Global existence of solutions of mixed sublinear-superlinear differential equations

Takaŝi KUSANO and William F. TRENCH

(Received November 16, 1985)

1. Introduction

Let L_n be the general disconjugate operator

$$(1) \qquad L_n = \frac{1}{p_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \frac{\cdot}{p_0(t)} \qquad (n \geq 2),$$

where $p_i: [0, \infty) \to (0, \infty)$, $0 \leq i \leq n$, are continuous. Let (1) be in canonical form [10] at ∞; i.e.,

$$(2) \qquad \int_0^\infty p_i(t)dt = \infty, \qquad 1 \leq i \leq n - 1.$$

Consider the mixed sublinear-superlinear differential equation

$$(3) \qquad L_n y + [a(t)|y|^\alpha + b(t)|y|^\beta] \operatorname{sgn} y = 0, \qquad t > 0,$$

where

$$(4) \qquad 0 < \alpha < 1, \qquad \beta > 1,$$

and $a, b: [0, \infty) \to R$ are continuous.

In Theorem 1 we give conditions which imply that (3) has a solution $\hat{y}$ which is defined on the entire interval $[0, \infty)$ and behaves as $t \to \infty$ like a prescribed solution of the unperturbed equation

$$(5) \qquad L_n y = 0, \qquad t > 0.$$

This type of global existence problem for equations of the form $L_n y + f(t, y) = 0$ has recently been studied by the present authors [9]. The theory developed in [9] covers the sublinear case $(b(t) \equiv 0)$ as well as the superlinear case $(a(t) \equiv 0)$, but not the mixed sublinear-superlinear case $(a(t) \not\equiv 0$ and $b(t) \not\equiv 0)$. In this paper a device is presented which enables us to demonstrate the existence of global solutions of the mixed sublinear-superlinear equation (3).

By means of a similar device we find conditions which imply that the mixed sublinear-superlinear elliptic equation

$$(6) \qquad \Delta u + \varphi(|x|)u^\lambda + \psi(|x|)u^\mu = 0, \qquad x \in R^N, \quad N \geq 3,$$

has a positive entire (i.e., defined for all x in $\boldsymbol{R}^N$) solution $\hat{u}$ such that

$$\lim_{|x|\to\infty} |x|^{N-2}\hat{u}(x) = c \ (>0).$$

Here $x = (x_1, ..., x_N)$, $|x|$ is the Euclidean length of x, $\varDelta$ is the N-dimensional Laplacian,

(7) $0 < \lambda < 1, \quad \mu > 1,$

and $\varphi, \psi: [0, \infty) \to (0, \infty)$ are continuous. The existence of decaying positive entire solutions for second order semilinear elliptic equations has been the subject of recent intensive investigations; see, e.g. [1–8]. However, the results obtained previously are not applicable to (6) in the mixed sublinear-superlinear case (7).

2. Ordinary differential equations

We define the iterated integrals

$$I_0 = 1,$$

$$I_j(t, s; q_j, ..., q_1) = \int_s^t q_j(r) I_{j-1}(r, s; q_{j-1}, ..., q_1) dr, \quad s, \quad t \geqq 0, j \geqq 1,$$

where $q_1, q_2, ...$ are locally integrable on $[0, \infty)$, and put

(8) $y_i(t) = p_0(t) I_{i-1}(t, 0; p_1, ..., p_{i-1}), \quad 1 \leqq i \leqq n,$

(9) $z_i(t) = p_n(t) I_{n-i}(t, 0; p_{n-1}, ..., p_i), \quad 1 \leqq i \leqq n.$

It is easily seen that $y_i(t)$, $1 \leqq i \leqq n$, form a fundamental system for (5) on $[0, \infty)$, and that $z_i(t)$, $1 \leqq i \leqq n$, are similarly related to the formal adjoint equation

$$L_n^* z = \frac{1}{p_0(t)} \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \frac{z}{p_n(t)} = 0.$$

We also introduce the notation:

$$\hat{J}_i(t; Q) = \int_t^\infty p_n(s) I_{n-i}(t, s; p_i, ..., p_{n-1}) Q(s) ds,$$

and

$$J_i(t; Q) = p_0(t) \hat{J}_i(t; Q) \quad \text{if} \quad i = 1,$$

$$J_i(t; Q) = p_0(t) I_1(t, 0; p_1 \hat{J}_i(\cdot; Q)) \quad \text{if} \quad i = 2,$$

$$J_i(t; Q) = p_0(t) I_{i-1}(t, 0; p_1, ..., p_{i-2}, p_{i-1} \hat{J}_i(\cdot; Q)) \quad \text{if} \quad 3 \leqq i \leqq n.$$

We need the following special case of Lemma 2 from [9]. It is here that (2) is required.

