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Stationary measures for an exclusion process on one-dimensional lattices with infinitely many hopping sites

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§ 0. Introduction

Since F. Spitzer introduced interacting Markovian particle systems in [11], Markov processes on the configuration space $S^{\mathbf{Z}^d}$ or $S^{\mathbf{R}^d}$ ($S=\{0, 1,\dots, n\}$ or $\{0, 1,\cdots\}$) have been investigated by many authors, and various results are obtained (see, for example, [4, 8, 9]). Those results are, in many cases, about the processes such that the time parameters are continuous and the number of sites in the configuration at which changes occur at the same time is finite. In this paper we consider a Markov process on $\mathscr{X}=\{0, 1\}^{\mathbf{Z}}$ such that the time parameter is discrete and the sites at which changes occur at each time are infinitely many. If we consider the important roles which discrete time stochastic processes play in the theory of probability, it seems worthwhile to investigate discrete time Markov processes in the field of interacting infinite particle systems ([2, 3]).

Let $x\equiv(\cdots x_{-1}x_0x_1\cdots)$ be an element of $\mathscr{X}$. According as $x_i=1$ or 0 we consider that the site i is occupied by a particle or not. Then $x\in\mathscr{X}$ represents a configuration of particles on $\mathbf{Z}$. We introduce a time evolution on $\mathscr{X}$ as follows (for details see §1). Suppose the configuration on $\mathbf{Z}$ at time t is $x\equiv(\cdots x_{-1}x_0x_1\cdots)$. Then as time increases from t to $t+1$ each particle of x moves to the left by one step with probability α $(0<\alpha<1)$ independently when its left site is unoccupied, that is, a particle at site i can move to the site $i-1$ only if $x_{i-1}=0$, and this transition of particle occurs independently in the configuration x. Therefore infinitely many particles can move simultaneously when $\#\{l\colon x_lx_{l+1}=01\}=\infty$. Getting a new configuration $x'\equiv(\cdots x'_{-1}x'_0x'_1\cdots)\in\mathscr{X}$ from x at time $t+1$, we then apply the same transition rule to x' and so on. Thus a time evolution is obtained as a Markov process on $\{0, 1\}^{\mathbf{Z}}$, which we call, following [6, 11], an exclusion process on $\mathbf{Z}$. It should be remarked here that our exclusion process can be thought of as a simple model of semiconductor which is in a (static) electric field if we regard $x_i=1$ and 0 as plus and minus charges respectively.

We define in §1 the transition probabilities of the Markov process stated above precisely and give a sufficient condition (Su) for a translation (=shift) invariant probability measure on $\mathscr{X}$ to be a stationary measure for the process

(Theorem 1). In §2 we define a family $\{\pi_\gamma: 0<\gamma<\infty\}$ of Gibbs states with nearest neighbor interaction on $\mathbf{Z}$, and show that each π_γ satisfies the condition (Su) (Theorem 2). In the one-dimensional case nearest neighbor Gibbs states are renewal measures. In §3 we show that the extreme points of the convex set of probability measures on $\mathscr{X}$ satisfying the condition (Su) are exhausted by $\{\pi_\gamma: 0<\gamma<\infty\}$ and trivial measures $\{\pi_0, \pi_\infty\}$ (Theorem 3). The structure of the set $\mathscr{I}$ of all stationary measures is discussed in §4 under the assumption that $0<\alpha\leqq 1/2$. It is proved that the totality of extreme points of $\mathscr{I}$ is $\{\pi_\gamma: 0\leqq\gamma\leqq\infty\}$ and $\{\Theta_n: n\in\mathbf{Z}\}$ (Theorem 4), where Θ_n is a Dirac measure concentrated at the point $\theta_n\equiv(x_i)_{i\in\mathbf{Z}}\in\mathscr{X}$, $x_i=1$ for $i\leqq n$ and $x_i=0$ for $i>n$. Then it follows that (Su) is also a necessary condition for a probability measure with zero mass on $\{\theta_n: n\in\mathbf{Z}\}$ to be stationary. Thus the structure of $\mathscr{I}$ is completely determined. In the last section we study the stochastic properties of a particle under the time evolution with respect to the stationary state π_γ. Suppose $r_0(t)$ is the random variable that represents the position of the particle at time t which was located at the origin at $t=0$. Then it is shown that $\{r_0(t), t=0, 1,\dots\}$ has homogeneous independent increments, i.e., $\{r_0(t)-r_0(t-1)\}_{t\in\mathbf{N}}$ is a Bernoulli sequence. Therefore the mean m_t and the variance σ_t^2 of $r_0(t)$ are calculated explicitly, and a central limit theorem is obtained (Theorem 5). In general it is not so easy to get the mean and the variance of a marked particle. A correspondence between the random variables $\{r_0(t), t=0, 1,\dots\}$ and the so called random walks is considered in Remark 4.

