HIROSHIMA MATH. J.
16 (1986), 539–552

On elliptic equations related to self-similar solutions for nonlinear heat equations

Yoshikazu GIGA*)

(Received September 20, 1985)

1. Introduction

This paper studies the existence and nonexistence of global solutions for

$$\Delta w - \left(\frac{1}{2} x \cdot \nabla w + \alpha w\right) + |w|^{p-1} w = 0 \tag{1.1}$$

in $\boldsymbol{R}^n$ for various $p > 1$, $\alpha \geqq 0$, where $x \cdot \nabla = \sum_{j=1}^{n} x_j \partial/\partial x_j$.

In [8] we studied the blow-up of solutions of the semilinear heat equation

$$u_t - \Delta u - |u|^{p-1} u = 0. \tag{1.2}$$

We have shown that the asymptotic behavior near the blow-up time is described by special solutions of (1.2) called *backward self-similar solutions*, i.e., functions of the form

$$u(x, t) = (-t)^{-1/(p-1)} w(x/(-t)^{1/2}) \tag{1.3}$$

which solve (1.2) in $\boldsymbol{R}^n \times (-\infty, 0)$; see also [7]. Plugging (1.3) in (1.2) yields an elliptic equation (1.1) for w with $\alpha = 1/(p-1)$.

In [8] we have proved that (1.1) has no bounded global solutions except constant solutions provided $\alpha = 1/(p-1)$ and $n/2 \leqq (p+1)/(p-1)$ (equivalently, $p \leqq (n+2)/(n-2)$ or $n \leqq 2$). In this paper α is considered a *parameter*. It turns out that $1/(p-1)$ is a 'bifurcation point', namely, there is a nonconstant bounded global solution to (1.1) provided $\alpha > 1/(p-1)$ and $n/2 < (p+1)/(p-1)$. For technical reasons we confine ourselves to radial functions, i.e., functions depending only on $r = |x|$. A radial function w is called *radially decreasing* if w is monotonically decreasing as a function of $r > 0$.

THEOREM 1. (*Existence*) *There is a positive radially decreasing solution w of* (1.1) *in $\boldsymbol{R}^n$ provided $\alpha > 1/(p-1)$ and $n/2 < (p+1)/(p-1)$.*

THEOREM 2. (*Asymptotic behavior*) *A positive radially decreasing so-*

---

*) The research was partially supported by the Educational Project for Japanese Mathematical Scientists and the Sakkokai Foundation. Most part of the research was done when the author was in the Courant Institute.

lution w of (1.1) *outside some ball with center zero satisfies the estimate*

$$0 < w(r) \leqq M/r^{2\alpha} \tag{1.4}$$

with some constant M independent of r. Moreover, there is a constant c_0 *such that*

$$\lim_{r\to\infty} w(r)r^{2\alpha} = c_0 > 0 \tag{1.5}$$

provided $\alpha+2\geqq n$.

THEOREM 3. (*Nonexistence*) *If* $\alpha\leqq 1/(p-1)$, *there are no positive radially decreasing solutions of* (1.1) *in* $\boldsymbol{R}^n$ *provided* $n/2\leqq(p+1)/(p-1)$. (*Note that the critical exponent is included.*)

The major difficulty comes from the second term in (1.1). If we drop this term, the equation becomes

$$\Delta u - u + |u|^{p-1}u = 0, \tag{1.6}$$

which is well studied. Using a variational method, Strauss [11] has constructed a global positive radially decreasing solution of (1.6) when $n/2<(p+1)/(p-1)$. In [2] Berestycki, Lions and Peletier give an ODE (ordinary differential equation) approach called the shooting method, to construct a positive solution. If we change the sign in front of $(x\cdot\nabla w/2+\alpha w)$ in (1.1), we get the equations related to forward self-similar solutions [7] of (1.2):

$$\Delta w + \frac{1}{2}x\cdot\nabla w + \alpha w + |w|^{p-1}w = 0. \tag{1.7}$$

This equation is first attacked by Haraux and Weissler [9] for $\alpha=1/(p-1)$ using an ODE approach; for more recent results see [10]. Recently, Escobedo and Kavian [5] extend their results by using a variational method. The results read: when $n/2<(p+1)/(p-1)$ there are always infinitely many rapidly decreasing solutions; however, the existence of a positive solution is proved only when $\alpha<n/2$. There are some results for (1.7) even if $n/2\geq(p+1)/(p-1)$, [1, 5, 9, 10]. The equation having an opposite sign in front of the nonlinear term in (1.7) is studied by Brezis, Peletier and Terman [3] for $\alpha=1/(p-1)$. Their results are extended by Escobedo and Kavian [5].

