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On real continuous kernels satisfying the semi-complete maximum principle

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§ 1. Introduction

According to the so-called Hunt theory, the complete maximum principle is an essential property for a continuous kernel V on a locally compact space X to possess a resolvent and further to be represented by a sub-markovian continuous semi-group $(T_t)_{t>0}$, that is, $Vf=\int_0^\infty T_t f dt$ for any $f \in C_K(X)$ (see, for example, [2] and [13]). While the logarithmic kernel on the 2-dimensional Euclidean space R^2 does not have this property, it satisfies the "*semi-complete maximum principle*" with respect to the Lebesgue measure ξ_2 (see [4]). Furthermore the logarithmic kernel possesses a resolvent and is represented by the 2-dimensional Gauss semi-group in the following sense:

$$\int_{R^2} \log|x-y| f(y) d\xi_2(y) = \int_0^\infty \int_{R^2} \frac{1}{4\pi t} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) d\xi_2(y) dt$$

for any $f \in C_K(R^2)$ with $\int f d\xi_2 = 0$. Recently, generalizing the logarithmic kernel, M. Itô [4]–[7] considered a real convolution kernel N of logarithmic type on a locally compact abelian group G. By definition, N is "*of logarithmic type*" if there exists a markovian convolution semi-group $(\alpha_t)_{t>0}$ such that $N*f = \int_0^\infty \alpha_t * f dt$ for any $f \in C_K(G)$ with $\int f d\xi = 0$, where ξ is a Haar measure on G. He showed in [4, Théorème A] that a real convolution kernel N is of logarithmic type if and only if

(L.0) N satisfies the semi-complete maximum principle with respect to ξ,

(L.1) $\inf_{x \in G} N*f(x) \leqq 0$ for any $f \in C_K(G)$ with $\int f d\xi = 0$,

(L.2) N is non-periodic,

(L.3) $\lim_{n\to\infty} \eta_{N,CK_n} = -\infty$, where $(K_n)_{n=1}^\infty$ is an exhaustion of G and η_{N,CK_n} is the N-reduced measure of N on CK_n.

In this paper, taking the above fact into consideration, we investigate a real continuous kernel V on a locally compact space X satisfying the semi-complete

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maximum principle with respect to a certain positive Radon measure m (see Definition 2) and conditions (A), (B), and (C) (in Theorem 7) which correspond to (L.1), (L.2) and (L.3). We shall construct a resolvent $(V_p)_{p>0}$ satisfying

$$Vf = V_p f + pV_p Vf \quad \text{for any} \quad f \in C_K^0(X, m)$$

(in section 3) and under some additional conditions, we shall also construct a continuous semi-group $(T_t)_{t>0}$ satisfying

$$Vf = \int_0^\infty T_t f dt \quad \text{for any} \quad f \in C_K^0(X, m)$$

(in section 4). Here $C_K^0(X, m) = \{f \in C_K(X); \int f dm = 0\}$. The results in section 3 are slight generalizations of the result announced in [17]. Remark that the resolvent associated with V is uniformly recurrent in the sense defined in [16]. We note in the final section that the Neumann kernel satisfies the semi-complete maximum principle with respect to its invariant measure.

Our study is also closely related to that of conditions of kernels to be "*weak potential operators*" for recurrent Markov processes in the probabilistic view point (see, for example, [10], [11], [12], [14] and [15] in which strong Feller kernels are studied).

§ 2. Definitions and preliminaries

Let X be a locally compact Hausdorff space with countable base. We denote by $C(X)$ the Fréchet spece of real continuous functions on X with the topology of compact convergence, by $C_K(X)$ the topological vector space of real continuous functions on X which have compact support with the usual inductive limit topology, by $M(X) = C_K(X)^*$ the topological vector space of real Radon measures on X with w^*-topology (i.e., vague topology), by $M_K(X) = C(X)^*$ the subspace of $M(X)$ consisting of measures with compact support. $C^+(X)$, $C_K^+(X)$, $M^+(X)$ and $M_K^+(X)$ denote their subsets of non-negative elements. We denote by $C_b(X)$ (resp. $C_o(X)$) the subset of $C(X)$ consisting of bounded functions (resp. functions tending to zero at infinity). For $m \in M^+(X)$, put $C_K^0(X, m) = \{f \in C_K(X); \int f dm = 0\}$ and put $M_K^0(X) = \{\mu \in M_K(X); \int d\mu = 0\}$, $M_b(X) = \{\mu \in M(X); \int d|\mu| < \infty\}$, where $|\mu|$ is the total variation of μ. Naturally, if X is compact, $C_K(X) = C_o(X) = C_b(X) = C(X)$ and $M_K(X) = M_b(X) = M(X)$.

An operator V: $C_K(X) \to C(X)$ is called a *real continuous kernel* on X if it is linear and continuous. If V is also positive, i.e., $Vf \in C^+(X)$ for $f \in C_K^+(X)$, we simply call it a *continuous kernel* on X.

For a real continuous kernel V, we denote by V^* its transposed operator $M_K(X) \to M(X)$, which is defined by

$$\int f dV^*\mu = \int Vf d\mu \qquad \text{for } f \in C_K(X) \text{ and } \mu \in M_K(X).$$

In general, a continuous linear operator from $M_K(X)$ into $M(X)$ is called a *real diffusion kernel* on X. Evidently, V^* is a real diffusion kernel.

The identity operator I on $C_K(X)$ is trivialy a continuous kernel. For the sake of simplicity, its transposed kernel I^* will be again denoted by I.

For a real continuous kernel V on X, we put

$$D(V^*) = \left\{\mu \in M(X); \int |Vf| d|\mu| < \infty \qquad \text{for any } f \in C_K(X)\right\}.$$

By the Banach-Steinhaus theorem, for each $\mu \in D(V^*)$, $C_K(X) \ni f \to \int Vfd\mu$ defines a Radon measure, which is denoted by $V^*\mu$. We write $D^0(V^*) = \{\mu \in D(V^*); \int d|\mu| < \infty$ and $\int d\mu = 0\}$ and $D^+(V^*) = D(V^*) \cap M^+(X)$.

We denote by ε_x the Dirac measure at $x \in X$. Let $(V^*\varepsilon_x)^+ - (V^*\varepsilon_x)^-$ be the Jordan decomposition of $V^*\varepsilon_x$. Then for any $f \in C_K^+(X)$,

$$\int fd(V^*\varepsilon_x)^{\pm} = \sup\{\pm Vg(x);\, g \in C_K(X),\, 0 \leqq g \leqq f\},$$

and hence $x \to \int fd|V^*\varepsilon_x|$ is a lower semi-continuous function on X. For a Borel function u on X and $x \in X$, we put $Vu(x) = \int udV^*\varepsilon_x$ and $|V|u(x) = \int ud|V^*\varepsilon_x|$ provided that they make sense. By an argument similar to that in [13, p. 176], we see that Vu and $|V|u$, when defined, are Borel measurable. Furthermore we can easily show

REMARK 1. *Let u be a Borel function and $\mu \in D(V^*)$. If $\int |V|\,|u| d|\mu| < \infty$, then $\int Vud\mu = \int udV^*\mu$.*

Let V_1 and V_2 be two real continuous kernels. We define the product operator V_1V_2 by $V_1V_2f(x) = \int V_2fdV_1^*\varepsilon_x$ provided that it makes sense for any $f \in C_K(X)$ and any $x \in X$.

A family $(V_p)_{p>0}$ of continuous kernels is called a *resolvent* if for any $p>0$, $q>0$, V_pV_q defines a continuous kernel and $V_p - V_q = (q-p)V_pV_q$. A family $(T_t)_{t>0}$ of continuous kernels is called a *continuous semi-group* if for any $t>0$, $s>0$, T_tT_s defines a continuous kernel, $T_tT_s = T_{t+s}$ and for each $f \in C_K(X)$, the mapping $[0, \infty) \ni t \to T_tf \in C(X)$ is continuous, where $T_0 = I$. We say that $(V_p)_{p>0}$ (resp. $(T_t)_{t>0}$) is *markovian* if for any $p>0$ and any $x \in X$, $p\int dV_p^*\varepsilon_x = 1$ (resp. for any $t>0$ and any $x \in X$, $\int dT_t^*\varepsilon_x = 1$).

DEFINITION 2. We say that a real continuous kernel V on X satisfies the *semi-complete maximum principle with respect to m* ($\in M^+(X)$) (resp. V satisfies the *complete maximum principle*) if for any $f \in C_K^0(X, m)$ (resp. for any $f \in C_K(X)$) and $a \in R$, $Vf \leqq a$ on $\operatorname{supp}(f^+)$ implies $Vf \leqq a$ on X.

Here R is the set of all real numbers, supp (g) is the support of g and $f^+(x)=\max\{f(x), 0\}$.

LEMMA 3. *Let V be a real continuous kernel on X and let $m\in M^+(X)$.*

(a) *If V satisfies the complete maximum principle, then V is positive, that is, V is a continuous kerenl.*

(b) *If V satisfies the semi-complete maximum principle with respect to m, then for $f\in C_K^0(X, m)$,*

$$\|Vf\|_\infty \leqq \sup_{x\in\mathrm{supp}(f)} |Vf(x)|,$$

where $\|f\|_\infty=\sup_{x\in X}|f(x)|$.