The structure of stationary measures for the simple exclusion processes with continuous time parameter such that the number of sites in the configuration at which changes occur at the same time is two was investigated to the full extent by T. M. Liggett in [7]. It is proved there that the extreme points of the translation invariant stationary measures are of the type of product measures, that is, Bernoulli measures. In our system, the extreme points of the translation invariant stationary measures are nearest neighbor Gibbs states. Because Bernoulli measures can be regarded as Gibbs states with no interaction potential, the mechanism of transition of particles of our system seems to be more natural (cf. Remark 1 in §1).

§1. Transition probabilities and a sufficient condition for stationary measures

In this section we define transition probabilities of the Markov process described in §0 and give a sufficient condition for a probability measure on $\{0, 1\}^{\mathbf{Z}}$ to be a stationary measure for the process.

Let $\mathscr{X}$ be $\{0, 1\}^{\mathbf{Z}}$, which is the state psace of our Markov process. An element of $\mathscr{X}$ is denoted by $(\cdots x_{-1}x_0x_1\cdots)$ or x shortly. We assume that the time parameter t takes its values on the set $\mathbf{T}=\{0, 1, 2,\dots\}$. For $i, j\in\mathbf{Z}$, $i\leqq j$, put

$$\mathscr{C}_{i,j} = \{{}_i[a_i a_{i+1} \cdots a_j]_j \colon a_\ell = 0 \text{ or } 1,\ i \leqq \ell \leqq j\},$$

where

$${}_i[a_i \cdots a_j]_j = \{(\cdots x_{-1} x_0 x_1 \cdots) \in \mathscr{X} \colon x_\ell = a_\ell,\ i \leqq \ell \leqq j\},$$

and $\mathscr{B}_{i,j} = \sigma(\mathscr{C}_{i,j})$, the σ-field generated by $\mathscr{C}_{i,j}$. Let $\mathscr{C} = \{\phi\} \cup (\cup_{i,j} \mathscr{C}_{i,j})$ and $\mathscr{B} = \sigma(\mathscr{C})$. The elements of $\mathscr{C}$ are called the basic cylinders. Subscripts i and j in ${}_i[\cdots]_j$ are sometimes omitted. The following lemma tells us that two probability measures on $(\mathscr{X}, \mathscr{B})$ which have the same values on $\mathscr{C}$ coincide.

Lemma 0. *Suppose $\tilde{\mu}$ is a nonnegative function on $\mathscr{C}$ satisfying*
(i) $\tilde{\mu}(\phi) = 0$,
(ii) $\tilde{\mu}({}_i[0]_i) + \tilde{\mu}({}_i[1]_i) = 1,\ i \in \mathbf{Z}$,
(iii) (*consistency condition*)

$$\begin{aligned}\tilde{\mu}({}_i[a_i \cdots a_j]_j) &= \tilde{\mu}({}_i[a_i \cdots a_j 0]_{j+1}) + \tilde{\mu}({}_i[a_i \cdots a_j 1]_{j+1}) \\ &= \tilde{\mu}({}_{i-1}[0 a_i \cdots a_j]_j) + \tilde{\mu}({}_{i-1}[1 a_i \cdots a_j]_j)\end{aligned}$$

$$\text{for all } [a_i \cdots a_j] \in \mathscr{C}_{i,j},\ i, j \in \mathbf{Z},\quad i \leqq j.$$

Then $\tilde{\mu}$ is extended uniquely to a probability measure μ on $(\mathscr{X}, \mathscr{B})$.