Compared with (1.6) and (1.7), our original problem (1.1) has different aspects. First, a bifurcation point for α is not $n/2$ but $1/(p-1)$. Second, the decay of solution is not exponential but of finite order. Recently, Peletier, Terman and Weissler [10] show that there are no H^1 solution for (1.1) if $\alpha<n/4$ and $n/2<(p+1)/(p-1)$. Their nonexistence result is compatible with our existence result, at least when $\alpha+2\geq n$, because (1.5) implies that w is not in $L^2(\boldsymbol{R}^n)$ if $\alpha<n/4$.

To show the existence we are forced to appeal to the shooting method since variational approaches [5, 11] apparently fail to work because of the lack of compactness. Our method is related to that of [2]. We rewrite (1.1) for positive radial functions and obtain an ODE:

$$w'' + \frac{n-1}{r} w' - \frac{rw'}{2} - \alpha w + w^p = 0, \quad ' = d/dr. \tag{1.8}$$

We try to find an initial value η giving a positive decreasing solution to (1.8) with

$$w(0) = \eta, \quad w'(0) = 0. \tag{1.9}$$

To carry out this idea we use Sturm's comparison lemma for oscillations since the method in [2] is not applicable.

In Section 2 we prove Theorem 1. The asymptotic behavior of solution is studied in Section 3. The proof of Theorem 2 is rather technical. In Section 4 we prove our nonexistence result. The key tool is a Pohozaev-type identity, a particular case of which is used in [8]. To avoid technical and notational complexity we do not attempt any possible generalizations for the nonlinear term $|w|^{p-1}w$.

I am grateful to Professor Robert Kohn and Professor Louis Nirenberg for valuable discussions and encouragements.

2. Existence

The goal of this section is to find the initial data η such that the corresponding solution $w(\eta, r)$ of (1.8–9) are positive and decreasing for all $r>0$. We begin by reviewing the idea of the shooting method [2]. We observe that the function $f(w) = -\alpha w + w^p$ changes its sign from negative to positive only at $k = \alpha^{1/(p-1)}$. For technical reasons we put $f(w)=0$ if $w \leq 0$. If $\eta > k$, $w(\eta, r)$ of (1.8–9) i.e. the solution of

$$w'' + \frac{n-1}{r} w' - \frac{rw'}{2} + f(w) = 0, \quad f(w) = -\alpha w + w^p \tag{2.1}$$

$$w(0) = \eta, \quad w'(0) = 0 \tag{2.2}$$

is decreasing on a sufficiently small interval $(0, \delta)$, $\delta > 0$, since w takes no local minimum larger than k. We classify initial data $\eta > k$ by the behavior of $w(\eta, r)$. Let I_+ be the set of initial data η such that $w(\eta, r)$ attains positive local minimum before it reaches zero:

$$I_+ = \{\eta > k;\ \exists r_0 > 0 \text{ such that } w'(\eta, r_0) = 0 \text{ and } w(\eta, r) > 0 \text{ for } 0 \leq r < r_0\}.$$

Let I_- be the set of η such that $w(\eta, r)$ reaches zero before it attains local minimum: Since w is decreasing on $(0, \delta)$ for some $\delta>0$, we have

$$I_- = \{\eta>k;\ \exists r_0>0 \text{ such that } w(\eta, r_0)=0 \text{ and } w'(\eta, r)<0 \text{ for } 0\le r<r_0\}.$$

The complement of the union of I_+ and I_- on (k, ∞) consists of initial data ζ we are looking for. ($w(\eta, r)$ exists globally for every $\eta>k$ as we shall prove in Proposition 1.) By the continuous dependence on initial data I_+ and I_- are open. Since (k, ∞) is connected and I_+ and I_- are mutually disjoint, there exists an initial data ζ such that $w(\zeta, r)>0$ and $w'(\zeta, r)>0$ for $r>0$, provided that both I_+ and I_- are nonempty.

It remains to show that I_+ and I_- are nonempty. Unfortunately, the method for (1.6) in [2] should not be applicable. To show $I_-\neq\phi$, we also use a variational approach, which requires the restriction on p, $n/2<(p+1)/(p-1)$. However, to get radially decreasing solutions extra difficulty arises since results in [6] are not applicable to (2.1). We are forced to use Sturm's comparison lemma. We construct a radially decreasing solution for the Dirichlet problem on a ball with sufficiently small radius. This process is carried out in Propositions 2 and 3. The proof for $I_+\neq\phi$ is substantially different from [2]. We apply Sturm's comparison lemma to compare the original problem with the linearized problem around k. We shall show in Proposition 4 that if α is larger than $1/(p-1)$, the solution $w(\eta, r)$ oscillates at least once provided $\eta>k$ is sufficiently close to k. If $\alpha\le 1/(p-1)$, I_+ should be empty, otherwise it would contradict the nonexistence results in Theorem 3.