(c) *If there exists a markovian resolvent $(V_p)_{p>0}$ such that for any $f\in C_K^0(X, m)$ (resp. for any $f\in C_K(X)$), $\lim_{p\to 0} V_pf=Vf$ in $C(X)$, then V satisfies the semi-complete maximum principle with respect to m (resp. V satisfies the complete maximum principle).*

(d) *If there exists a markovian continuous semi-group $(T_t)_{t>0}$ such that for any $f\in C_K^0(X, m)$ (resp. for any $f\in C_K(X)$), $\lim_{t\to\infty}\int_0^t T_sfds=Vf$ in $C(X)$, then the same conclusion as above is obtained.*

In fact, (a) and (b) are clear from the definition. It is known that, for a markovian resolvent $(V_p)_{p>0}$, each V_p satisfies the complete maximum principle (see, e.g., [2]). Let $f\in C_K^0(x, m)$ (resp. $f\in C_K(X)$) and $a\in R$. If $Vf\leqq a$ on supp (f^+), then for any $\varepsilon>0$, there exists $p_\varepsilon>0$ such that $V_pf\leqq a+\varepsilon$ on supp (f^+) and so on X for any $0<p<p_\varepsilon$. Letting $p\downarrow 0$ and $\varepsilon\downarrow 0$ we have (c). For (d), put $V_p=\int_0^\infty e^{-pt}T_tdt$ $(p>0)$. Then $(V_p)_{p>0}$ is a markovian resolvent. Since

$$\begin{aligned} Vf(x) - V_pf(x) &= Vf(x) - p\int_0^\infty e^{-pt}\Big(\int_0^t T_sf(x)ds\Big)dt \\ &= p\int_0^\infty e^{-pt}(Vf(x) - \int_0^t T_sf(x)ds)dt \end{aligned}$$

and since $\lim_{t\to\infty}\int_0^t T_sfds=Vf$ in $C(X)$, for any compact set K in X and any $\varepsilon>0$, there exist $T>0$ and $M>0$ such that $\left|\int_0^t T_sf(x)ds\right|\leqq M$ on K for any $t>0$ and $\left|\int_0^t T_sf(x)ds-Vf(x)\right|<\varepsilon$ on K for any $t\geqq T$. Therefore

$$|Vf(x)-V_pf(x)| \leqq p\int_0^T e^{-pt}2Mdt + p\int_T^\infty \varepsilon e^{-pt}dt$$

on K. Letting $p\to 0$ and $\varepsilon\to 0$, we see $\lim_{p\to 0} V_pf=Vf$ uniformly on K, so that (c) gives (d).

In the same manner as in [4, Remarque 5 and Proposition 11], we obtain the following

PROPOSITION 4. *Let V satisfy the semi-complete maximum principle with respect to $m\in M^+(X)$ and let $c\geqq 0$. Then we have*:

(a) *For any $f\in C_K^0(X, m)$ and $a\in R$, $(V+cI)f\leqq a$ on* $\mathrm{supp}(f^+)$ *implies $Vf\leqq a$ on X.*

(b) *V^*+cI satifies the semi-balayage principle relative to V^*; that is, for any $\mu\in M_K^+(X)$ and any relatively compact open set $\omega\neq\emptyset$ in X, there exist $\mu'_\omega\in M_K^+(X)$ and $a'_{\mu,\omega}\in R$ such that*

(SB.1) $\int d\mu'_\omega = \int d\mu$,

(SB.2) $\mathrm{supp}(\mu'_\omega)\subset\bar{\omega}$,

(SB.3) $(V^*+cI)\mu'_\omega + a'_{\omega,\mu}m = V^*\mu$ *in* ω,

(SB.4) $(V^*+cI)\mu'_\omega + a'_{\omega,\mu}m \leqq V^*\mu$ *on* X.

We say that μ'_ω (resp. $a'_{\mu,\omega}$) is a *semi-balayaged measure* (resp. a *semi-balayage constant*) *of μ on ω with respect to* (V^*+cI, V^*).

A real continuous kernel V on X is said to be *strong Feller* if for any bounded Borel function g on X with compact support, $Vg(x)=\int gdV^*\varepsilon_x$ is continuous.

REMARK 5. *Let V, m and c be as in Proposition 4. Assume that V is strong Feller. Then for any bounded Borel function g with compact support and $\int gdm=0$, and for any $a\in R$, $(V+cI)g\leqq a$ on $\{x; g(x)>0\}$ implies $Vg\leqq a$ on X.*

In fact, if $(V+cI)g\leqq a$ on $\{x; g(x)>0\}$, $Vg\leqq a$ on the same set. Since Vg is continuous for any $\varepsilon>0$ there exists a relatively compact open set ω_ε such that $\{x; g(x)>0\}\subset\omega_\varepsilon$ and $Vg\leqq a+\varepsilon$ on $\bar{\omega}_\varepsilon$. For $x\in X$, let $\varepsilon'_{x,\varepsilon}$ and $a'_{x,\varepsilon}$ be a semi-balayaged measure and a semi-balayage constant of ε_x on ω_ε with respect to (V^*, V^*). Then we have

$$\begin{aligned} Vg^+(x) &= \int g^+dV^*\varepsilon_x = \int g^+d(V^*\varepsilon'_{x,\varepsilon}+a'_{x,\varepsilon}m) \\ &= \int Vg^+d\varepsilon'_{x,\varepsilon} + a'_{x,\varepsilon}\int g^+dm \leqq \int(Vg^-+a+\varepsilon)d\varepsilon'_{x,\varepsilon} + a'_{x,\varepsilon}\int g_-dm \\ &= \int g^-d(V^*\varepsilon'_{x,\varepsilon}+a'_{x,\varepsilon}m) + a + \varepsilon \leqq Vg^-(x) + a + \varepsilon, \end{aligned}$$

where $g^-=g^+-g$. Letting $\varepsilon\downarrow 0$, we see $Vg(x)\leqq a$ for all $x\in X$.

DEFINITION 6 (see [16, Definition 1]). We say that a resolvent $(V_p)_{p>0}$ is *uniformly recurrent* if there exist a family $(u_p)_{p>0}$ in $C(X)$ and $p_o>0$ satisfying the following:

(a) $u_p>0$ on X for all $p>0$.

(b) $\lim_{p\to 0} u_p(x)=0$ for all $x\in X$.

(c) For any $f \in C_K^+(X)$, $(u_p V_p f)_{p_o > p > 0}$ forms a normal family on any compact set in X.

(d) For any $x \in X$, there exists $f \in C_K^+(X)$ such that $\inf_{p_o > p > 0} u_p V_p f(x) > 0$.

We also say that a continuous semi-group $(T_t)_{t>0}$ is *uniformly recurrent* if its resolvent defined by $V_p = \int_0^\infty e^{-pt} T_t dt$ is uniformly recurrent.

§3. The resolvent associated with a real continuous kernel

The purpose of this section is to show the following theorems, which generalize the result in [17].

THEOREM 7. *Let m be a positive Radon measure on X whose support is equal to X and let V be a real continuous kernel which satisfies the semi-complete maximum principle with respect to m. We assume:*

(A) *There exists a constant c_V such that for $\mu \in M_K^0(X)$ and $a \in R$, $V^*\mu \geqq am$ implies $a \leqq c_V \int d|\mu|$.*

(B) *If $(V^* + cI)\mu = am$ for $\mu \in D^0(V^*)$, $c > 0$ and $a \in R$, then $\mu = 0$ and $a = 0$.*

(C) *For any $f \in C_K^+(X)$ with $f \neq 0$, $\lim_{x \to \delta} Vf(x) = -\infty$, where δ is the Alexandrov point of X.*

Then there exists a markovian resolvent $(V_p)_{p>0}$ which has the following properties:

(1) *For any $x \in X$ and any $p > 0$, $V^*\varepsilon_x = V_p^*\varepsilon_x + pV^*V_p^*\varepsilon_x + a_{x,p}m$ with some constant $a_{x,p}$. In particular,*

$$Vf = V_p f + pV_p Vf \quad \text{for any} \quad f \in C_K^0(X, m).$$

(2) *$(V_p)_{p>0}$ is uniformly recurrent.*

(3) *For any $p > 0$, $m \in D(V_p^*)$ and $pV_p^* m = m$. Furthermore if $\mu \in D^+(V_p^*)$ and $pV_p^*\mu \leqq \mu$, then $\mu = cm$ with some constant $c \geqq 0$.*

By the condition (B), a markovian resolvent $(V_p)_{p>0}$ satisfying (1) is uniquely determined. We call it the *resolvent associated with V.*

THEOREM 8. *Let V and m be as in Theorem 7 and let $(V_p)_{p>0}$ be the resolvent associated with V. Assume further that*

(D) *for any $f \in C_K^0(X, m)$, $Vf \in C_o(X)$.*

Then for $f \in C_K^0(X, m)$, we have:

(1) *If $\int dm = \infty$, $\lim_{p \downarrow 0} V_p f = Vf$ uniformly on X.*

(2) *If $\int dm < \infty$, the above equality holds if and only if $\int Vfdm = 0$.*

(3) *If X is compact, $\lim_{p \downarrow 0} V_p f = Vf - (\int dm)^{-1} \int Vfdm$ uniformly on X.*

REMARK 9. *If V is strong Feller then the condition* (B) *is satisfied.*

In fact, let $c>0$. Remark 5 and the proof of [11, theorem 5.1] show that for any $f\in C_K(X)$, there exist a sequence $(g_n)_{n=1}^{\infty}$ of bounded Borel functions with compact support and $\int g_n dm=0$ and a sequence $(a_n)_{n=1}^{\infty}$ of constants such that $f=\lim_{n\to\infty}((V+cI)g_n+a_n)$ uniformly on X. Thus if $(V^*+cI)\mu=am$ with $\int d\mu=0$, then

$$\int f d\mu = \lim_{t\to\infty}((V+cI)g_n+a_n)d\mu$$

$$= \lim_{n\to\infty}\int g_n d(V^*+cI)\mu = \lim_{n\to\infty} a\int g_n dm = 0,$$

which implies $\mu=0$ and hence $a=0$.