LEMMA 1. *If $Q \in C[0, \infty)$ and*

(10)
$$\int^{\infty} z_i(s)|Q(s)|ds < \infty,$$

then $J_i(t; Q)$ is defined for $t \geq 0$ and

(11)
$$L_n J_i(t; Q) = - Q(t);$$

moreover,

(12)
$$|J_i(t; Q)| \leq y_i(t) \int_0^{\infty} z_i(s)|Q(s)|ds, \quad t \geq 0,$$

and

(13)
$$J_i(t; Q) = o(y_i(t)), \quad t \to \infty.$$

The following theorem is the main result of this section.

THEOREM 1. *Suppose that*

(14)
$$A = \int_0^{\infty} |a(t)|y_i^{\alpha}(t)z_i(t)dt < \infty$$

and

(15)
$$B = \int_0^{\infty} |b(t)|y_i^{\beta}(t)z_i(t)dt < \infty$$

for some i, $1 \leq i \leq n$, and define

(16)
$$\gamma = A^{(\beta-1)/(\beta-\alpha)}B^{(1-\alpha)/(\beta-\alpha)}\left[\left(\frac{\beta-1}{1-\alpha}\right)^{(1-\alpha)/(\beta-\alpha)} + \left(\frac{1-\alpha}{\beta-1}\right)^{(\beta-1)/(\beta-\alpha)}\right].$$

Suppose also that $\gamma < 1$, choose θ so that

(17)
$$\gamma < \frac{\theta}{1+\theta} < 1,$$

and define

(18)
$$c(\theta) = \frac{1}{1+\theta}\left[\frac{(1-\alpha)A}{(\beta-1)B}\right]^{1/(\beta-\alpha)}.$$

Then, if c is a positive constant sufficiently close to $c(\theta)$, (3) has a solution $\hat{y}$ on $[0, \infty)$ such that

$$|\hat{y}(t)-cy_i(t)| \leq \theta cy_i(t), \quad t \geq 0,$$

and

(19) $$\lim_{t\to\infty} \frac{\hat{y}(t)}{y_i(t)} = c.$$

PROOF. The set of functions

$$Y = \{y \in C[0, \infty): |y(t)-cy_i(t)| \leqq \theta c y_i(t), \quad t \geqq 0\},$$

where $c>0$ is a constant to be specified later, is a closed convex subset of the Fréchet space $C[0, \infty)$ of continuous functions on $[0, \infty)$ with the topology of uniform convergence on compact subintervals of $[0, \infty)$. Define the mapping $\mathscr{F}$ by

$$\mathscr{F}y(t) = cy_i(t) + J_i(t; [a|y|^\alpha + b|y|^\beta]\text{sgn } y).$$

Now suppose that $y \in Y$. Then

$$|a|y|^\alpha + b|y|^\beta| \leqq |a| [c(1+\theta)y_i]^\alpha + |b| [c(1+\theta)y_i]^\beta;$$

hence (14) and (15) imply (10) with

(20) $$Q = [a|y|^\alpha + b|y|^\beta]\text{sgn } y.$$

Therefore $\mathscr{F}y$ is defined on $[0, \infty)$ and satisfies the inequality

$$|\mathscr{F}y(t)-cy_i(t)| \leqq [c^\alpha(1+\theta)^\alpha A + c^\beta(1+\theta)^\beta B]y_i(t), \quad t \geqq 0.$$

(see (12) with Q as in (20).)   Consequently, $\mathscr{F}$ maps Y into itself if

$$c^\alpha(1+\theta)^\alpha A + c^\beta(1+B)^\beta B \leqq \theta c,$$

or, equivalently, if

$$c^{\alpha-1}(1+\theta)^\alpha A + c^{\beta-1}(1+\theta)^\beta B = f(c) \leqq \theta.$$

It is elementary to verify that if (4) holds, then $f(c)$ assumes the minimum value

$$f(c(\theta)) = \gamma(1+\theta) < \theta$$

(cf. (16) and (17)) at the point $c(\theta)$ defined in (18).   Therefore, $\mathscr{F}(Y)\subset Y$ if c is sufficiently close to $c(\theta)$.