PROPOSITION 1. *For every $\eta>0$, there is a unique global solution of* (2.1–2).

PROOF. Although (2.1) is singular at $r=0$, there is a unique local solution w since $w'(0)=0$. Let $[0, r_\eta)$ be the maximal interval on which the solution w is defined. We shall prove below that $r_\eta=\infty$. Let $\theta(r)=\exp(-r^2/2)$. Multiplying (2.1) by $\theta w'$ yields

$$\frac{1}{2}(\theta|w'|^2)' + \frac{n-1}{r}\theta|w'|^2 + \theta F(w)' = 0,$$

where $F(w)=\int_0^w f(s)ds$. Integrating this equation over $(0, r)$ gives

$$\frac{1}{2}\theta(r)|w'|^2(r) + \int_0^r \frac{n-1}{s}\theta|w'|^2 ds + \int_0^r \theta F(w)' ds = 0$$

since $w'(0)=0$. Integrating by parts, we have

$$\text{(2.3)}\quad \frac{1}{2}\theta(r)|w'|^2(r) + \int_0^r \frac{n-1}{s}\theta|w'|^2 ds + \theta(r)F(w(r)) + \int_0^r s\theta F(w) ds = F(\eta).$$

Suppose that $r_\eta < \infty$. We may assume that $w(r) \to \infty$ as $r \to r_\eta$ since otherwise $w'(r)$ would be bounded near r_η which contradicts the maximality of r_η. Since $w(r) \to \infty$ as $r \to r_\eta$, the second term in (2.3) tends to $+\infty$ while the first two terms are positive. Thus the third term should tend to $-\infty$. However, this is impossible because $F(w)$ is bounded below. Therefore we have proved $r_\eta = \infty$ which completes the proof. □

To show $I_- \neq \phi$, we consider the boundary value problem

$$\Delta w - \frac{1}{2} x \cdot \nabla w - \alpha w + w^p = 0 \quad \text{in} \quad B(R) = \{|x| < R\} \tag{2.4}$$

$$w|_{\partial B(R)} = 0. \tag{2.5}$$

We would like to get positive radially decreasing solutions. If we consider the same problem for (1.6) we know by [6] that all positive solutions are radially decreasing. For (2.4–5), results in [6] are not applicable, so we directly find a positive radially decreasing solution. We first construct a positive radial solution by a variational method (cf. [11]).

PROPOSITION 2. *Assume $n/2 < (p+1)/(p-1)$. Then* (2.4–5) *has a positive radial solution.*

PROOF. Consider a minimizing problem for

$$I(w) = \frac{1}{2} \int_{B(R)} (|\nabla w|^2 + \alpha |w|^2) \rho dx$$

under constraints:

$$\int_{B(R)} |w|^{p+1} \rho dx = 1, \quad w|_{\partial B(R)} = 0, \quad w \textit{ is radial},$$

where $\rho(x) = \exp(-|x|^2/4)$. If $n/2 < (p+1)/(p-1)$, a minimizing sequence converges to some radial function w_0 strongly in L^{p+1} since the inclusion $H_0^1 \subset L^{p+1}$ is compact. We may assume that w_0 is nonnegative by taking $|w_0|$ if necessary. The function w_0 solves the Euler-Lagrange equation:

$$\int_{B(R)} \rho (\nabla w_0 \cdot \nabla \varphi + \alpha w_0 \cdot \varphi) - \mu \int_{B(R)} \rho w_0^p \varphi dx = 0 \tag{2.6}$$

with some constant μ for all *radial* $\varphi \in H_0^1(B(R))$. Since $\nabla \cdot (\rho \nabla w_0)$ is radial, integrating by parts yields

$$\frac{1}{\rho} \nabla \cdot (\rho \nabla w_0) - \alpha w_0 + \mu w_0^p = 0 \quad \text{in} \quad B(R).$$

We see the multiplier μ should be positive by plugging $\varphi=w_0$ in (2.6) and noting $I(w_0)>0$. The function $w=\mu^{1/(p-1)}w_0$ is a nontrivial nonnegative radial solution of (2.4–5). Since all nonnegative solution must be positive in $B(R)$ by the maximum principle, w is the desired positive solution of (2.4–5). □

The solution constructed in Proposition 2 may not be radially decreasing. We shall apply Sturm's comparison lemma to prove that all radial solutions are monotone provided the radius is sufficiently small. We give a version of Sturm's lemma and its proof for completeness.