REMARK 10. *If X is compact, then the conditions* (A) *and* (B) *are always satisfied.*

In fact, putting $c_V=\|V1\|_\infty$, we have $a\leqq\int 1dV^*\mu\leqq c_V\int d|\mu|$, and hence (A) is satisfied. As for (B), in the same manner as in [12, Lemma 3.1] (considering the space $C_K^0(X, m)$ in place of $N(m)$ there) we see that for any $f\in C(X)$, there exist $g\in C_K^0(X, m)$ and $a\in R$ such that $f=(V+cI)g+a$ on X. Then, we obtain (B) as in Remark 9.

EXAMPLE 11. Let R^n be the n-dimensional Euclidean space and let ξ_n be the Lebesgue measure on R^n $(n=1, 2)$. The real continuous kernels $G_{1,a}$, G_2 and P defined by

$$G_{1,a}f(x) = -\frac{1}{2}\int(|x-y|+a(x-y))f(y)d\xi_1(y),\ f\in C_K(R^1) \quad (0\leqq a<1),$$

$$G_2 f(x) = -\int \log|x-y| f(y)d\xi_2(y),\ f\in C_K(R^2),$$

$$Pf(x) = -\int \log|x-y| f(x)d\xi_1(y),\ f\in C_K(R^1),$$

satisfy the semi-complete maximum principle with respect to the Lebesgue measure (see, e.g., [4] and [11]). Furthermore, they all satisfy the conditions (A), (B) and (C). In fact, for (A), see [5, Théorème 52′] (acturally we may take $c_V=0$). Since they are all strong Feller, Remark 9 gives (B). (C) is clear. For another examples, which are not convolution kernels, see [10] and the section 5 of this paper.

To prove Theorem 7, we prepare the following

LEMMA 12. *Let V be a real continuous kernel satisfying the semi-complete* *maximum principle with respect to* $m \in M^+(X)$. *Suppose that the condition* (C) *in Theorem 7 is fulfiled. If* $\mu_n \in D^+(V^*)$, $\int d\mu_n \leqq 1$ $(n=1, 2,\dots)$, μ_n *and* $V^*\mu_n$ *converge vaguely to* μ *and* ν *respectively as* $n \to \infty$, *then*

(a) $\mu \in D^+(V^*)$,

(b) $\lim_{n\to\infty} \int d\mu_n = \int d\mu$,

(c) $\nu = V^*\mu + am$ *for some constant* $a \leqq 0$.

PROOF. Let K be any compact set in X with non-empty interior and let $f_o \in C_K^+(X)$ with $\operatorname{supp}(f_o) \subset K$ and $\int f_o dm = 1$. Since $\operatorname{supp}((Vf_o)^+)$ is compact (by (C))

$$-\infty < \int f_o d\nu = \lim_{n\to\infty} \int Vf_o d\mu_n \leqq \int Vf_o d\mu \leqq \int (Vf_o)^+ d\mu < \infty$$

and hence $\int |Vf_o| d\mu < \infty$. By the continuity of V, there exists a constant $c_K > 0$ such that $\max_{x\in K} |Vf(x)| \leqq c_K \|f\|_\infty$ for any $f \in C_K(X)$ with $\operatorname{supp}(f) \subset K$. We put $a_f = \int f dm$. Then

$$|Vf - V(a_f f_o)| \leqq c_K(\|f\|_\infty + |a_f| \, \|f_o\|_\infty) \text{ on } K.$$

Since $\operatorname{supp}(f - a_f f_o) \subset K$ and $f - a_f f_o \in C_K^0(X, m)$, the semi-complete maximum principle implies that the above inequality holds on X, and hence

$$\int |Vf| d\mu \leqq (m(K) \int |Vf_o| d\mu + c_K + c_K m(K) \|f_o\|_\infty) \|f\|_\infty ,$$

because $\int d\mu \leqq 1$ and $|a_f| \leqq m(K) \|f\|_\infty$. Consequently we have (a).

Evidently, $\liminf_{n\to\infty} \int d\mu_n \geqq \int d\mu$. Let $f_o \in C_K^+(X)$ with $f_o \neq 0$. Since $(Vf_o)^+ \in C_K^+(X)$ by (C),

$$\int (Vf_o)^- d\mu_n = \int (Vf_o)^+ d\mu_n - \int Vf_o d\mu_n \longrightarrow \int (Vf_o)^+ d\mu - \int f_o d\nu < \infty \quad (n \to \infty).$$

Hence there is $M \geqq 0$ such that $\int (Vf_o)^- d\mu_n \leqq M$ for all n. On the other hand, by (C), for any $\varepsilon < 0$ there is a compact set K_ε such that $(Vf_o)^-(x) > 1/\varepsilon$ for $x \in CK_\varepsilon$. Thus,

$$\int d\mu_n \leqq \varepsilon \int_{CK_\varepsilon} (Vf_o)^-(x) d\mu_n + \int_{K_\varepsilon} d\mu_n \leqq \varepsilon M + \int_{K_\varepsilon} d\mu_n \longrightarrow \varepsilon M + \int_{K_\varepsilon} d\mu \quad (n \to \infty).$$

Since ε is arbitrary, it follows that $\limsup_{n\to\infty} \int d\mu_n \leqq \int d\mu$, which shows (b).

An argument as in the proof of (a) leads to $\nu \leqq V^*\mu$. Since for any $f \in C_K^0(X, m)$, $Vf \in C_b(X)$ (see Lemma 3 (b)), (b) shows

$$\int f d\nu = \lim_{n\to\infty} \int Vf d\mu_n = \int Vf d\mu = \int f dV^*\mu.$$

It follows from these facts that $\nu = V^*\mu + am$ with some $a \leqq 0$. This completes the proof.

Using the above lemma, we shall show the following, which is called the *semi-balayability* in the case when V is a convolution kernel (cf. [7]).

PROPOSITION 13. *Let V and m be as in Theorem 7 and let $c \geqq 0$. Then for any $\mu \in M_K^+(X)$ and any open set $\omega \neq \emptyset$ in X, there exist $\mu'_\omega \in D^+(V^*)$ and $a'_{\mu,\omega} \in R$ satisfying* (SB.1), (SB.2), (SB.3) *and* (SB.4) *in Proposition* 4. *μ'_ω and $a'_{\mu,\omega}$ are called a semi-balayaged measure and a semi-balayage constant of μ on ω with respect to $(V^* + cI, V^*)$. Furthermore, $a'_{\mu,\omega} \leqq 2c_V \int d\mu$ with c_V given in condition* (A).

PROOF. We may assume that $\int d\mu = 1$. If ω is relatively compact, the assertion has already been shown in Proposition 4. Hence we may assume that X is non-compact and ω is not relatively compact. Let $(\omega_n)_{n=1}^\infty$ be an exhausition of ω, that is, a sequence of relatively compact open sets in X satisfying $\bar{\omega}_n \subset \omega_{n+1}$ $(n \geqq 1)$ and $\cup_{n=1}^\infty \omega_n = \omega$. By Proposition 4 there exist $\mu'_n \in M_K^+(X)$ and $a'_n \in R$ such that $\int d\mu'_n = 1$, $\operatorname{supp}(\mu'_n) \subset \bar{\omega}_n$, $V^*\mu = (V^* + cI)\mu'_n + a'_n m$ in ω_n and $V^*\mu \geqq (V^* + cI)\mu'_n + a'_n m$ on X. Since $(\mu'_n)_{n=1}^\infty$ is vaguely bounded, we may assume that $\lim_{n\to\infty} \mu'_n$ exists in $M^+(X)$, which is denoted by μ'_ω. Then $\operatorname{supp}(\mu'_\omega) \subset \bar{\omega}$. Since $V^*(\mu - \mu'_n) \geqq a'_n m$ and $\mu - \mu'_n \in M_K^0(X)$, condition (A) gives $a'_n \leqq 2c_V$ for all $n \geqq 1$. Let $f \in C_K^+(X)$ with $\int f dm = 1$ and $\operatorname{supp}(f) \subset \omega_1$. Then

$$a'_n = \int Vf d\mu - \int (V + cI) f d\mu'_n \geqq \int Vf d\mu - \int ((Vf)^+ + cf) d\mu'_n .$$

Since $(Vf)^+ \in C_K^+(X)$, $(a'_n)_{n=1}^\infty$ is bounded below, so that it is bounded. Hence we may assume that a'_n converges to a_o $(\leqq 2c_V)$ and $V^*\mu'_n$ converges vaguely as $n \to \infty$. By Lemma 12, we see that $\int d\mu'_\omega = 1 = \int d\mu$ and $\lim_{n\to\infty} V^*\mu'_n = V^*\mu'_\omega + am$ with some $a \leqq 0$. Putting $a'_{\mu,\omega} = a + a_o$, we obtain that $V^*\mu = (V^* + cI)\mu'_\omega + a'_{\mu,\omega} m$ in ω and $V^*\mu \geqq (V^* + cI)\mu'_\omega + a'_{\mu,\omega} m$ on X. Since $a_o \leqq 2c_V$ and $a \leqq 0$, we have $a'_{\mu,\omega} \leqq 2c_V = 2c_V \int d\mu$. Thus Proposition 13 is shown.

