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Asymptotic behavior of periodic nonexpansive evolution operators in uniformly convex Banach spaces

Dedicated to Professor Isao Miyadera on his sixtieth birthday

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1. Introduction

Let $\{C_t\}_{t\geq 0}$ be a family of nonempty closed convex subsets of a Banach space X and let $U=\{U(t, s): 0\leq s\leq t\}$ be a nonexpansive evolution operator constrained in $\{C_t\}$, i.e. U is a family of mappings $U(t, s): C_s\to C_t$ such that

$$U(t, s)U(s, r) = U(t, r), \quad U(r, r) = I,$$

$$\|U(t, s)x - U(t, s)y\| \leq \|x-y\|$$

for $0\leq r\leq s\leq t$ and $x, y\in C_s$. Such an evolution operator U is said to be T-*periodic* $(T>0)$ if

$$C_{t+T} = C_t \quad \text{and} \quad U(t+T, s+T) = U(t, s) \qquad \text{for} \quad 0\leq s\leq t.$$

The objective of this paper is to study the asymptotic behavior as $t\to\infty$ of bounded orbits $U(t, 0)x$ defined by a T-periodic nonexpansive evolution operator U. We shall show under appropriate conditions on the space X that if $U(nT+t, 0)x$ is bounded in n for $x\in C_0$ and $t\in[0, T]$, then the sequence $\{U(nT+t, 0)x\}_{n\geq 1}$ is weakly or strongly almost convergent to some T-periodic trajectory $U(t, 0)z$, where z is a point of C_0 with $U(T, 0)z=z$.

In the case of Hilbert spaces this problem was considered for the evolution operator U associated with an initial value problem of the form

$$du(t)/dt + Au(t)\ni f(t), \quad t\geq 0, \quad u(0) = x,$$

by Baillon and Haraux [2], Baillon [1] and Brezis [4], where A is a maximal monotone operator and f is a T-periodic function.

To state our results we recall that X is said to be of type (F) if the norm of X is Fréchet differentiable, namely, for each $x\in X\setminus\{0\}$ the quotient $t^{-1}(\|x+ty\|-\|x\|)$ converges as $t\to 0$ uniformly for $y\in X$ with $\|y\|\leq 1$. It is known that the space L^p is uniformly convex and of type (F) whenever $1<p<\infty$. Further, a sequence $\{x_n\}$ in X is said to be weakly (resp. strongly) almost convergent to x, if $n^{-1}\sum_{i=0}^{n-1} x_{i+k}$ converges weakly (resp. strongly) to x as $n\to\infty$ and the convergence is uniform in $k \in N \equiv \{0, 1, 2,...\}$. We denote by $\mathscr{F}(S)$ the set of all fixed points of a mapping S.

In what follows, let U be a T-periodic nonexpansive evolution operator constrained in $\{C_t\}$ and set

$$u_n(t) = U(nT+t, 0)x \qquad \text{for} \quad x \in C_0, \quad t \in [0, T], \quad n \in N.$$

Then we have:

THEOREM 1. *Let X be a uniformly convex Banach space over the real field $\boldsymbol{R}$. Suppose that X is of type (F) and $U(t, 0)x_0$ is bounded in $t \geq 0$ for some $x_0 \in C_0$. Then, for each $x \in C_0$, there exists a point $z \in \mathscr{F}(U(T, 0))$ such that $\{u_n(t)\}$ is weakly almost convergent to $U(t, 0)z$ for each $t \in [0, T]$.*

THEOREM 2. *Let X be a uniformly convex Banach space over $\boldsymbol{R}$. Suppose that $U(t, 0)x_0$ is bounded in $t \geq 0$ for some $x_0 \in C_0$. Let $x \in C_0$ be such that $\lim_{n\to\infty} \|u_{n+k}(t) - u_n(t)\|$ exists uniformly in $k \in N$ for each $t \in [0, T]$. Then, there exists a point $z \in \mathscr{F}(U(T, 0))$ such that $\{u_n(t)\}$ is strongly almost convergent to $U(t, 0)z$ for each $t \in [0, T]$.*