LEMMA 1 (Sturm's comparison). *Suppose that u and v solve differential equations*

$$(\sigma u')' + \sigma q_1 u = f_1 \tag{2.7}$$

$$(\sigma v')' + \sigma q_2 v = -f_2 \tag{2.8}$$

on an interval (a, b), where $\sigma>0$, q_1, q_2 are continuous functions and $f_1, f_2\geq 0$. Suppose that $u>0$ on (a, b) and $u(b)=0$. At a we assume either $u(a)=v(a)=0$, $v'(a)>0$ or $u'(a)=v'(a)=0$, $v(a)>0$. If $q_2\geq q_1$, then v has zero in (a, b) unless $q_1\equiv q_2, f_1\equiv f_2\equiv 0$.

PROOF. Suppose v had no zero in (a, b). Then $v>0$ on (a, b). Computing $v\cdot(2.7)-u\cdot(2.8)$ yields

$$v(\sigma u')' - u(\sigma v')' + \sigma(q_1-q_2)uv = vf_1 + uf_2.$$

Integrating this over (a, b) gives

$$\sigma(vu'-uv')\,|_a^b + \int_a^b \sigma(q_1-q_2)uvdr - \int_a^b (vf_1+uf_2)dr = 0.$$

Since $u'(b)\geqq 0$, the boundary condition yields

$$\sigma(vu'-uv')\,|_a^b \leq 0.$$

This leads to a contradiction since the other two terms are strictly negative unless $q_1=q_2, f_1=f_2=0$. Thus, the proof is completed. □

We shall use Lemma 1 to compare (2.1) with its linearized equation around non-zero equilibrium $k=\alpha^{1/(p-1)}$. We recall some properties of eigenvalues for

$$w'' + \frac{n-1}{r}w' - \frac{r}{2}w' + \lambda w = 0$$

or

$$(\sigma w')' + \lambda\sigma w = 0 \quad \text{with} \quad \sigma = r^{n-1}\exp(-r^2/4). \tag{2.9}$$

We consider (2.9) on (a, b) with $w(a)=w(b)=0$, where a is positive, and denote the first eigenvalue by $\lambda(a, b)$. If $b=\infty$, we understand that there is no condition at infinity except that $\int^{\infty}|w|^2\sigma dr$ is finite. When we consider (2.9) on $(0, b)$ with $w'(0)=w(b)=0$, the first eigenvalue is denoted by $\lambda(b)$.

LEMMA 2. (i) *If* $b_1>b_2>0$, *then* $\lambda(b_2)>\lambda(b_1)$. *Also* $\lambda(b)\to\infty$ *as* $b\to 0$.
(ii) *Relation* $(a_1, b_1)\subset(a_2, b_2)$ *implies* $\lambda(a_1, b_1)>\lambda(a_2, b_2)$ *unless* $a_1=a_2$, $b_1=b_2$. *Also* $\lambda(a, b)\to\infty$ *as* $b\to a$.
(iii) *For* $a<b$ *we have* $\lambda(b)<\lambda(a, b)$.
(iv) $\lambda(\sqrt{2n})=1$, $\lambda(\sqrt{2n}, \infty)\leqq 1$.

PROOF. The first three results are standard (e.g. [4]) since $\lambda(b)$ and $\lambda(a, b)$ are, respectively

$$\lambda(b)=\inf_{w(b)=0}\left(\int_0^b \sigma|w'|^2 dr/\int_0^b \sigma|w|^2 dr\right)$$

$$\lambda(a, b)=\inf_{w(a)=w(b)=0}\left(\int_a^b \sigma|w'|^2 dr/\int_a^b \sigma|w|^2 dr\right).$$

It remains to prove (iv). The function r^2-2n solves (2.9) with $\lambda=1$. Since $\sqrt{2n}$ is the only zero of r^2-2n on $(0, \infty)$ and positive eigenfunctions correspond to the first eigenvalue, we obtain $\lambda(\sqrt{2n})=1$. The variational definition for $\lambda(a, \infty)$ immediately yields $\lambda(\sqrt{2n}, \infty)\leq 1$ by plugging $w=r^2-2n$. (It is not difficult to check $\lambda(\sqrt{2n}, \infty)=1$; however, we skip it since we do not use it in the sequel.)

PROPOSITION 3. (i) *All radial solutions of* (2.4–5) *are radially decreasing provided R is sufficiently small.*
(ii) *The set* I_- *for* (2.1–2) *is nonempty provided* $n/2<(p+1)/(p-1)$.