REMARK 14. *If $\omega = X$, and $c > 0$, then the condition* (B) *shows that μ'_ω and $a'_{\mu,\omega}$ are uniquely determined.*

We shall turn to the proof of Theorem 7. From now on, let V and m be the same as in Theorem 7. We devote ourselves to the case that X is non-compact; the case X is compact is similar and simpler (note Remark 10).

Let $p > 0$ be fixed. We can define a linear operator V_p on $C_K(X)$ by

$$V_p f(x) = \frac{1}{p} \int f d\varepsilon'_{x,p}, \quad x \in X,$$

where $\varepsilon'_{x,p}$ is the semi-balayaged measure of ε_x on X with respect to $(V^*+p^{-1}I, V^*)$. We may write $V_p^*\varepsilon_x=p^{-1}\varepsilon'_{x,p}$. Then $p\int dV_p^*\varepsilon_x=1$ and $(pV^*+I)V_p^*\varepsilon_x+a_{x,p}m=V^*\varepsilon_x$ with some constant $a_{x,p}\leqq 2c_V$. Thus, we have

LEMMA 15. *$(V_p)_{p>0}$ possesses property* (1) *in Theorem* 7.

Furthermore we have

LEMMA 16. *The mapping V_p is a continuous kernel on X.*

PROOF. Clearly V_p is positive. Hence it is sufficient to show that $V_pf\in C(X)$ for any $f\in C_K(X)$. It is then sufficient to see that for any $(x_n)_{n=1}^\infty\subset X$ with $\lim_{n\to\infty}x_n=x\in X$,

$$\lim_{n\to\infty} V_p^*\varepsilon_{x_n} = V_p^*\varepsilon_x \quad \text{vaguely.}$$

We have $V^*\varepsilon_{x_n}=(pV^*+I)V_p^*\varepsilon_{x_n}+a'_nm$ with constants $a'_n\leqq 2c_V$. Let $f\in C_K^+(X)$ with $\int fdm=1$. Then

$$\begin{aligned} a'_n &= Vf(x_n) - p\int VfdV_p^*\varepsilon_{x_n} - \int fdV_p^*\varepsilon_{x_n} \\ &\geqq Vf(x_n) - \left(\|(Vf)^+\|_\infty + \frac{1}{p}\|f\|_\infty\right), \end{aligned}$$

so that the relative compactness of $(x_n)_{n=1}^\infty$ implies that $(a'_n)_{n=1}^\infty$ is bounded. Let λ be any vague accumulation point of $(V_p^*\varepsilon_{x_n})_{n=1}^\infty$. There is a subsequence of (x_n), which is again denoted by (x_n), such that $V_p^*\varepsilon_{x_n}\to\lambda$ vaguely. We may assume that a'_n, and hence $V^*V_p^*\varepsilon_{x_n}$, converges as $n\to\infty$. By Lemma 12, we see that $\lambda\in D^+(V^*)$, $p\int d\lambda=1$ and $V^*\varepsilon_x=(pV^*+I)\lambda+a'm$ with some constant a'. On the other hand, since $V^*\varepsilon_x=(pV^*+I)V_p^*\varepsilon_x+a'_xm$, condition (B) gives $\lambda=V_p^*\varepsilon_x$. Since λ is an arbitrary vague accumulation point, we conclude that $\lim_{n\to\infty}V_p^*\varepsilon_{x_n}=V_p^*\varepsilon_x$ vaguely. Thus Lemma 16 is shown.

LEMMA 17. (1) *If we write $V^*\varepsilon_x=V_p^*\varepsilon_x+pV^*V_p^*\varepsilon_x+a_xm$, then $x\to a_x$ is lower semi-continuous and bounded above.*

(2) *If $\mu\in D^+(V^*)$, then $\int d\mu<\infty$, $\mu\in D^+(V_p^*)$ and $V_p^*\mu\in D^+(V^*)$. Furthermore, $pV_p^*\mu$ and $\int a_xd\mu(x)$ are a semi-balayaged measure and a semi-balayage constant of μ on X with respect to $(V^*+p^{-1}I, V^*)$.*

PROOF. (1): By Proposition 13, $a_x\leqq 2c_V$ for any $x\in X$. Let $f\in C_K^+(X)$ with $\int fdm=1$. Then $a_x=Vf(x)-V_pf(x)-pV_pVf(x)$. Since V_p is a continuous kernel and supp$((Vf)^+)$ is compact, V_pVf is upper semi-continuous so that $x\to a_x$ is lower semi-continuous.

(2): Let $\mu\in D^+(V^*)$ and let $f\in C_K^+(X)$ with $f\neq 0$. By definition $\int|Vf|d\mu<\infty$ and hence condition (C) gives $\int d\mu<\infty$. Since $p\int dV_p^*\varepsilon_x=1$ for any $x\in X$, we see $M_b(X)\subset D(V_p^*)$ so that $\mu\in D^+(V_p^*)$. Next we take a sequence $(\mu_n)_{n=1}^{\infty}\subset M_K^+(X)$ which converges increasingly to μ. Then $-\infty<\int a_x d\mu_n(x)\leqq 2c_V\int d\mu<\infty$ for all $n\geqq 1$ and hence we see $V^*\mu_n\in D^+(V^*)$ and $V^*\mu_n=V_p^*\mu_n+pV^*V_p^*\mu_n+(\int a_x d\mu(x))m$. Since

$$
\begin{aligned}
p\int|Vf|dV_p^*\mu &= \sup\nolimits_{n\geqq 1} p\int|Vf|dV_p^*\mu_n\\
&= \sup\nolimits_{n\geqq 1}\left(-p\int VfdV_p^*\mu_n+2p\int(Vf)^+dV_p^*\mu_n\right)\\
&\leqq \int|Vf|d\mu+\int(V_pf+\|2(Vf)^+\|_\infty)d\mu+\int fdm\cdot 2c_V\int d\mu<\infty,
\end{aligned}
$$

we see $V_p^*\mu\in D^+(V^*)$ and $\lim_{n\to\infty}pV^*V_p^*\mu_n=pV^*V_p^*\mu$ vaguely. This also implies $\lim_{n\to\infty}\int a_x d\mu_n(x)=\int a_x d\mu(x)>-\infty$. Thus we have $V^*\mu=V_p^*\mu+pV^*V_p^*\mu+(\int a_x d\mu(x))m$, which shows (2).

To see that $(V_p)_{p>0}$ is a resolvent, we shall show the following

LEMMA 18. *For any $p>0$, $q>0$ and $\mu\in M_K^+(X)$, we have*

$$V_p^*\mu-V_q^*\mu=(q-p)V_p^*V_q^*\mu \quad (\textit{the resolvent equation}).$$

PROOF. Let a_p' and a_q' be the semi-balayage constants of μ on X with respect to $(V^*+p^{-1}I,\ V^*)$ and to $(V^*+q^{-1}I,\ V^*)$, respectively. Then

$$
\begin{aligned}
&\left(V^*+\frac{1}{q}I\right)(V_p^*\mu-V_q^*\mu)\\
&=\left(V^*+\frac{1}{p}I\right)V_q^*\mu-\left(\frac{1}{p}-\frac{1}{q}\right)V_p^*\mu-\left(V^*+\frac{1}{q}I\right)V_p^*\mu\\
&=\frac{1}{p}(V^*\mu-a_p'm)-\left(\frac{1}{p}-\frac{1}{q}\right)V_p^*\mu-\frac{1}{q}(V^*\mu-a_q'm)\\
&-\left(\frac{1}{p}-\frac{1}{q}\right)(V^*\mu-V_p^*\mu)+\left(\frac{1}{q}a_q'-\frac{1}{p}a_p'\right)m.
\end{aligned}
$$

We also denote by $a_{p,q}'$ the semi-balayage constant of $q^{-1}V_p^*\mu$ on X with respect to $(V^*+q^{-1}I,\ V^*)$ (cf. Lemma 17). Then

$$
\begin{aligned}
\left(V^*+\frac{1}{q}I\right)(V_q^*V_p^*\mu) &= \frac{1}{q}V^*V_p^*\mu-a_{p,q}'m\\
&=\frac{1}{pq}(V^*\mu-V_p^*\mu)-\left(\frac{1}{pq}a'+a_{p,q}'\right)m,
\end{aligned}
$$

and hence

$$\left(V^*+\frac{1}{q}I\right)(V_p^*\mu - V_q^*\mu-(q-p)V_q^*V_p^*\mu)$$
$$=\left\{\frac{1}{q}a'_q-\frac{1}{p}a'+(q-p)\left(\frac{1}{pq}a'_p+a'_{p,q}\right)\right\}m.$$

Since $\int d(V_p^*\mu - V_q^*\mu-(q-p)V_q^*V_p^*\mu)=(1/p-1/q-(q-p)/pq)\int d\mu=0$, we obtain the desired equality by condition (B). This completes the proof.

