Theorem 9, we have only to show that $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for any $\mathfrak{p}\in Dp_1(R)$. Suppose the contrary. Then there exists $\mathfrak{p}\in Ass_R(\overline{R}/R)$. Delete a direct summand $R\oplus\cdots\oplus R$ of M if necessary. We may assume that for any $f\in M_\mathfrak{p}^*, f(M_\mathfrak{p})\subsetneq R_\mathfrak{p}$. Then we have $\mathscr{R}(M_\mathfrak{p}^*)\supsetneq R_\mathfrak{p}$. [Indeed, let $f_1,\dots,f_n$ be generators of $M_\mathfrak{p}^*$ and put $I=f_1(M_\mathfrak{p})+\cdots+f_n(M_\mathfrak{p})\subseteq\mathfrak{p}R_\mathfrak{p}$. Take a non-zero element $a\in I$ with $I\not\subseteq aR_\mathfrak{p}$. Then there exists $b\in R_\mathfrak{p}$ / $aR_\mathfrak{p}$ such that $bI\subseteq aR_\mathfrak{p}$ because $\mathfrak{p}\in Dp_1(R)$. Thus $b/a\in K$ / $R_\mathfrak{p}$ and $(b/a)I\subseteq R$. Hence $(b/a)M_\mathfrak{p}^*\subseteq M_\mathfrak{p}^*$. So

$b/a \in \mathscr{R}(M_{\mathfrak{p}}^{*}) / R_{\mathfrak{p}}$.] But M^* is LCM-stable over R and hence $M_{\mathfrak{p}}^{*}$ is LCM-stable over $R_{\mathfrak{p}}$, which is absurd (Proposition 5). Q.E.D.

11. COROLLARY. *Assume that $\bar{R}$ is a finite R-module. If both M and M^* are LCM-stable over R, then $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in Dp_1(R)$.*

PROOF. By Proposition 10, M is reflexive. The conclusion follows from Theorem 9. Q.E.D.

Now we make preparations for Theorem 15 below which was our main target.

12. Let A be a ring extension of R. The following ideal is introduced in [3]:

$$\mathscr{F}_R(A) := \{a \in R \mid a \neq 0,\ A[1/a] \text{ is flat over } R[1/a]\} \cup \{0\}.$$

This ideal is called the obstruction ideal of flatness.

13. An integral domain A is said to be a locally simple extension of R if for each prime ideal $\mathfrak{p}$ of R, there exists an element α of A such that $A_{\mathfrak{p}} = R_{\mathfrak{p}}[\alpha]$.

14. PROPOSITION ([3]). *Let A be a finite extension of R. If A is locally simple over R, then each prime divisor of $\mathscr{F}_R(A)$ is of depth one, i.e., depth $R_{\mathfrak{p}} = 1$ for any prime divisor of $\mathscr{F}_R(A)$.*

Combining Proposition 14 with Theorem 9, we have the following result:

15. THEOREM. *Assume that $\bar{R}$ is a finite R-module. Let A be a finite, locally simple extension of R. Then if A is reflexive and LCM-stable over R, A is flat over R.*

References

[1] R. Gilmer, Finite element factorization in group rings, Lec. Notes in Pure and Applied Math. **7**, Marcel Dekker, Inc., New York, 1974.

[2] H. Matsumura, Commutative Algebra, W. A. Benjamin, New York, 1970.

[3] S. Oda and K. Yoshida, Obstruction ideals of flatness and etale, Math. Rep. Toyama Univ. **7** (1984), 67–78.

[4] F. Richman, Generalized quotient rings, Proc. Amer. Math. Soc., **16** (1965), 794–799.

[5] K. Yoshida, LCM-stability and flatness, Bull. Fac. Sci., Ibaraki Univ. **17** (1985), 45–47.

[6] ———, On birational-integral extension of rings and prime ideals of depth one, Japan J. Math. **8** (1982), 49–70.

Uji-Yamada High School
(Uraguchi, Ise, Mie)

and

Department of Applied Mathematics,
Okayama University of Science