PROOF. (i) We begin by rewriting (2.1) by $w=k-u$, $k=\alpha^{1/(p-1)}$:

$$(\sigma u')' + (p-1)\alpha\sigma u = h(u), \tag{2.10}$$

where $h(u)>0$ unless $u=0$ and $h(u)=o(|u|)$. Suppose that w were not monotone decreasing on $(0, R)$. Then there would exist $(0, b)$ or (a, b) $(a>0)$ in $(0, R)$ such that $u>0$ on $(0, b)$ or (a, b), respectively, with $u'(0)=u(b)=0$ (resp. $u(a)=u(b)=0$). Applying Lemma 1 to (2.10) and

$$(\sigma v')' + (p-1)\alpha\sigma v = 0,$$

we obtain that $\lambda(b)$, $\lambda(a, b)\leq(p-1)\alpha$ by Lemma 2 (i) (ii). If R is small enough, by Lemma 2 (i), we have $\lambda(R)>(p-1)\alpha$. This implies that $\lambda(R)>\lambda(b)$ or $\lambda(a, b)$ for some $0<a<b<R$ which is absurd by Lemma 2 (i) (ii). We thus conclude that

w is monotone decreasing provided R is sufficiently small.

(ii) Proposition 2 and (i) imply that I_- is nonempty.

PROPOSITION 4. *If $\alpha > 1(p-1)$, then I_+ contains $(k, k+\delta)$ for some small $\delta > 0$. In particular, I_+ is nonempty.*

PROOF. As in Proposition 3, plugging $w = v + k$ in (2.1) gives

$$(\sigma v')' + (p-1)\alpha\sigma v + h(v) = 0, \quad v(0) > 0, \quad v'(0) = 0 \tag{2.11}$$

where $h(v) > 0$ unless $v = 0$ and $h(v) = o(|v|)$. Since $\alpha(p-1) > 1$, applying Lemma 1 to (2.11) and

$$(\sigma u')' + \sigma u = 0, \quad u'(0) = 0, \quad u(\sqrt{2n}) = 0,$$
$$u > 0 \quad \text{in} \quad (0, \sqrt{2n}) \quad \text{(cf. Lemma 2 (iv))}$$

shows that the first zero r_* of v is less than $\sqrt{2n}$.

Let δ be a small positive number such that $1 + \delta < \alpha(p-1)$. Since $\lambda(\sqrt{2n}, \infty) \leqq 1$, applying Lemma 2 (ii) yields that there is R_0 such that $\lambda(\sqrt{2n}, R_0) = 1 + \delta$. Since $\lambda(r_*, R_0) < \lambda(\sqrt{2n}, R_0)$, there is $R < R_0$ such that $\lambda(r_*, R) = 1 + \delta$; here, we again apply Lemma 2 (ii). In other words,

$$(\sigma u')' + (1+\delta)\sigma u = 0 \quad \text{in} \quad (r_*, R) \tag{2.12}$$

with $u(r_*) = u(R) = 0$ has a positive solution.

There is a constant ε such that

$$\alpha(p-1)\sigma z + h(z) = (1+\delta)\sigma z + g(z) \qquad \text{for} \quad |z| \leqq \varepsilon$$

with $g(z) \cdot z > 0$ (unless $z = 0$). We compare

$$(\sigma v')' + (1+\delta)\sigma v + g(v) = 0 \quad \text{in} \quad (r_*, R)$$
$$v(r_*) = 0, \quad v'(r_*) < 0$$

with (2.12), where $v + k = w$ solves (2.1) on $(0, R)$. Since R_0 is independent of initial data $v(0)$, we may assume $|v| \leqq \varepsilon$ on $(0, R_0)$ by taking initial data $v(0) = w(0) - k$ sufficiently small. Applying Lemma 1 by taking $v = -v$, we see that v should have a zero in (r_*, R). In particular w has a local minimum in (r_*, R). This means that all initial data $> k$ sufficiently close to k belong to I_+, provided $\alpha > 1/(p-1)$. This completes the proof.

PROOF OF THEOREM 1. If $\alpha > 1/(p-1)$ and $n/2 < (p+1)/(p-1)$, both I_+ and I_- are nonempty by Proposition 4 and Proposition 3 (ii). Since I_+ and I_- are open (cf. [2]), the complement of the union of I_+ and I_- in (k, ∞) is nonempty. By Proposition 1, this implies that there exists an initial data ζ such that $w(\zeta, r) > 0$ and $w'(\zeta, r) < 0$ for $r > 0$ where w solves (2.1–2). Since (2.1–2) is (1.1) for radial functions, the proof is completed.