LEMMA 19. *Let $(\mu_n)_{n=1}^{\infty}\subset M^+(X)$ with $\lim_{n\to\infty}\int d\mu_n=0$ and let $(p_n)_{n=1}^{\infty}\subset R$ with $p_n>0$ and $\lim_{n\to\infty}p_n=0$. If $V_{p_n}^*\mu_n$ converges vaguely as $n\to\infty$, then the vague limit is of the form cm with some $c\geqq 0$.*

PROOF. Let $\lambda=\lim_{n\to\infty}V_{p_n}^*\mu_n$. For any $f\in C_K^0(X, m)$, since $Vf\in C_b(X)$ and $V_{p_n}f=Vf-p_nV_{p_n}Vf$, we have

$$\int fd\lambda=\lim_{n\to\infty}\int fdV_{p_n}^*\mu_n=\lim_{n\to\infty}\int V_{p_n}fd\mu_n$$
$$=\lim_{n\to\infty}p_n\int\left(Vf(x)-p_n\int VfdV_{p_n}^*\varepsilon_x\right)d\mu_n(x)=0,$$

which implies that $\lambda=cm$ with some $c\geqq 0$.

LEMMA 20. *The family $(V_p)_{p>0}$ is a uniformly recurrent markovian resolvent.*

PROOF. By Lemmas 16 and 18, we see that $(V_p)_{p>0}$ is a resolvent. Clearly it is markovian. To see the uniform recurrence, we first show that for any $p>0$ and any $x\in X$, $\mathrm{supp}(V_p^*\varepsilon_x)=X$. Let x be fixed. By the resolvent equation, we see that $\mathrm{supp}(V_p^*\varepsilon_x)$ is independent of $p>0$. Since $(qV_q^*\varepsilon_x)_{q>0}$ is vaguely bounded, there exist $(q_n)_{n=1}^{\infty}\subset R$ and $\lambda\in M^+(X)$ with $\int d\lambda\leqq 1$ such that $q_n>0$, $\lim_{n\to\infty}q_n=0$ and $\lim_{n\to\infty}q_nV_{q_n}^*\varepsilon_x=\lambda$ vaguely. By Lemma 19, we see $\lambda=cm$ with some $c\geqq 0$. Therefore if $\lambda\neq 0$ we see $\mathrm{supp}(V_p^*\varepsilon_x)\subset\mathrm{supp}(\lambda)=X$ so that $\mathrm{supp}(V_p^*\varepsilon_x)=X$. In case that $\lambda=0$, we put $\mathrm{supp}(V_p^*\varepsilon_x)=X_o$ and suppose that $X_o\neq X$. Let $f_o\in C_K^+(X)$ with $\mathrm{supp}(f_o)\subset X\backslash X_o$ and $\int f_odm=1$. Then for any $f\in C_K^+(X)$, $V(f-a_ff_o)\in C_b(X)$ shows that $(q_n\int V(f-a_ff_o)dV_{q_n}^*\varepsilon_x)_{n=1}^{\infty}$ is bounded, where $a_f=\int fdm$. By Lemma 15,

$$V_{q_n}f(x)=V_{q_n}(f-a_ff_o)(x)=V(f-a_ff_o)(x)-q_n\int V(f-a_ff_0)dV_{q_n}^*\varepsilon_x,$$

so that $(V_{q_n}f(x))_{n=1}^{\infty}$ is also bounded. Hence the equality

$$Vf(x)-V_{q_n}f(x)=q_n\int VfdV_{q_n}^*\varepsilon+a'_na_f$$

with $a'_n\leqq 2c_V$ implies that $(q_n\int VfdV_{q_n}^*\varepsilon_x)_{n=1}^{\infty}$ is bounded below. On the other hand, since $\lim_{n\to\infty} q_n V^*_{q_n}\varepsilon_x=0$ vaguely and $q_n\int dV^*_{q_n}\varepsilon_x=1$,

$$\lim_{n\to\infty} q_n\int VfdV^*_{q_n}\varepsilon_x = -\infty$$

(see the proof of Lemma 12 (b)). This contradiction shows that $\operatorname{supp}(V^*_p\varepsilon_x)=X$ for any $p>0$ also in case $\lambda=0$.

Now let $f_1\in C^+_K(X)$ with $\int f_1dm=1$. We see that $V_pf_1>0$ on X for any $p>0$ and $V_pf_1(x)$ increases as $p\downarrow 0$ for any $x\in X$ (by the resolvent equation). Remark that $\lim_{p\to 0} V_pf_1(x)=\infty$ for all $x\in X$. In fact, if $\lim_{p\to 0} V_pf_1(x)<\infty$ for some $x\in X$, then the equality $Vf_1(x)-V_pf_1(x)=p\int Vf_1dV^*_p\varepsilon_x+a'_{x,p}$ with $a'_{x,p}\leqq 2c_V$ implies $(p\int Vf_1dV^*_p\varepsilon_x)_{p>0}$ is bounded below and hence by the same manner as above we have a contradiction. For any $p>0$, we put

$$u_p(x) = \frac{1}{V_pf_1(x)}\,.$$

We shall show that $(u_p)_{p>0}$ is a family defining the uniform recurrence of $(V_p)_{p>0}$. It is clear that $(u_p)_{p>0}$ satisfies conditions (a), (b) and (d) in Definition 6. Furthermore the Dini theorem shows $\lim_{p\downarrow 0} u_p=0$ in $C(X)$. Let $g\in C^+_K(X)$. For any sequence $(u_{p_n}V_{p_n}g)_{n=1}^\infty$ in $(u_pV_pg)_{1\geqq p>0}$, if $(p_n)_{n=1}^\infty$ has a subsequence $(q_j)_{j=1}^\infty$ with $\lim_{j\to\infty} q_j=p_o\neq 0$, then by the Dini theorem $\lim_{j\to\infty} u_{q_j}V_{q_j}g=u_{p_o}V_{p_o}g$ in $C(X)$. Hence to verify condotion (c) in Definition 6 it is sufficient to show that for any $g\in C^+_K(X)$ with $\int gdm=1$, any compact set K in X and any $\varepsilon>0$, there exists $r_o>0$ such that

$$|u_pV_pg-u_qV_qg| < \varepsilon \quad \text{on} \quad K$$

for any $0<p,\ q<r_o$. Put $h_g=f_1-g\in C^0_K(X,\ m)$. Then $\|Vh_g\|_\infty<\infty$ and

$$|u_p(x)V_pg(x)-u_q(x)V_qg(x)| = \left|\frac{V_pg(x)-V_pf_1(x)}{V_pf_1(x)} - \frac{V_qg(x)-V_qf_1(x)}{V_qf_1(x)}\right|$$

$$\leqq u_p(x)\,|V_ph_g(x)| + u_q(x)\,|V_qh_g(x)|\,.$$

Lemma 15 gives $\|V_ph_g\|_\infty\leqq 2\|Vh_g\|_\infty$. Hence we may assume that $\|Vh_g\|_\infty\neq 0$. By the fact that $\lim_{p\to 0} u_p=0$ in $C(X)$, there exists $r_o>0$ such that for any $0<p<r_o$, $u_p<\varepsilon/4\|Vh_g\|_\infty$ on K. Then $|u_pV_pg-u_qV_qg|<\varepsilon$ on K for any $0<p,\ q<r_o$. Thus $(u_pV_pg)_{1\geqq p>0}$ forms a normal family on K. This completes the proof of Lemma 20.

LEMMA 21. *For each* $p>0$, $\{\mu\in D^+(V^*_p);\ pV^*_p\mu\leqq\mu\}=\{cm;\ c\geqq 0\}$.

PROOF. Put $S(pV^*_p)=\{\mu\in D^+(V^*_p);\ pV^*_p\mu\leqq\mu\}$. By [16, Proposition 5] $S(pV^*_p)=\{\mu\in D^+(V^*_p);\ pV^*_p\mu=\mu\}$, and by [16, Cororally 13 and Lemma 22], we see that $S(pV^*_p)$ is one-dimensional. Hence, to complete the proof, it is sufficient to show that $m \in S(pV_p^*)$. Let f_1 and $(u_p)_{p>0}$ be as in the proof of Lemma 20. Since $(u_p(x)V_p^*\varepsilon_x)_{1 \geqq p>0}$ is vaguely bounded (by (c) in Definition 6) and $u_q(x)\int f_1 dV_q^*\varepsilon_x = \int f_1 dm = 1$ $(q>0)$, Lemma 19 implies $\lim_{q\to 0} u_q(x)V_q^*\varepsilon_x = m$. Letting $q \downarrow 0$ in the equation

$$u_q(x)V_q^*\varepsilon_x - u_q(x)V_p^*\varepsilon_x = (p-q)V_p^*(u_q(x)V_q^*\varepsilon_x),$$

we obtain $m \in D^+(V_p^*)$ and $m \geqq pV_p^*m$. Thus Lemma 21 is shown.

By Lemmas 15, 16, 20 and 21, we have Theorem 7.

We now give the proof of Theorem 8.