It is not difficult to show that $\mathscr{F}$ is continuous and $\mathscr{F}(Y)$ is relatively compact with respect to the topology of $C[0, \infty)$.   The Schauder-Tychonoff fixed point theorem then implies that $\mathscr{F}$ has a fixed point $\hat{y}$ in Y.   Since

$$\hat{y}(t) = cy_i(t) + J_i(t; [a|\hat{y}|^\alpha + b|\hat{y}|^\beta]\text{sgn } \hat{y}),$$

(11) with $Q=[a|\hat{y}|^\alpha + b|\hat{y}|^\beta]\text{sgn } \hat{y}$ implies that $\hat{y}$ satisfies (3), and (13) implies (19).   This completes the proof.

REMARK 1. If (17) is replaced by

$$\gamma = \frac{\theta}{1+\theta} < 1$$

(i.e., if $\theta = \gamma/(1-\gamma)$), then the argument given above, with $c = c(\theta)$, shows that (3) has a solution $\hat{y}$ on $[0, \infty)$ such that

$$|\hat{y}(t) - (1-\gamma)Ky_i(t)| \leq \gamma Ky_i(t), \qquad t \geq 0,$$

and

$$\lim_{t \to \infty} \frac{\hat{y}(t)}{y_i(t)} = (1-\gamma)K,$$

where

$$K = \left[\frac{(1-\alpha)A}{(\beta-1)B} \right]^{1/(\beta-\alpha)}.$$

REMARK 2. If $\gamma < 1/2$, then choosing θ so that $\gamma \leq \theta/(1+\theta) < 1/2$ yields a solution $\hat{y}$ which is positive on $[0, \infty)$ if $i = 1$, or on $(0, \infty)$ if $2 \leq i \leq n$.

If $p_0 = \cdots = p_n = 1$, then (3) reduces to

(21) $$y^{(n)} + [a(t)|y|^\alpha + b(t)|y|^\beta]\text{sgn } y = 0,$$

while (8) and (9) reduce to

$$y_i(t) = \frac{t^{i-1}}{(i-1)!}, \quad z_i(t) = \frac{t^{n-i}}{(n-i)!}.$$

Theorem 1 implies the following corollary, in which we have taken $j = i-1$ for convenience.

COROLLARY 1. *Consider the equation* (21) *with* $a, b: [0, \infty) \to \mathbf{R}$ *and* α, β *as in* (4). *Suppose that*

$$C = \int_0^\infty t^{n-j-1+j\alpha}|a(t)|dt < \infty$$

and

$$D = \int_0^\infty t^{n-j-1+j\beta}|b(t)|dt < \infty$$

for some j, $0 \leq j \leq n-1$. *Define*

$$\delta = C^{(\beta-1)/(\beta-\alpha)}D^{(1-\alpha)/(\beta-\alpha)}\left[\left(\frac{\beta-1}{1-\alpha}\right)^{(1-\alpha)/(\beta-\alpha)} + \left(\frac{1-\alpha}{\beta-1}\right)^{(\beta-1)/(\beta-\alpha)} \right].$$

Suppose also that

(22) $\delta < j!(n-j-1)!$,

choose θ so that

$$\frac{\delta}{j!(n-j-1)!} < \frac{\theta}{1+\theta} < 1,$$

and define

$$c^*(\theta) = \frac{1}{1+\theta}\left[\frac{(1-\alpha)C}{(\beta-1)D}\right]^{1/(\beta-\alpha)}.$$

Then, if c is a positive constant sufficiently close to c*(θ), (21) has a solution ŷ on [0, ∞) such that

(23) $|\hat{y}(t)-ct^j| \leq \theta ct^j, \qquad t \geq 0,$

and

(24) $\lim_{t\to\infty}\frac{\hat{y}(t)}{t^j} = c.$

REMARK 3.   If (22) is replaced by

$$\delta < \frac{j!(n-j-1)!}{2},$$

then (21) has solutions which are positive on (0, ∞) and satisfy (23) and (24) for suitable $c > 0$.

3.   Partial differential equations

We now consider the elliptic partial differential equation (6), and show that a technique similar to the one used in the preceding section can be applied to give conditions implying that (6) has an entire radially symmetric solution which is positive for all x in R^N and decays to zero as $|x| \to \infty$.

In the following, we use the notation

$$\sigma(t) = \max\{t, t^{N-1}\}, \quad \rho(t) = \min\{1, t^{2-N}\} \quad (\rho(0) = 1).$$

We remind the reader of our earlier stated assumptions on φ, ψ, λ, μ and N, which apply in the following theorem.