3. Asymptotic behavior

This section is devoted to the proof of Theorem 2.

PROPOSITION 5. *Suppose that* $w>0$ *solves* (2.1) *in* (a, ∞) *and is decreasing, where* $a \geq 0$. *Then* $\lim_{r \to \infty} w(r)=0$.

PROOF. Let q be the limit of w as $r \to \infty$. We first observe that $q<k=\alpha^{1/(p-1)}$, since otherwise $w'' \leq 0$ for sufficiently large r which contradicts $w \geq k$ (unless $w=k$).

Divide (2.1) by r to get

$$\frac{w''}{r}+\frac{n-1}{r^2}w'-\frac{w'}{2}=-\frac{f(w)}{r}. \tag{3.1}$$

Since $q<k$, there is r_0 such that $f(w)(r) \leq 0$ on (r_0, ∞).

We shall claim that the left hand side of (3.1)

$$g=\frac{w''}{r}+\frac{n-1}{r^2}w'-\frac{w'}{2}$$

is integrable on (r_0, ∞). Since w' is integrable on (r_0, ∞) the second two terms of g are integrable. Since $g \geq 0$ on (r_0, ∞), it remains to prove that there is a sequence $r_j \to \infty$ such that

$$\int_{r_0}^{r_j} \frac{w''}{r} dr \quad \text{exists as} \quad j \to \infty.$$

Integrating by parts yields

$$\int_{r_0}^{r} \frac{w''}{r} dr = \frac{w'}{r}\Big|_{r_0}^{r}+\int_{r_0}^{r} \frac{w'}{r^2} dr.$$

The integrand of the right hand side (RHS) is integrable on (r_0, ∞). Since w' is integrable, there is a sequence $r_j \to \infty$ such that $w'(r_j) \to 0$ as $j \to \infty$. We have thus proved that g is integrable.

Since g is integrable on (r_0, ∞) so is $f(w)/r$ by (3.1). Thus, $q<k$ should equal zero since otherwise $f(w)/r$ would not be integrable on (r_0, ∞).

PROPOSITION 6. *Suppose that* $w>0$ *solves* (2.1) *in* (a, ∞) *and is monotone decreasing, where* $a \geq 0$. *Then, for a given* $\theta<\alpha$

$$w(r) \leq \frac{C}{r^{2\theta}} \tag{3.2}$$

with a constant C *independent of* r.

PROOF. Since $w(r)\to 0$ by Proposition 5 and $w'<0$, (2.1) gives

$$w'' - \frac{rw'}{2} - \mu w \geq 0,$$

where $\mu<\alpha$. The function $W=Mr^{-2\theta}$, $M>0$, $\theta<\mu$ solves

$$W'' - \frac{rW'}{2} - \left(\theta + \frac{2\theta(2\theta+1)}{r^2}\right) W = 0.$$

For a large r, say $r\geq r_0$, W satisfies

$$W'' - \frac{rW'}{2} - \mu W \leq 0.$$

Take M large so that $W(r_0)>w(r_0)$. By comaprison we conclude $w\leq W$ for $r\geq r_0$, which is the same as (3.2). □

The estimate (3.2) is not sharp. In fact we may replace θ in (3.2) by α.

PROPOSITION 7. *Suppose that $w>0$ solves* (2.1) *in* (a, ∞) *and is decreasing, where $a\geq 0$. Then*

$$w(r) \leq \frac{C}{r^{2\alpha}} \tag{3.3}$$

with a constant C independent of r.

PROOF. We transform the dependent variable by $w=r^{-2\alpha}z$. Since

$$w' = \left(-\frac{2\alpha z}{r} + z'\right) r^{-2\alpha} \tag{3.4}$$

$$w'' = \left(\frac{2\alpha(2\alpha+1)}{r^2} z - \frac{4\alpha z'}{r} + z''\right) r^{-2\alpha},$$

(2.1) can be written as

$$z'' - \frac{1}{2} rz' + \left(\frac{n-1-4\alpha}{r}\right) z' + \frac{2\alpha(\alpha+2-n)}{r^2} z + \frac{z^p}{r^{2\alpha(p-1)}} = 0. \tag{3.5}$$

The estimate (3.2) yields for every δ

$$z(r) \leq Cr^{\delta}, \quad r > a \tag{3.6}$$

with $C=C(\delta)$. Since $w'\leq 0$, (3.6) together with (3.4) yields

$$z'(r) \leq C'r^{\delta-1} \tag{3.7}$$

with $C'=C'(\delta, \alpha)$. Applying (3.6) and (3.7) to (3.5), we have for small $\varepsilon>0$

$$\left|\frac{z''(r)}{r}-\frac{z'(r)}{2}\right|\leq\frac{M}{r^{1+\varepsilon}}\ r>a \tag{3.8}$$

for some constant $M=M(\varepsilon, \alpha, p)$.