Now let us define transition probabilities of our Markov process using Lemma 0. We denote $x \rhd y$ if $y \equiv (\cdots y_{-1} y_0 y_1 \cdots)$ is obtained from $x \equiv (\cdots x_{-1} x_0 x_1 \cdots)$ by substituting some of (possibly infinitely many) $x_\ell x_{\ell+1} = 01$'s in x with 10's. Note that the replacement $x_\ell x_{\ell+1} = 01$ by $y_\ell y_{\ell+1} = 10$ corresponds to the transition of particle at $\ell + 1$ to ℓ. Given two basic cylinders ${}_{i-1}[x_{i-1} x_i \cdots x_j x_{j+1}]_{j+1}$ and ${}_i[a_i \cdots a_j]_j$ we will write $[x_{i-1} \cdots x_{j+1}] \rhd [a_i \cdots a_j]$ if we can choose a_{i-1} and a_{j+1}, which are uniquely determined, such that $a_{i-1} a_i \cdots a_j a_{j+1}$ is obtained from $x_{i-1} \cdots x_{j+1}$ by substituting some of $x_\ell x_{\ell+1} = 01$'s with 10's $(i-1 \leqq \ell \leqq j)$. Let $0 < \alpha < 1$ be a fixed constant. For $x \equiv (\cdots x_{-1} x_0 x_1 \cdots) \in \mathscr{X}$ and $A \equiv [a_i \cdots a_j] \in \mathscr{C}_{i,j}$, define

$$\tilde{P}(x, A) = \begin{cases} \prod_{\ell=i-1}^{j} \alpha^{\chi_\ell^{(1)}} \cdot (1-\alpha)^{\chi_\ell^{(2)}} & \text{if } [x_{i-1} \cdots x_{j+1}] \rhd [a_i \cdots a_j] \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\chi_\ell^{(k)} \equiv \chi_\ell^{(k)}([x_{i-1} \cdots x_{j+1}], [a_i \cdots a_j])$$

$$\underset{(k=1)}{=} \begin{cases} 1 & \text{if } \ x_\ell x_{\ell+1} = 01 \\ & \text{and } \ a_\ell a_{\ell+1} = 10 \\ 0 & \text{otherwise,} \end{cases} \qquad \underset{(k=2)}{=} \begin{cases} 1 & \text{if } \ x_\ell x_{\ell+1} = 01 \\ & \text{and } \ a_\ell a_{\ell+1} = 01 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{P}(x,\cdot)$ satisfies the assumption of Lemma 0, we can extend $\tilde{P}(x,\cdot)$ to a probability measure $P(x,\cdot)$ on $(\mathscr{X}, \mathscr{B})$ uniquely. The measurability of $\tilde{P}(\cdot, A)$, $A \in \mathscr{C}$, and the fact that $\sigma(\mathscr{C})=\mathscr{B}$ imply the measurability of $P(\cdot, A)$, $A \in \mathscr{B}$. Thus transition probabilities $P(x, A)$, $x \in \mathscr{X}$, $A \in \mathscr{B}$, which determine a Markov process (MP) on $\{0, 1\}^{\mathbf{Z}}$, are defined. It is not so hard to check that $P(x, A)$ describes our exclusion process. Indeed, by the definition of $\tilde{P}(x, A)$, the transition of particles occurs independently at each site of x with probability α when $x_\ell x_{\ell+1}=01$; and further, infinitely many particles can move simultaneously if $\#\{\ell : x_\ell x_{\ell+1}=01\}=\infty$.