THEOREM 2.   *Suppose that*

(25) $\int^{\infty} t^{N-1-\lambda(N-2)}\varphi(t)dt < \infty$

and

(26)
$$\int^{\infty} t^{N-1-\mu(N-2)}\psi(t)dt < \infty,$$

and put

(27)
$$\Phi = \int_0^{\infty} \sigma(t)\rho^{\lambda}(t)\varphi(t)dt, \quad \Psi = \int_0^{\infty} \sigma(t)\rho^{\mu}(t)\psi(t)dt,$$

and

(28)
$$\eta = \Phi^{(\mu-1)/(\mu-\lambda)}\Psi^{(1-\lambda)/(\mu-\lambda)}\left[\left(\frac{1-\lambda}{\mu-1}\right)^{(\mu-1)/(\mu-\lambda)} + \left(\frac{\mu-1}{1-\lambda}\right)^{(1-\lambda)/(\mu-\lambda)}\right].$$

Then (6) has a radially symmetric solution û which is defined and positive for all x in $\mathbf{R}^N$, and has the asymptotic behavior

$$\lim_{|x|\to\infty} |x|^{N-2}\hat{u}(x) = c(>0),$$

provided that

(29)
$$\eta \leqq N - 2.$$

Note that (25) and (26) imply that Φ and Ψ are finite. The proof of this theorem rests on the observation that if $y = y(t)$ satisfies the ordinary differential equation

(30)
$$(t^{N-1}y')' + t^{N-1}\varphi(t)y^{\lambda} + t^{N-1}\psi(t)y^{\mu} = 0, \quad t > 0, \quad y'(0) = 0,$$

then the function $u(x) = y(|x|)$ is a radially symmetric solution of (6). Therefore, it suffices to show that (30) has a positive solution $\hat{y}$ on $[0, \infty)$ such that

(31)
$$\lim_{t\to\infty} t^{N-2}\hat{y}(t) = c > 0.$$

The following lemma will help us to accomplish this.

LEMMA 2. *Let $h: [0, \infty) \to [0, \infty)$ be a continuous function such that*

$$\int_0^{\infty} t^{N-1}h(t)dt < \infty$$

and define

$$\mathcal{M}h(t) = \frac{1}{N-2}\left[\int_0^t \left(\frac{s}{t}\right)^{N-2} sh(s)ds + \int_t^{\infty} sh(s)ds\right], \quad t \geqq 0.$$

Then

$$[t^{N-1}(\mathcal{M}h)'(t)] = -t^{N-1}h(t), \quad t \geqq 0,$$

and

(32) $I_1(h)\rho(t) \leqq \mathcal{M}h(t) \leqq I_2(h)\rho(t), \qquad t \geqq 0,$

where

$$I_1(h) = \frac{1}{N-2} \int_0^\infty \min\{s, s^{N-1}\}h(s)ds,$$

$$I_2(h) = \frac{1}{N-2} \int_0^\infty \max\{s, s^{N-1}\}h(s)ds.$$

Moreover,

$$\lim_{t\to\infty} t^{N-2}\mathcal{M}h(t) = \frac{1}{N-2} \int_0^\infty s^{N-1}h(s)ds.$$

We omit the routine proof of this lemma.

PROOF OF THEOREM 2.   Let

$$Y = \{y \in C[0, \infty): c_1\rho(t) \leqq y(t) \leqq c_2\rho(t), \quad t \geqq 0\},$$

where c_1 and c_2 are positive constants to be specified later.   Notice that Y is a closed convex subset of $C[0, \infty)$ with the topology of uniform convergence on compact subintervals.   Define the mapping $\mathcal{F}$ by

(33) $\mathcal{F}y(t) = \mathcal{M}(\varphi y^\lambda + \psi y^\mu)(t), \qquad t \geqq 0.$

If $y \in Y$, then

(34) $\mathcal{F}y(t) \leqq \mathcal{M}(\varphi c_2^\lambda \rho^\lambda)(t) + \mathcal{M}(\psi c_2^\mu \rho^\mu)(t)$

$$\leqq c_2^\lambda I_2(\varphi\rho^\lambda)\rho(t) + c_2^\mu I_2(\psi\rho^\mu)\rho(t), \qquad t \geqq 0,$$

where we have invoked the second inequality in (32) with $h = \varphi\rho^\lambda$ and $h = \psi\rho^\mu$. From (27) and the definition of $I_2(h)$, (34) can be rewritten as

$$\mathcal{F}y(t) \leqq \frac{1}{N-2} [c_2^\lambda \Phi + c_2^\mu \Psi]\rho(t), \qquad t \geqq 0.$$

Therefore,

(35) $\mathcal{F}y(t) \leqq c_2\rho(t), \qquad t \geqq 0,$

if

(36) $c_2^{\lambda-1}\Phi + c_2^{\mu-1}\Psi = g(c_2) \leqq N - 2.$

A routine computation shows that η as defined in (28) is the minimum value of $g(c_2)$, and that it is attained when

$$c_2 = \left[\frac{(1-\lambda)\varPhi}{(\mu-1)\varPsi} \right]^{1/(\mu-\lambda)}.$$

With this choice of c_2, (29) implies (36).