REMARKS. (a) The author was informed by Professor W. J. Davis that Theorem 1 had been obtained independently by Bruck [8] with a different proof. It would be interesting to compare our proof with that given in [8]. (b) By a fixed point theorem due to Browder and Petryshyn [5] we can easily prove that $\mathscr{F}(U(T, 0)) \neq \emptyset$ if and only if $U(t, 0)x_o$ is bounded in t for some $x_o \in C_0$. In this case $U(t, 0)x$ is also bounded in t for all $x \in C_0$. Moreover, it should be mentioned that if $z \in \mathscr{F}(U(T, 0))$ then $U(t, 0)z$ is periodic in t with period T. Therefore, Theorem 1 states that any bounded orbit given by U is weakly almost convergent to a periodic trajectory. (c) If $\{U(t, 0)x : t \geq 0\}$ is precompact in X, then it can be shown (see [10]) that for each $t \in [0, T]$, $\lim_{n\to\infty} \|u_{n+k}(t) - u_n(t)\|$ exists uniformly in $k \in N$.

2. Proofs of the theorems

Let Γ denote the set of strictly increasing, continuous and convex functions $\gamma: [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$. Let D be a closed convex subset of X. According to Bruck [6] we say that a mapping $S: D \to X$ is *of type* (γ) if $\gamma \in \Gamma$ and for $x, y \in D$ and $\alpha \in [0, 1]$ we have

$$\gamma(\|S(\alpha x + (1-\alpha)y) - \alpha Sx - (1-\alpha)Sy\|) \leq \|x-y\| - \|Sx - Sy\|.$$

It is known ([6, Lemma 1.1]) that if D is bounded, then one can construct a $\gamma \in \Gamma$ such that every nonexpansive mapping $S: D \to X$ is of type (γ). Further, we denote by J the duality mapping of X.

LEMMA 3. *Put* $U_t = U(T+t, t)$ *for* $t \in [0, T]$. *Then we have*:

(*a*) *For each* $t \in [0, T]$ *and* $w \in \mathscr{F}(U_t)$ *there exists* $z \in \mathscr{F}(U_0)$ *such that* $U(t, 0)z = w$.

(*b*) *For each* $t \in [0, T]$ *and* $z \in \mathscr{F}(U_0)$, $U(t, 0)z \in \mathscr{F}(U_t)$.

PROOF. Let $t \in [0, T]$ and $w \in \mathscr{F}(U_t)$. We first note that $U_{t+T} = U_t$. Put $z = U(T, t)w$. Then, $U_0 z = U_T z = U(2T, T+t)U_t w = U(T, t)w = z$ and $U(t, 0)z = U(T+t, T)z = U(T+t, T)U(T, t)w = U_t w = w$. This proves (*a*). Next, let $t \in [0, T]$ and $z \in \mathscr{F}(U_0)$. Then $U_t U(t, 0)z = U(T+t, T)U_0 z = U(t, 0)z$ and so $U(t, 0)z \in \mathscr{F}(U_t)$. Thus (*b*) is proved.

LEMMA 4. *Let X be a uniformly convex Banach space over* $\boldsymbol{R}$. *Suppose that X is of type* (*F*) *and* $\mathscr{F}(U_0) \neq \emptyset$. *Then, for* $x \in C_0$ *and* $z_1, z_2 \in \mathscr{F}(U_0)$, *the limit*

$$h(t) = \lim_{n\to\infty} (u_n(t) - U(t, 0)z_1, J(U(t, 0)z_1 - U(t, 0)z_2)) \tag{1}$$

exists for $t \in [0, T]$ *and* $h(t)$ *is nonincreasing in t*.

PROOF. Let $t \in [0, T]$, $x \in C_0$ and $z_1, z_2 \in \mathscr{F}(U_0)$. For $\alpha \in (0, 1]$ and $n \in \boldsymbol{N}$, we put

$$a_n(t, \alpha) = \|\alpha u_n(t) + \beta U(t, 0)z_1 - U(t, 0)z_2\|,$$

where $\beta = 1-\alpha$. Then, $u_n(t) = U_t^n U(t, 0)x$, U_t is a nonexpansive mapping of C_t into itself and $U(t, 0)z_i \in \mathscr{F}(U_t)$, $i = 1, 2$, by Lemma 3 (*b*). Hence, by [6, Lemmas 2.1 and 2.2], $\lim_{n\to\infty} a_n(t, \alpha)$ exists. Moreover, $u_n(t)$ is bounded in n since $\mathscr{F}(U_t) \neq \emptyset$. Since X is of type (*F*), it follows (see [9]) that