PROOF OF THEOREM 8. Let $(x_n)_{n=1}^{\infty} \subset X$ and $(p_n)_{n=1}^{\infty} \subset R$ with $\lim_{n\to\infty} p_n = 0$ $(p_n > 0)$. Since $(p_n V_{p_n}^* \varepsilon_{x_n})_{n=1}^{\infty}$ is vaguely bounded, Lemma 19 shows that its any vaguely accumulation point is cm with some $c \geqq 0$. It is clear that if $\int dm = \infty$ then $c=0$ and if X is compact then $c = 1/\int dm$. The equality $Vf(x_n) - V_{p_n}f(x_n) = \int Vfd(p_n V_{p_n}^* \varepsilon_{x_n})$ and the fact $Vf \in C_o(X)$ show (1), (3) and the "if" part of (2). On the other hand, the equality $pV_p^*m = m$ $(p<0)$ implies the "only if" part of (2). This completes the proof.

§4. The continuous semi-group associated with a real continuous kernel

We shall show the following

THEOREM 22. *Let V be a real continuous kernel on X and let m be a positive Radon measure on X whose support is equal to X. Suppose that V satisfies the semi-complete maximum principle with respect to m and conditions* (A), (B), (C) *and* (D) *in Theorems 7 and 8. We further assume:*

(B_0) *For any $\mu \in D^0(V^*)$ and $a \in R$, $V^*\mu = am$ implies $\mu = 0$ and $a = 0$.*

(D_0) *If $\int dm < \infty$, then $\int Vfdm = 0$ for any $f \in C_K^0(X, m)$.*

Then there exists a uniquely determined uniformly recurrent markovian continuous semi-group $(T_t)_{t>0}$ such that for any $f \in C_K^0(X, m)$ and $t>0$,

$$Vf(x) = \int_0^t T_s f(x)ds + T_t Vf(x) \quad (x \in X).$$

We call the above $(T_t)_{t>0}$ the *continuous semi-group associated with V*

REMARK 23. *In the case that X is compact, D. Revuz* [14, *p.* 258] *discussed similar results under the assumption that V satisfies the semi-complete maximum principle with respect to m, V is a compact operator on $C_K^0(X, m)$ into itself and*

(B_0') *the image $V[C_K^0(X, m)]$ is dense in $C_K^0(X, m)$.*

It is easily seen that (B_0') implies (B_0).

Before the proof of Theorem 22, we recall a characterization of Hunt kernels. A continuous kernel V on X is called a *Hunt kernel* if there exists a continuous semi-group $(T_t)_{t>0}$ such that $C_K(X) \ni f \to \int_0^\infty T_t f dt$ defines a continuous kernel and and $Vf = \int_0^\infty T_t f dt$. Remark that $(T_t)_{t>0}$ is uniquely determined. It is known ([4, Proposition 1]) that V is a Hunt kernel if and only if V possesses a *resolvent* (i.e., there exists a resolvent $(V_p)_{p>0}$ such that for any $f \in C_K(X)$, $\lim_{p\to 0} V_p f = Vf$ in $C(X)$) and V is *non-degenerate* (i.e., for any x, $y \in X$ with $x \neq y$, $V^*\varepsilon_x$ is not proportional to $V^*\varepsilon_y$).

LEMMA 24. *Let V and m be as in Theorme 22 and let $(V_p)_{p>0}$ be the resolvent associated with V. Then there exists a uniquely determined markovian continuous semi-group $(T_t)_{t>0}$ such that for any $p>0$ and any $f \in C_K(X)$*

$$V_p f = \int_0^\infty e^{-pt} T_t f dt.$$

PROOF. By Lemma 18, V_p possesses the resolvent $(V_{p+q})_{q>0}$. On the other hand, the equality $V^*\varepsilon_x = V_p^*\varepsilon_x + pV^*V_p^*\varepsilon_x + a_x m$ and condition (B_0) implies that V_p is non-degenerate. Therefore V_p is a Hunt kernel such that there exists a continuous semi-group $(T_{p,t})_{t>0}$ such that $V_p f = \int_0^\infty T_{p,t} f dt$ ($f \in C_K(X)$). By the unicity of $(T_{p,t})_{t>0}$ and the fact that $(V_p)_{p>0}$ is a markovian resolvent, we see that there exists a uniquely determined markovian continuous semi-group $(T_t)_{t>0}$ such that $T_{p,t} = e^{-pt} T_t$ ($t>0$). This completes the proof.

REMARK 25. *If V further satisfies*

(A_s) *there exists a constant c_V such that for any $\mu \in D^0(V^*)$ and $a \in R$, $V^*\mu \geqq am$ implies $a \leqq c_V \int d|\mu|$,*

then each V_p is a weakly regular Hunt kernel on X in the sense given in [2] (*see* [17, *Lemme* 18] *for a proof*).

PROOF OF THEOREM 22. By Theorem 8 and condition (D_0), $\lim_{p\to 0} V_p f = Vf$ uniformly on X for any $f \in C_K^0(X, m)$. For the continuous semi-group $(T_t)_{t>0}$ given in Lemma 24, we see easily that

$$T_t V_p f = e^{pt} V_p f - e^{pt} \int_0^t e^{-ps} T_s f ds$$

for any $t>0$, $p>0$ and $f \in C_K(X)$. Letting $f \in C_K^0(X, m)$ and $p \downarrow 0$, we immediately obtain the desired equality. The uniform recurrence follows from the definition. This completes the proof.

It is well-known (see, e.g., [10]) that the continuous semi-groups associated with the real continuous kernels $G_{1,0}$, G_2, and P in Example 11 are the 1-*dimensional Gauss semi-group* $((4\pi t)^{-1/2} \exp(-(x-y)^2/4t) d\xi_1(y))_{t>0}$, the

2-dimensional Gauss semi-group $((4\pi t)^{-1}\exp(-|x-y|^2/4t)d\xi_2(y))_{t>0}$ and the *1-dimensional Poisson semi-group* $(t/(t^2+(x-y)^2)d\xi_1(y))_{t>0}$, respectively. These kernels satisfy

$$Vf(x) = \frac{1}{p}\sum_{n=1}^{\infty}(pV_p)^n f(x) \quad (x \in X)$$

and

$$Vf(x) = \int_0^{\infty} T_t f(x)dt \quad (x \in X),$$

for any $f \in C_K^0(X, m)$. Unfortunately in our general case, an additional assumption is needed to show the above equalities.

We begin with the following preparation.

LEMMA 26. *Let $(T_t)_{t>0}$ be the semi-group given in Theorem 22. Then, $\mu \in D^+(T^*)$ and $T_t^*\mu \leqq \mu$ for all $t>0$ if and only if $\mu = cm$ with some constant $c \geqq 0$. Furthermore $T_t^* m = m$ for a.e. $t>0$.*

PROOF. The "only if" part follows from Lemma 21. Let $f \in C_K^+(X)$. Then

$$\int f dm = p\int f dV_p^* m = p\int_0^{\infty} e^{-pt}\left(\int T_t^* f dm\right)dt$$

and hence from the injectivity of the Laplace transform it follows that $T_t^* m = m$ for a.e. $t>0$. Since $(0, \infty) \ni t \to \int f dT_t^* m$ is lower semi-continuous, we see $T_t^* m \leqq m$ for all $t>0$. Thus Lemma 26 is shown.

We now denote by $L^p(m)$ $(1 \leqq p \leqq \infty)$ the usual real L^p-space on X with respect to m and by $\|\cdot\|_p$ its norm. For measurable functions u and v, put $(u, v)_m = \int uv dm$ provided that the right hand side makes sense.

Let T be a continuous kernel on X such that $\int dT^*\varepsilon_x \leqq 1$ for any $x \in X$ and let $m \in D^+(T^*)$ and $T^* m \leqq m$. Then for $f \in C_K(X)$

$$\int (Tf)^2 dm = \int\left(\int f dT^*\varepsilon_x\right)^2 dm(x) \leqq \int\left(dT^*\varepsilon_x\right)\left(\int f^2 dT^*\varepsilon_x\right)dm(x) \leqq \int f^2 dm.$$

This implies that $Tf \in L^2(m)$ for any $f \in C_K(X)$ and T can be extended to a positive contraction operator on $L^2(m)$. We denote by $\tilde{T}$ its extension and by $\tilde{T}^*$ the adjoint operator of $\tilde{T}$. Clearly, $\tilde{T}^*$ is positive and contractive. Furthermore we see easily

LEMMA 27. (a) *If $u \in L^2(m)$, $(\tilde{T}^* u)dm = dT^*(um)$ as Radon measures on X.*

(b) *If T is symmetric, that is, $(g, Tf)_m = (Tg, f)_m$ for any $f, g \in C_K(X)$, then $\tilde{T} = \tilde{T}^*$.*

(c) *Let $(T_t)_{t>0}$ be a markovian continuous semi-group on X with $m \in$*

$D^+(T_t^*)$ and $T_t^* m \leqq m$ for all $t>0$. Then for $t, s>0$

$$\tilde{T}_t \tilde{T}_s = \tilde{T}_{t+s} \quad \text{and} \quad \tilde{T}_t^* \tilde{T}_s^* = \tilde{T}_{t+s}^*.$$

(d) *Let $(V_p)_{p>0}$ be a markovian resolvent on X with $m \in D^+(V_p^*)$ and $pV_p^* m \leqq m$ for all $p>0$. Then for $p, q>0$*

$$\tilde{V}_p - \tilde{V}_q = (q-p)\tilde{V}_p \tilde{V}_q \quad \text{and} \quad \tilde{V}_p^* - \tilde{V}_q^* = (q-p)\tilde{V}_p^* \tilde{V}_q^*,$$

where $\tilde{V}_p = \frac{1}{p}(p\tilde{V}_p)$ and $\tilde{V}_p^* = \frac{1}{p}(p\tilde{V}_p)^*$.