A probability measure μ on $(\mathscr{X}, \mathscr{B})$ is called a stationary measure for the Markov process (MP) if

$$\text{(Eq)} \qquad \int_{\mathscr{X}} d\mu(x) f(x) = \int_{\mathscr{X}} d\mu(x) \int_{\mathscr{X}} P(x, dx') f(x')$$

for all bounded measurable functions f. The set of stationary measures for (MP) is denoted by $\mathscr{S}$. Lemma 0 implies that if (Eq) holds for all indicator functions χ_A, $A \in \mathscr{C}$, that is,

$$\text{(Eq)}' \quad \mu({}_i[a_i \cdots a_j]_j) = \sum_{[b_{i-1}\cdots b_{j+1}] \triangleright [a_i \cdots a_j]} \alpha^m (1-\alpha)^n \mu({}_{i-1}[b_{i-1}\cdots b_{j+1}]_{j+1}),$$

$$\left(m = \sum_{\ell=i-1}^{j} \chi_\ell^{(1)}([b_{i-1}\cdots b_{j+1}], [a_i \cdots a_j]),\ n = \sum_{\ell=i-1}^{j} \chi_\ell^{(2)}\right),$$

then $\mu \in \mathscr{S}$.

A probability measure μ on $(\mathscr{X}, \mathscr{B})$ is said to be translation invariant if $\mu(A+\ell)=\mu(A)$ for any $A \in \mathscr{B}$ and $\ell \in \mathbf{Z}$ where

$$A + \ell = \left\{ y \equiv (\cdots y_{-1} y_0 y_1 \cdots) \in \mathscr{X} \;\middle|\; \begin{array}{l} \text{there exists } x \equiv (\cdots x_{-1} x_0 x_1 \cdots) \in A \\ \text{such that } y_i = x_{i+\ell} \text{ for all } i \in \mathbf{Z} \end{array} \right\}.$$

The set of translation invariant probability measures on $(\mathscr{X}, \mathscr{B})$ is denoted by $\mathscr{T}$. Notice that the translation invariance of μ allows us not to specify the coordinates of cylinders in measuring the elements of $\mathscr{C}$, that is, to wirte $\mu[010101]$, for example, has its meaning. A sufficient condition of stationary measures is now stated as follows.

THEOREM 1. *Suppose a probability measure μ on $(\mathscr{X}, \mathscr{B})$ satisfies the condition*

$$\text{(Su)} \quad \begin{cases} \text{(i)} & \mu \textit{ is translation invariant,} \\ \text{(ii)} & (1-\alpha)^{\#_{01}[a_i\cdots a_j]}\mu[a_i\cdots a_j] = (1-\alpha)^{\#_{01}[b_i\cdots b_j]}\mu[b_i\cdots b_j] \\ & \textit{for } {}_i[a_i\cdots a_j]_j \textit{ and } {}_i[b_i\cdots b_j]_j \textit{ in } \mathscr{C} \textit{ with} \\ & \qquad a_i = b_i,\ a_j = b_j \textit{ and } \sum_{\ell=i}^{j} a_\ell = \sum_{\ell=i}^{j} b_\ell, \end{cases}$$

where

$$\#_{01}[a_i\cdots a_j] = \#\{\ell : a_\ell a_{\ell+1} = 01,\ i \leqq \ell \leqq j-1\}.$$

Then μ is a stationary measure for the Markov process (MP).