$$\begin{aligned}(u_n(t) - U(t, 0)z_1, &J(U(t, 0)z_1 - U(t, 0)z_2)) \\ &= \lim_{\alpha\downarrow 0} (2\alpha)^{-1}\{a_n(t, \alpha)^2 - \|U(t, 0)z_1 - U(t, 0)z_2\|^2\}\end{aligned}$$

and the convergence is uniform in n. Hence the formula (1) is obtained via the relation

$$\begin{aligned} h(t) &= \lim_{n\to\infty} \lim_{\alpha\downarrow 0} (2\alpha)^{-1}\{a_n(t, \alpha)^2 - \|U(t, 0)z_1 - U(t, 0)z_2\|^2\} \\ &= \lim_{\alpha\downarrow 0} \lim_{n\to\infty} (2\alpha)^{-1}\{a_n(t, \alpha)^2 - \|z_1 - z_2\|^2\}, \end{aligned} \tag{2}$$

where we used the fact that $\|z_1 - z_2\| = \|U(T, 0)z_1 - U(T, 0)z_2\| \leq \|U(t, 0)z_1 - U(t, 0)z_2\| \leq \|z_1 - z_2\|$.

Next we show that $h(t)$ is nonincreasing in t. Since $\|u_n(0) - z_1\| \geq \|U_0 u_n(0) - U_0 z_1\| = \|u_{n+1}(0) - z_1\|$, $\{\|u_n(0) - z_1\|\}$ must converge. Take an $R > 0$ so that $\sup_n \|u_n(0)\| \leq R$ and $\|z_1\|$, $\|z_2\| \leq R$, and put $D = C_0 \cap \{u \in X : \|y\| \leq R\}$. Then, there exists $\gamma \in \Gamma$ such that $U(t, 0)|_D$ (the restriction of $U(t, 0)$ to D) is of type (γ)

for all $t \in [0, T]$. Hence

$$\begin{aligned} &\| U(t, 0)(\alpha u_n(0)+\beta z_1)-\alpha U(t, 0)u_n(0)-\beta U(t, 0)z_1 \| \\ &\quad \leq \gamma^{-1}(\|u_n(0)-z_1\| - \|U(t, 0)u_n(0)-U(t, 0)z_1\|) \\ &\quad \leq \gamma^{-1}(\|u_n(0)-z_1\| - \|u_{n+1}(0)-z_1\|) \end{aligned}$$

for $\alpha \in (0, 1]$ and $\beta = 1-\alpha$. Since $u_n(t) = U(nT+t, nT)u_n(0) = U(t, 0)u_n(0)$, we have

$$\begin{aligned} a_n(t, \alpha) \leq\ &\| U(t, 0)(\alpha u_n(0)+\beta z_1) - U(t, 0)z_2 \| \\ &+ \gamma^{-1}(\|u_n(0)-z_1\| - \|u_{n+1}(0)-z_1\|). \end{aligned}$$

Now let $0 \leq s \leq t$. Then the first term on the right side of the above inequality is estimated as

$$\begin{aligned} &\| U(s, 0)(\alpha u_n(0)+\beta z_1) - U(s, 0)z_2 \| \\ &\quad \leq \|\alpha U(s, 0)u_n(0)+\beta U(s, 0)z_1 - U(s, 0)z_2\| \\ &\qquad + \|U(s, 0)(\alpha u_n(0)+\beta z_1)-\alpha U(s, 0)u_n(0)-\beta U(s, 0)z_1\| \\ &\quad \leq a_n(s, \alpha) + \gamma^{-1}(\|u_n(0)-z_1\| - \|u_{n+1}(0)-z_1\|). \end{aligned}$$

Consequently, we obtain

$$a_n(t, \alpha) \leq a_n(s, \alpha) + 2\gamma^{-1}(\|u_n(0)-z_1\| - \|u_{n+1}(0)-z_1\|).$$

From this relation, (2) and the fact that $\lim_{n\to\infty} \|u_n(0)-z_1\|$ exists, we conclude that $h(t) \leq h(s)$. Thus the lemma is proved.