Given T as above, consider the subset of $L^2(m)$ on which all powers of both operators $\tilde{T}$ and $\tilde{T}^*$ act as isometries:

$$I(T) = \{u \in L^2(m);\ \|u\|_2 = \|\tilde{T}^n u\|_2 = \|\tilde{T}^{*n} u\|_2 \text{ for all } n \geqq 1\}.$$

The following is an essential tool in our argument.

LEMMA 28 (*see* [1, *pp.* 85–88]). (a) *If $u \in I(T)$, then $|u| \in I(T)$.*

(b) *$I(T)$ is an invariant subspace of $\tilde{T}$ and $\tilde{T}^*$, and furthermore*

$$I(T) = \{u \in L^2(m);\ u = \tilde{T}^n \tilde{T}^{*n} u = \tilde{T}^{*n} \tilde{T}^n u \text{ for all } n \geqq 1\}.$$

(c) *For $v \in L^2(m)$, any weak accumulation point of $(\tilde{T}^n v)_{n=1}^{\infty}$ or $(\tilde{T}^{*n} v)_{n=1}^{\infty}$ belongs to $I(T)$.*

(d) *If $v \perp I(T)$ (i.e., for any $u \in I(T)$, $(u, v)_m = 0$), then*

$$\lim_{n\to\infty} \tilde{T}^n v = \lim_{n\to\infty} \tilde{T}^{*n} v = 0 \quad \text{weakly in} \quad L^2(m).$$

LEMMA 29. *Let $(T_t)_{t>0}$ and $(V_p)_{p>0}$ be as in Lemma* 27 (c) *and* (d), *respectively. Then:*

(a) *For any $s>0$,*

$$\begin{aligned} I(T_s) &= \{u \in L^2(m);\ \|u\|_2 = \|\tilde{T}_t u\|_2 = \|\tilde{T}_t^* u\|_2 \text{ for all } t>0\} \\ &= \{u \in L^2(m);\ u = \tilde{T}_t \tilde{T}_t^* u = \tilde{T}_t^* \tilde{T}_t u \text{ for all } t>0\}. \end{aligned}$$

(b) *For any $p>0$, if $u \in I(pV_p)$, then $u = p\tilde{V}_p u = p\tilde{V}_p^* u$.*

PROOF. Let $u \in I(T_s)$. For given $t>0$, we choose n such that $t \leqq ns$. Then

$$\|u\|_2 = \|\tilde{T}_s^n u\|_2 = \|\tilde{T}_{ns} u\|_2 = \|\tilde{T}_{ns-t} \tilde{T}_t u\|_2 \leqq \|\tilde{T}_t u\|_2 \leqq \|u\|_2$$

and hence $\|\tilde{T}_t u\| = \|u\|_2$. Similarly $\|\tilde{T}_t^* u\|_2 = \|u\|_2$. Conversely if $\|u\|_2 = \|\tilde{T}_t u\|_2 = \|\tilde{T}_t^* u\|_2$ for all $t>0$, then taking $t = ns$ we see $u \in I(T_s)$. The second equality is an easy consequence of the Schwartz inequality (see [1, p. 85]).

Next, let $u \in I(pV_p)$ and let $q>p$. By Lemma 27 (d) and Lemma 28 (b),

$u=p\tilde{V}_p p\tilde{V}_p^* u=p\tilde{V}_q p\tilde{V}_p^* u+p(q-p)\tilde{V}_q\tilde{V}_p p\tilde{V}_p^* u$. Thus

$$\|u\|_2 \leqq \|p\tilde{V}_q p\tilde{V}_p^* u\|_2 + (q-p)\|\tilde{V}_q u\|_2 \leqq p\|\tilde{V}_q u\|_2 + (q-p)\|\tilde{V}_q u\|_2 \leqq \|u\|_2,$$

which implies $q\tilde{V}_q u=q\tilde{V}_q p\tilde{V}_p^* u=u$. Since $p\tilde{V}_p u\in I(pV_p)$ (by Lemma 28 (b)), we also see $q\tilde{V}_q u=q\tilde{V}_q(p\tilde{V}_p^* p\tilde{V}_p u)=(q\tilde{V}_q p\tilde{V}_p^*)p\tilde{V}_p u=p\tilde{V}_p u$. Hence $u=p\tilde{V}_p u$. Similarly $u=p\tilde{V}_p^* u$. This completes the proof.

We say that a real continuous kernel V on X is *absolutely continuous with respect to* m if $V^*\varepsilon_x$ is absolutely continuous with respect to m for any $x\in X$.

LEMMA 30. *Let V and m be as in Theorem 22 and let $(V_p)_{p>0}$ be the resolvent associated with V. Then*

(a) *for any $p>0$ and $x\in X$, $V_p^*\varepsilon_x$ is not singular with respect to m,*

(b) *if V is absolutely continuous with respect to m then so is V_p for any $p>0$.*

Assertion (a) is shown in the same manner as in [6, Théorème 1.8], so we omit the proof (we do not use this fact later). Assertion (b) follows directly from the equality $V^*\varepsilon_x=V_p^*\varepsilon_x+pV^*V_p^*\varepsilon_x+a_x m$ $(x\in X)$.

THEOREM 31. *Let V and m be as in Theorem 22 and let $(V_p)_{p>0}$ be the resolvent associated with V. Let $p>0$ be fixed. Then for any $f\in C_K^0(X, m)$, we have*

$$(Vf, g)_m = \frac{1}{p}\sum_{n=1}^{\infty}((pV_p)^n f, g)_m$$

for any $g\in C_K(X)$. Furthermore if V is absolutely continuous with respect to m, then

$$Vf(x) = \frac{1}{p}\sum_{n=1}^{\infty}(pV_p)^n f(x) \quad (x\in X).$$

For the proof, we first show the following

LEMMA 32. *For any $p>0$, $I(pV_p)=\{0\}$ if $\int dm=\infty$ and $I(pV_p)=\{const.\}$ if $\int dm<\infty$. In particular, for any $f\in C_K^0(X, m)$ and any $q>0$ $\lim_{n\to\infty}(pV_p)^n V_q f=\lim_{n\to\infty}(pV_p)^n f=0$ weakly in $L^2(m)$.*

PROOF. Let $u\in I(pV_p)$. By Lemma 28 (a), we may assume that $u>0$. By Lemma 29 (b) and Lemma 27 (a), the positive Radon measure um satisfies $pV_p^*(um)=um$ and hence Lemma 21 tells us $u=$const.. Since $u\in L^2(m)$, $u=0$ if $\int dm=\infty$. Hence the second assertion follows from Lemma 28 (d) if $\int dm=\infty$. If $\int dm<\infty$, Lemma 28 (c) and the facts that $\int(pV_p)^n V_q f dm=\int fd((pV_p^*)^n V_q^*)m=q^{-1}\int fdm=0$ and $\int(pV_p)^n fdm=0$ together imply the second assertion.

PROOF OF THEOREM 31. Let $f \in C_K^0(X, m)$. The equality $Vf = V_p f + pV_p Vf$ implies

$$Vf = \frac{1}{p} \sum_{n=1}^{N} (pV_p)^n f + (pV_p)^N Vf$$

for all $N \geqq 1$. Hence it is sufficient to show that $\lim_{N\to\infty}((pV_p)^N Vf, g)_m = 0$ for any $g \in C_K(X)$. Since $\lim_{p\to 0} V_p f = Vf$ uniformly on X and $pV_p 1 = 1$, we have

$$\lim_{q\to 0} \lim_{N\to\infty} ((pV_p)^N V_q f, g)_m = \lim_{N\to\infty} ((pV_p)^N Vf, g)_m.$$

By Lemma 32 we see the left hand side is equal to 0 and hence $\lim_{N\to\infty} ((pV_p)^N Vf, g)_m = 0$.

For the second assertion, we first remark that for any $q > 0$, V_q is absolutely continuous with respect to m (Lemma 30). Let $x \in X$. By the same reason as above, it is sufficient to show that

$$\lim_{N\to\infty} (pV_p)^N V_q f(x) = 0 \quad \text{for any} \quad q > 0.$$

There exists $u_{q,x} \in L^1(m)$ such that $V_q^* \varepsilon_x = u_{q,x} dm$. Since $\|(pV_p)^N f\|_\infty \leqq \|f\|_\infty$, Lemma 32 shows $\lim_{N\to\infty} \int ((pV_p)^N f) u_{q,x} dm = 0$. Since $(pV_p)^N V_q f(x) = V_q (pV_p)^N \cdot f(x)$, we obtain therefore that $\lim_{N\to\infty} (pV_p)^N V_q f(x) = 0$. This completes the proof.