Returning now to (33), we note that if $y \in Y$, then

$$\mathscr{F}y(t) \geqq \mathscr{M}(\varphi c_1^\lambda \rho^\lambda)(t) + \mathscr{M}(\psi c_1^\mu \rho^\mu)(t)$$

$$\geqq c_1^\lambda I_1(\varphi \rho^\lambda)\rho(t) + c_1^\mu I_1(\psi \rho^\mu)\rho(t), \qquad t \geqq 0,$$

where we have invoked the first inequality in (32) with $h = \varphi \rho^\lambda$ and $h = \psi \rho^\mu$. Therefore,

(37) $$\mathscr{F}y(t) \geqq c_1 \rho(t), \qquad t \geqq 0,$$

if

(38) $$c_1^{\lambda-1} I_1(\varphi \rho^\lambda) + c_1^{\mu-1} I_1(\psi \rho^\mu) \geqq 1.$$

Since $\lambda < 1$, the left side of (38) tends to ∞ as $c_1 \to 0+$; therefore, we can certainly choose c_1 so that $0 < c_1 < c_2$ and (38) holds.

Now (35) and (37) imply that $\mathscr{F}$ maps Y into itself. The continuity of $\mathscr{F}$ and the relative compactness of $\mathscr{F}(Y)$ are easily proved. Therefore, $\mathscr{F}\hat{y} = \hat{y}$ for some $\hat{y}$ in Y, by the Schauder-Tychonoff theorem. Since

$$\hat{y} = \mathscr{M}(\varphi \hat{y}^\lambda + \psi \hat{y}^\mu),$$

Lemma 2, with $h = \varphi \hat{y}^\lambda + \psi \hat{y}^\mu$ and $\mathscr{M}h = \hat{y}$, implies that $\hat{y}$ satisfies (30) and (31). Therefore, the function $\hat{u}(x) = \hat{y}(|x|)$ satisfies the requirements of the theorem, and the proof is complete.

REMARK 4. It would be of interest to give conditions which guarantee the existence of decaying positive entire solutions of non-radial equations of the form

$$\varDelta u + \varphi(x)u^\lambda + \psi(x)u^\mu = 0, \qquad x \in R^N,$$

where $0 < \lambda < 1$, $\mu > 1$, and φ, ψ: $R^N \to (0, \infty)$ are locally Hölder continuous.

References

[1] W.-Y. Ding and W.-M. Ni, On the elliptic equation $\varDelta u + Ku^{(n+2)/(n-2)} = 0$ and related topics, Duke Math. J. **52** (1985), 485–506.

[2] N. Fukagai, Existence and uniqueness of entire solutions of second order sublinear elliptic equations, Funkcial. Ekvac. (to appear)

[3] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. **34** (1981), 525–598.

[4] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal. **49** (1973), 241–269.

[5]   N. Kawano, J. Satsuma and S. Yotsutani,   On the positive solutions of an Emden-type
        elliptic equation,   Proc. Japan Acad. Ser. A. **61** (1985), 186–189.
[6]   T. Kusano and M. Naito,   Positive entire solutions of superlinear elliptic equations,
        Hiroshima Math. J. **16** (1986), 361–366.
[7]   T. Kusano and M. Naito,   Oscillation theory of entire solutions of second order super-
        linear elliptic equations, Funkcial Ekvac. (to appear)
[8]   T. Kusano and C. A. Swanson,   Entire positive solutions of singular semilinear elliptic
        equations,   Japan. J. Math. **11** (1985), 144–155.
[9]   T. Kusano and W. F. Trench,   Global existence of nonoscillatory solutions of perturbed
        disconjugate equations, submitted for publication.
[10]   W. F. Trench,   Canonical forms and principal systems for general disconjugate equations,
        Trans. Amer. Math. Soc. **189** (1974), 319–327.

Department of Mathematics,
Faculty of Science,
Hiroshima University

and

Department of Mathematics and Computer Science,
Drexel University,
Philadelphia, PA 19104, U. S. A.[*)]

*) The present address of the second author is as follows: Department of Mathematics,
    Trinity University, San Antonio, Texas 78284, U. S. A.