Integrating by parts yields

$$\int_{r_0}^{r}\left(\frac{z''}{s}-\frac{z'}{2}\right)ds=\frac{z'}{s}\bigg|_{r_0}^{r}+\int_{r_0}^{r}\frac{z'}{s^2}ds-\frac{1}{2}(z(r)-z(r_0)).$$

By (3.7) the first two terms of RHS converge as $r\to\infty$. This implies that $\lim_{r\to\infty} z(r)$ exists since LHS converges as $r\to\infty$ by (3.8). In particular z is bounded which means that $w\cdot r^{2\alpha}$ is bounded. Thus, we have completed the proof.

PROPOSITION 8. *Suppose that $w>0$ solves (2.1) in (a, ∞) and is decreasing, where $a\geq 0$. Then there is a positive constant c_0 such that*

$$w(r)r^{2\alpha}\longrightarrow c_0 \tag{3.9}$$

as $r\to\infty$ provided $\alpha+2\geq n$.

PROOF. We shall claim z in the Proof of Proposition 7, is monotone increasing provided $\alpha+2\geq n$. We argue by contradiction. Suppose that z were not monotone increasing. Since $\alpha+2\geq n$, (3.5) says that there are no positive minima of z. We may assume $z'<0$ on some interval (r_0, ∞) since there is at most one point where z' changes its sign. We may also assume

$$\frac{r}{2}-\frac{n-1-4\alpha}{r}>0 \quad \text{on} \quad [r_0, \infty)$$

by taking r_0 large. There is a point $r_1>r_0$ where $z''(r_1)>0$, otherwise $z''\leq 0$ on (r_0, ∞) which contradicts $z>0$. Since $z'(r_1)<0$ and $r_1>r_0$, (3.5) implies $z''(r_1)<0$, which leads again to a contradiction. We thus conclude that z is monotone increasing.

Since z is bounded by (3.3) and monotone increasing, $c_0=\lim_{r\to\infty} z(r)$ exists and is *positive*. This is the same as (3.9).

REMARK. If $\alpha+2<n$, a positive solution of (3.5) may tend to zero, so the asymptotic behavior would be much more complicated. Some logarithmic decay for z is likely; however, we do not pursue this problem here.

PROOF OF THEOREM 2. The estimate (1.4) is the same as (3.3) and (1.5) is the same as (3.9).

4. Nonexistence

The essence of our analysis in this section is a simple integral identity called a Pohozaev-type identity for w of (1.1). Proposition 8 in [8] gives an integral identity for $\alpha=1/(p-1)$ which is easily extended to general α.

PROPOSITION 9. *If $w(x)$ is a bounded solution of* (1.1) *in $\boldsymbol{R}^n$ and $|\nabla w|$ grows at most polynomially in $|x|$, then*

$$(4.1)\quad \left(\frac{n}{p+1}+\frac{2-n}{2}\right)\int|\nabla w|^2\rho dx+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)\int|x|^2|\nabla w|^2\rho dx + n\gamma(\alpha, p)\int|w|^2\rho dx-\frac{\gamma(\alpha, p)}{2}\int|x|^2|w|^2\rho dx=0,$$

where $\rho=\exp(-|x|^2/4)$,

$$\gamma(\alpha, p)=\frac{(1-p)\alpha+1}{2(p+1)}$$

and the integrals are over $\boldsymbol{R}^n$.

PROOF. The proof is found in [8, Proposition 2] with trivial modifications. However, for the reader's convenience, we present here an outline. We shall obtain (4.1) as a linear combination of three other identities. The first is

$$(4.2)\quad \int|\nabla w|^2\rho dx+\alpha\int|w|^2\rho dx-\int|w|^{p+1}\rho dx=0,$$

obtained formally by multiplying (1.1) by $-w\rho$, intergrating over $\boldsymbol{R}^n$, and using integration by parts. This proceduce is easily justified since ρ decreases exponentially as $|x|\to\infty$, while w and $|\nabla w|$ grow polynomially in $|x|$ by hypothesis; it suffices to do the integration by parts on a ball of radius R and then let $R\to\infty$.

The second identity is

$$(4.3)\quad \int|x|^2|\nabla w|^2\rho dx+\left(\alpha+\frac{1}{2}\right)\int|x|^2|w|^2\rho dx - n\int|w|^2\rho\, dx-\int|x|^2|w|^{p+1}\rho dx=0.$$

It is obtained by multiplying (1.1) by $-|x|^2w\rho$, integrating over $\boldsymbol{R}^n$, and using integration by parts; see [8].