PROOF OF THEOREM 1. Let $x \in C_0$ and $t \in [0, T]$. As seen before $\mathscr{F}(U_0) \neq \emptyset$ by the assumption, and hence $\mathscr{F}(U_t) \neq \emptyset$. Since $u_n(t) = U_t^n U(t, 0)x$, it follows from [6, Theorem 2.1] (cf. [9, Theorem 3.1]) that $\{u_n(t)\}$ is weakly almost convergent to a point $z(t) \in \mathscr{F}(U_t)$. Since $u_n(T) = u_{n+1}(0)$ for $n \in \boldsymbol{N}$, we have $z(T) = w\text{-}\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} u_i(0) = z(0)$. Now, take arbitrary elements z_1, z_2 in $\mathscr{F}(U_0)$. By Lemma 4, we have

$$\begin{aligned} h(t) &= \lim_{n\to\infty} n^{-1} \textstyle\sum_{i=0}^{n-1} (u_i(t) - U(t, 0)z_1, g) \\ &= \lim_{n\to\infty} (n^{-1} \textstyle\sum_{i=0}^{n-1} u_i(t) - U(t, 0)z_1, g) \\ &= (z(t) - U(t, 0)z_1, g), \end{aligned}$$

where $g = J(U(t, 0)z_1 - U(t, 0)z_2)$. This, together with the relation $z(0) = z(T)$, shows that $h(0) = h(T) \leq h(t) \leq h(0)$. Therefore, $h(t) \equiv h(0)$ for $t \in [0, T]$. We now take $z_1 = z(0) \in \mathscr{F}(U_0)$. Then we have

$$(3) \qquad (z(t)-U(t,0)z_1, J(U(t,0)z_1-U(t,0)z_2)) \\ = (z(0)-z_1, J(z_1-z_2)) = 0 \quad \text{for} \quad t\in[0,T].$$

We then demonstrate that $z(t)=U(t,0)z_1$ for $t\in[0,T]$. Suppose to the contrary that there exists $t_o\in[0,T]$ such that $z(t_o)\neq U(t_o,0)z_1$. Since $z(t_o)\in\mathscr{F}(U_{t_o})$, Lemma 3 ($a$) would imply that there exists $z_2\in\mathscr{F}(U_0)$ satisfying $U(t_o,0)z_2=z(t_o)$. Hence (3) would give $-\|U(t_o,0)z_1-U(t_0,0)z_2\|^2=0$ and $U(t_o,0)z_1=U(t_o)z_2$. This contradicts the fact that $z(t_o)\neq U(t_o,0)z_1$. Consequently, we have $z(t)=U(t,0)z_1$ for all $t\in[0,T]$. Thus the proof of Theorem 1 is complete.

PROOF OF THEOREM 2. Since $u_n(t)=U_t^nU(t,0)x$ and $\lim_{n\to\infty}\|U_t^{n+k}U(t,0)x-U_t^nU(t,0)x\|$ exists uniformly in k by the assumption, it follows from [10, Theorem 1] that $\{u_n(t)\}$ is strongly almost convergent to a point $z(t)\in\mathscr{F}(U_t)$. Let $\sup_n\|u_n(0)\|\leq R$ and put $D=C_0\cap\{y\in X: \|y\|\leq R\}$. Then, by [7, Theorem 2.1], there exists $\gamma\in\Gamma$, depending only upon R and the modulus of uniform convexity of X, such that

$$\gamma(\|U(t,0)(n^{-1}\textstyle\sum_{i=0}^{n-1}y_i) - n^{-1}\sum_{i=0}^{n-1}U(t,0)y_i\|) \\ \leq \max_{0\leq i,j\leq n-1}\{\|y_i-y_j\|-\|U(t,0)y_i - U(t,0)y_j\|\}$$

for any $y_0,\ldots,y_{n-1}\in D$, $t\in[0,T]$ and any $n\geq 1$. Putting $y_i=u_{i+k}(0)$ in the above inequality and noting that $u_n(t)=U(t,0)u_n(0)$, we have

$$\|U(t,0)(n^{-1}\textstyle\sum_{i=0}^{n-1}u_{i+k}(0)) - n^{-1}\sum_{i=0}^{n-1}u_{i+k}(t)\| \leq \gamma^{-1}(\varepsilon_{k,n})$$

for any k and n, where

$$\varepsilon_{k,n} = \max_{0\leq i,j\leq n-1}\{\|u_{i+k}(0)-u_{j+k}(0)\|-\|u_{i+k+1}(0)-u_{j+k+1}(0)\|\}.$$