THEOREM 33. *Let V and m be as in Theorem 22 and $(T_t)_{t>0}$ be the continuous semi-group associated with V. Suppose that for any $t > 0$, T_t is symmetric. Then for any $f \in C_K^0(X, m)$ we have*

$$(Vf, g)_m = \int_0^\infty (T_s f, g)_m ds$$

for any $g \in C_K(X)$. Furthermore if V is absolutely continuous with respect to m, then

$$Vf(x) = \int_0^\infty T_s f(x) ds \quad (x \in X).$$

PROOF. In Theorem 22, we have already shown that $Vf(x) = \int_0^t T_s f(x) ds + T_t Vf(x)$ $(x \in X)$ for any $t > 0$. Hence it is sufficient to show that $\lim_{t\to\infty} (T_t Vf, g)_m = 0$ for any $g \in C_K(X)$. Assume, to the contrary, that there exist $g \in C_K(X)$ and a sequence $(t_n)_{n=1}^\infty$ with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} (T_{t_n} Vf, g)_m \neq 0$. We may assume that there exists $\varepsilon > 0$ such that $(T_{t_n} Vf, g)_m > \varepsilon$ for all $n \geqq 1$. For $t < t'$

$$|(T_t Vf, g)_m - (T_{t'} Vf, g)_m| \leqq \int_t^{t'} |(T_s Vf, g)_m| ds \leqq (t'-t) \|Vf\|_\infty (1, |g|)_m.$$

This implies that the function $(0, \infty) \ni t \to (T_t Vf, g)_m$ is uniformly continuous and hence there exists $t_o > 0$ such that $T_{t_o}^* m = m$ and

$$\limsup_{t\to\infty} (T_{nt_o}Vf, g)_m > \varepsilon/2.$$

Since $\lim_{q\to 0} V_q f = Vf$ uniformly on X, there exists $q_o>0$ such that $\limsup_{n\to\infty} (T_{nt_o}V_{q_o}f, g)_m > \varepsilon/4$. On the other hand, by the condition that each T_t is symmetric and by Lemma 29 (a), we see

$$I(T_{t_o}) = \{u \in L^2(m);\ \tilde{T}_t^* u = u \text{ for any } t>0\}.$$

Then, it follows from Lemma 26 that $I(T_{t_o})=\{0\}$ if $\int dm=\infty$ and $I(T_{t_o})=\{\text{const.}\}$ if $\int dm<\infty$. So in the same manner as in Lemma 32, we have $\lim_{n\to\infty} (T_{nt_o} V_{q_o}f, g)_m = 0$, which is a contradiction.

The second assertion can be shown in the same manner as the corresponding part of Theorem 31. This completes the proof.

REMARK 34. *In the case that T_t, $t>0$, are all absolutely continuous with respect to m, the assumption that T_t, $t>0$, are symmetric can be removed in the above theorem.*

In fact, in the above proof, we used the symmetrictiy only to show that $I(T_t)=\{0\}$ if $\int dm=\infty$ and $I(T_t)=\{\text{const.}\}$ if $\int dm<\infty$ for $t>0$. However if T_t is absolutely continuous with respect to m, [1, p. 52, Theorem A] shows that there exists an $m\times m$-measurable function $\rho_t(x, y)$ on $X\times X$ such that for any $f\in C_K(X)$

$$T_t f(x) = \int \rho_t(x, y) f(y) dm(y) \quad m - a.e. \quad x\in X.$$

Since $(T_t)_{t>0}$ is uniformly recurrent, we may consider that $\tilde{T}_t$ is a Harris proccess (see [1, p. 58]) and hence $I_o=\{A;\ \chi_A\in I(T_t)\}$ is atomic (see [1, p. 58, Theorem D and p. 87, Theorem B]), where χ_A is the characteristic function of A. Let A be an atom in I_o. Then the argument in [1. p. 90] shows that either $\tilde{T}_t^n\chi_A$, $n=0, 1,\dots$, are all distinct, or there exists an integer $k\geqq 1$ with $\tilde{T}_t^{*k}\chi_A=\tilde{T}_t^k\chi_A=\chi_A$. But the Hopt maximal ergodic lemma [1, p. 11, (2.1)] shows that the first case does not occur. Remarking that $I(T_t)$, and hence I_o, is independent of $t>0$, we see that for t, $t'>0$ with t/t' irrational, there exist n, $m\geqq 1$ such that $\tilde{T}_{nt}^*\chi_A=\tilde{T}_{mt'}^*\chi_A=\chi_A$. This implies that $\{s\in[0, \infty);\ T_s^*(\chi_A m)=\chi_A m\}$ is dense in $[0, \infty)$. Since $s\to \int T_s f d\chi_A m$ $(f\in C_K^+(X))$ is lower semi-continuous, $T_s^*(\chi_A m)\leqq \chi_A m$ for every $s\geqq 0$. By Lemma 26, we see $I_o=\{\emptyset\}$ if $\int dm=\infty$ and $I_o=\{X\}$ if $\int dm<\infty$. Since I_o generates $I(T_t)$ ([1, p.87, Theorem B]), we have $I(T_t)=\{0\}$ if $\int dm=\infty$ and $I(T_t)=\{\text{const.}\}$ if $\int dm<\infty$.

§5. Neumann kernels as our examples

In this section we shall discuss the Neumann kernel as an example of a continuous kernel satisfying the semi-complete maximum principle (cf. [10, Example 5]). We consider the same setting as in S. Itô's paper [9]. Let D be a relatively compact subdomain of n-dimensional orientable C^∞-manifold whose boundary $S=\bar{D}-D$ consists of a finite number of $(n-1)$-dimensional simple hypersurfaces of class C^2. Let A be an elliptic differential operator of the form:

$$Au(x) = \frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{a(x)}\left(a^{ij}(x)\frac{\partial u(x)}{\partial x^j} - b^i(x)u(x)\right)\right)$$

for $u\in C^2(D)$, where $\|a^{ij}(x)\|$ and $\|b^i(x)\|$ $(1\leq i, j\leq n)$ are contravariant tensors of class C^2 on $\bar{D}$, $\|a^{ij}(x)\|$ is symmetric and strictry positive definite and $a(x)=\det\|a_{ij}(x)\|=\det\|a^{ij}(x)\|^{-1}$. We denote by dx and dS_ξ respectively the volume element in D and the hypersurface element on S with respect to the Riemannian metric defined by $\|a_{ij}(x)\|$. We also denote by $\frac{\partial u(\xi)}{\partial n_\xi}$ and $\beta(\xi)$ respectively the outer normal derivative of $u(x)$ and the outer normal component of the vector $\|b^i(x)\|$ at the point $\xi\in S$. The adjoint differential operator A^* of A is defined as follows:

$$A^*u(x) = \frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{a(x)}\,a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right)+b^i(x)\frac{\partial u(x)}{\partial x^i}$$

for $u\in C^2(D)$. Let $U(t, x, y)$ be the fundamental solution (for definition, see [8]) of the initial-boundary value problem of the parabolic equation:

$$\frac{\partial u}{\partial t} = Au+f\ (t>0,\ x\in D),\ u|_{t=0} = u_o \text{ and } \frac{\partial u}{\partial n} - \beta u = \psi\ (t>0,\ x\in S).$$

Then $U(T, x, y)$ is also the fundamental solution of the adjoint initial-boundary value problem:

$$\frac{\partial u}{\partial t} = A^*u+f\ (t>0,\ x\in D),\ u|_{t=0} = u_o \text{ and } \frac{\partial u}{\partial n} = \psi\ (t>0,\ x\in S).$$

The family of continuous kernels $(U_t)_{t>0}$ on $X=\bar{D}$ defined by

$$U_tf(y) = \int U(t, x, y)f(x)dx, \quad f\in C(X)$$

is a markovian continuous semi-group. In [9], it is shown that there exists a function $\omega(x)>0$ on X satisfying

$$\int\omega(y)U(t, y, x)dy = \omega(x) \quad\text{and}\quad \int\omega(x)dx = 1$$

and that

$$K(y, x) = \int_0^\infty (U(t, y, x)-\omega(x))dt$$

is well-difined whenever $x, y \in X$ and $x \neq y$, and is a kernel function of the boundary value problem (Neumann problem)

$$Au(x) = f(x) \quad \text{in} \quad D \quad \text{and} \quad \frac{\partial u(\xi)}{\partial n_\xi} - \beta(\xi)u(\xi) = \psi(\xi) \quad \text{on} \quad S$$

and also the adjoint problem

$$A^*u(x) = f(x) \quad \text{in} \quad D \quad \text{and} \quad \frac{\partial u(\xi)}{\partial n_\xi} = \psi(\xi) \quad \text{on} \quad S.$$

The real continuous kernel K on X defined by

$$Kf(x) = \int K(y, x)f(y)dy, \quad f \in C(X)$$

satisfies the semi-complete maximum principle with respect to $\omega(x)dx$. In fact, for any $f \in C_K^0(X, \omega dx)$,

$$\lim_{t\to\infty} \int_0^t U_s f(y)ds = \lim_{t\to\infty} \int_0^t (U(s, x, y) - \omega(x))f(x)dxds = Kf(y)$$

and the convergence is uniform on X (see [9, Theorem 2 and p. 27, (3.10)]), and hence Remark 3 (d) shows our assertion. We also see that $(U_t)_{t>0}$ is uniformly recurrent.

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