The third identity is

$$(4.4)\quad \frac{1}{2}(2-n)\int|\nabla w|^2\rho\, dx-\frac{1}{2}n\alpha\int|w|^2\rho dx+\frac{n}{p+1}\int|w|^{p+1}\rho dx + \frac{1}{4}\int|x|^2|\nabla w|^2\rho dx+\frac{1}{4}\alpha\int|x|^2|w|^2\rho dx-\frac{1}{2(p+1)}\int|x|^2|w|^{p+1}\rho dx=0.$$

It can be obtained by multiplying (1.1) by $-(x\cdot\nabla)w\rho$ and using integration by parts. All procedure is justified since $|\nabla^2 w|$ grows at most polynomially in $|x|$; the estimate follows from the boundedness of w and $|\nabla w|$ and a priori estimates for Δ. An attractive derivation of (4.4) is found in [8].

To complete the proof, we eliminate the terms involving $|w|^{p+1}$ and $|x|^2|w|^{p+1}$ by taking linear combination

$$\frac{n}{p+1}(4.2)-\frac{1}{2(p+1)}(4.3)+(4.4)=0,$$

which is the same as (4.1).

PROOF OF THEOREM 3. Suppose w were a positive global radial decreasing solution of (1.1). Since w is bounded and solves (2.1), w' grows at most polynomially in r. Thus, we may apply Proposition 9 to our w.

We first observe

$$n\int|w|^2\rho dx-\int\frac{|x|^2}{2}|w|^2\rho dx=-\frac{d}{d\lambda}\Big|_{\lambda=1}\int|w(\lambda x)|^2\rho(x)dx.$$

Rewriting (4.1) by using this relation, we obtain

$$\left(\frac{n}{p+1}+\frac{2-n}{2}\right)\int|\nabla w|^2\rho dx+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)\int|\nabla w|^2|x|^2\rho dx$$
$$-\gamma(\alpha,p)\frac{d}{d\lambda}\Big|_{\lambda=1}\int|w(\lambda x)|^2\rho(x)dx=0.$$

Note that the condition $\alpha\le 1/(p-1)$ is equivalent to $\gamma\ge 0$. If $n/2\le(p+1)/(p-1)$ the above identity yields

$$\frac{d}{d\lambda}\Big|_{\lambda=1}\int|w(\lambda x)|^2\rho(x)dx\geqq 0.$$

This inequality excludes radially decreasing function. Thus the proof is completed.

REMARK. If $\alpha=1/(p-1)$, in [8] the nonexistence is shown in the class of bounded solutions. We do not know whether the same type of the nonexistence is true even for $\alpha<1/(p-1)$.

References

[1] F. V. Atkinson and L. A. Peletier, Sur les solutions radiales de l'équation $\Delta u+\frac{1}{2}x\cdot\nabla u+\frac{1}{2}\lambda u+|u|^{p-1}u=0$, preprint.
[2] H. Berestycki, P. L. Lions and L. A. Peletier, An ODE approach to the existence of

positive solutions for semilinear problems in $\boldsymbol{R}^N$, Indiana Univ. Math. J., **30** (1981), 141–157.

[3] H. Brezis, L. A. Peletier and D. Terman, A very singular solution of the heat equation with absorption, Univ. P. & M. Curie, Preprint #84015.

[4] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I, 1st English Edition, Interscience, New York-London, 1953, Chap VI.

[5] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equations, Univ. P. & M. Curie, preprint #85003.

[6] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys., **68** (1979), 209–243.

[7] Y. Giga, Self-similar solutions for semilinear parabolic equations, In "Nonlinear Systems of Partial Differential Equations in Applied Mathematics," (eds. B. Nicolaenko, D. Holm, J. Hyman), Lectures in Applied Math. 23, part 2, 247–251, AMS, Providence RI., 1986.

[8] Y. Giga and R. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math., **38** (1985), 297–319.

[9] A. Haraux and F. Weissler, Non-uniqueness for a semilinear initial value problem, Indiana Univ. Math. J., **31** (1982), 167–189.

[10] L. A. Peletier, D. Terman and F. Weissler, On the equation $\Delta u+\frac{1}{2}x\cdot\nabla u+f(u)=0$, Arch. Rational Mech. Anal., **94** (1986), 83–99.

[11] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. **55** (1977), 149–162.

Department of Mathematics,
*Nagoya University**

* Present address: Depratment of Mathematics, Faculty of Science, Hokkaido University.