Fix any $\varepsilon>0$ and take $m=m(\varepsilon,t)\geq 1$ such that $\|m^{-1}\sum_{i=0}^{m-1}u_{i+k}(s)-z(s)\|<\varepsilon/2$ for $s\in\{0,t\}$ and $k\in\boldsymbol{N}$. Then we have

$$\|U(t,0)z(0)-z(t)\| \\ \leq \|U(t,0)z(0)-U(t,0)(m^{-1}\textstyle\sum_{i=0}^{m-1}u_{i+k}(0))\| \\ + \|z(t)-m^{-1}\textstyle\sum_{i=0}^{m-1}u_{i+k}(t)\| \\ + \|U(t,0)(m^{-1}\textstyle\sum_{i=0}^{m-1}u_{i+k}(0))-m^{-1}\sum_{i=0}^{m-1}u_{i+k}(t)\| \\ \leq \varepsilon/2+\varepsilon/2+\gamma^{-1}(\varepsilon_{k,m}) \qquad \text{for all} \quad k\in\boldsymbol{N}.$$

Since $\|u_{n+p}(0)-u_n(0)\|$ is nonincreasing in n for each p, it follows that $\lim_{k\to\infty}\varepsilon_{k,m}=0$ and $\|U(t,0)z(0)-z(t)\|\leq\varepsilon$. Since $\varepsilon>0$ is arbitrary, we infer that $z(t)=U(t,0)z(0)$ with $z(0)\in\mathscr{F}(U_0)$. The proof of Theorem 2 is thereby complete.

3. Periodic forcing

Let A be an m-accretive operator in a Banach space X over $\boldsymbol{R}$ and $f \in L^1_{loc}(0, \infty: X)$ be T-periodic. It is well-known that for each $s \geq 0$ and $x \in cl\, D(A)$ (the closure of the domain of A), there exists a unique integral solution $u(t; s, x)$ of

$$du(t)/dt + Au(t) \ni f(t), \quad t \in [s, \infty) \tag{4}$$

with initial condition $u(s) = x$. (See [3] for the notion of m-accretive operator, the notion of integral solution and existence theorems for integral solutions of (4).) By setting $U(t, s)x = u(t; s, x)$, we see that $\{U(t, s): 0 \leq s \leq t\}$ forms a T-periodic nonexpansive evolution operator constrained in $C_t \equiv cl\, D(A)$. In this case, Theorem 1 implies the following result.

COROLLARY 5. *Let X be uniformly convex and of type* (F). *Let $x \in cl\, D(A)$ and $u_n(t) = u(nT + t; 0, x)$. If $u(t; 0, x)$ is bounded in t, then there is a T-periodic integral solution $\omega(t)$ of equation* (4) *with $s = 0$ and a T-periodic forcing term f such that $\{u_n(t)\}$ is weakly almost convergent to $\omega(t)$ for each $t \in [0, T]$.*

REMARK. Baillon [1] proved this corollary in the case where X is a Hilbert space.

Furthermore, assume in Corollary 5 that X is a Hilbert space, $A = \partial\phi$, the subdifferential of a proper convex lower semicontinuous function $\phi: X \to (-\infty, \infty]$, and that $f \in L^2_{loc}(0, \infty; X)$. Then, applying the same argument as in [2, Theorem 5] we can choose a subsequence $\{n(k)\}$ such that

$$\int_0^T \|du_{n(k)}(t)/dt - d\tilde{\omega}(t)/dt\|^2 dt \longrightarrow 0$$

as $n(k) \to \infty$, where $\tilde{\omega}(t)$ is an arbitrary T-periodic solution of (4) with $s = 0$ and $A = \partial\phi$. But

$$\begin{aligned} \|u_{n+1}(t) - u_n(t)\| &= \|U(t, 0)u_n(T) - U(t, 0)u_n(0)\| \\ &\leq \|u_n(T) - u_n(0)\| \\ &\leq \int_0^T \|(d/dt)(u_n(t) - \tilde{\omega}(t))\| dt \\ &\leq T^{1/2} \Big(\int_0^T \|(d/dt)(u_n(t) - \tilde{\omega}(t))\|^2 dt \Big)^{1/2}, \end{aligned}$$

and hence $\|u_{n(k)+1}(t) - u_{n(k)}(t)\| \to 0$ as $n(k) \to \infty$. Since $\|u_{n+1}(t) - u_n(t)\|$ is non-increasing in n, we conclude that $\lim_{n \to \infty} \|u_{n+1}(t) - u_n(t)\| = 0$, which is the so-called

Tauberian condition for almost convergent series (see [6, Section 3]). Therefore, in this case, $u_n(t)$ itself converges weakly as $n \to \infty$ to some T-periodic solution of (4) with $s=0$. This is nothing but the result given in [2].

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