The proof of Theorem 1 is elementary and straightforward. Therefore we only give the outline of it. For the proof it suffices to check that (Eq)′ holds for all $[a_i\cdots a_j]\in\mathscr{C}$ under the assumption (Su). If $\#_{01}[a_i\cdots a_j]+\#_{10}[a_i\cdots a_j]=0$ it is verified directly, such as:

$$\begin{aligned}
\int_{\mathfrak{X}} \mu(dx)P(x,\, {}_i[00]_{i+1}) &= \sum_{[b_{i-1}b_ib_{i+1}b_{i+2}]\triangleright[00]} \alpha^m(1-\alpha)^n\mu[b_{i-1}b_ib_{i+1}b_{i+2}]. \\
&= \mu({}_i[000]_{i+2}) + (1-\alpha)\mu({}_i[001]_{i+2}) \\
&\qquad + \alpha\mu({}_{i-1}[0100]_{i+2}) + \alpha(1-\alpha)\mu({}_{i-1}[0101]_{i+2}) \\
&= \mu({}_i[00]_{i+1}) - \alpha\mu({}_i[001]_{i+2}) + \alpha\mu({}_{i-1}[0010]_{i+2}) + \alpha\mu({}_{i-1}[0011]_{i+2}) \\
&\qquad \text{(by (Su)–(ii))} \\
&= \mu({}_i[00]_{i+1}) \qquad \text{(by (iii) of Lemma 0 and (Su)–(i))}.
\end{aligned}$$

For the general $[a_i\cdots a_j]$ we use the following notation:

$$N(i,j;k) = \mu({}_i[0\cdots 00\overbrace{11\cdots 1}^{k}]_j), \quad R(i,j;k) = \mu({}_i[0\cdots 00\overbrace{11\cdots 1}^{k}0]_j),$$

$$L(i,j;k) = \mu({}_i[10\cdots 00\overbrace{11\cdots 1}^{k-1}]_j), \quad B(i,j,k) = \mu({}_i[\overbrace{1\cdots 11}^{k}00\cdots 0]_j).$$

If k represents $\sum_{\ell=i}^{j} a_\ell$ for $[a_i\cdots a_j]\in\mathscr{C}_{i,j}$ then

$$\begin{cases}
(1-\alpha)^{\#_{01}[0a_{i+1}\cdots a_{j-1}1]-1}\mu[0a_{i+1}\cdots a_{j-1}1] = N(i,j;k), \\
(1-\alpha)^{\#_{01}[0a_{i+1}\cdots a_{j-1}0]-1}\mu[0a_{i+1}\cdots a_{j-1}0] = R(i,j;k), \\
(1-\alpha)^{\#_{01}[1a_{i+1}\cdots a_{j-1}1]-1}\mu[1a_{i+1}\cdots a_{j-1}1] = L(i,j;k), \\
(1-\alpha)^{\#_{01}[1a_{i+1}\cdots a_{j-1}0]}\mu[1a_{i+1}\cdots a_{j-1}0] = B(i,j;k)
\end{cases}$$

by (Su)–(ii) provided $(\#_{01}+\#_{10})([a_i\cdots a_j])>0$. The consistency property of μ also implies

$$\begin{cases}
N(i,j+1;k+1) + R(i,j+1;k) = N(i,j;k), \\
(1-\alpha)\{R(i,j;k) - R(i,j+1;k)\} = N(i,j+1;k+1)
\end{cases}$$

and so on.

Now suppose $\#_{01}+\#_{10}>0$ for $[a_i\cdots a_j]$ and $k=\sum_{\ell=i}^{j} a_\ell$. Then (Eq)′ for μ is proved as follows:

$$(1-\alpha)^{\#_{01}[a_i\cdots a_j]-1} \sum_{[b_{i-1}\cdots b_{j+1}]\rhd[a_i\cdots a_j]} \alpha^m(1-\alpha)^n\mu[b_{i-1}\cdots b_{j+1}]$$

$$= \begin{cases} N(i,j;k) + (\alpha/(1-\alpha))N(i-1,j;k+1) & \text{if } (a_i, a_j)=(0,0) \\ (1-\alpha)N(i,j;k) + \alpha N(i,j+1;k) \\ \quad + \alpha N(i-1,j;k+1) + (\alpha^2/(1-\alpha))N(i-1,j+1;k+1) \\ & \text{if } (a_i, a_j)=(0,1) \\ \{\alpha N(i,j;k) + (1-\alpha)N(i-1,j;k) \\ \quad + B(i-1,j+1;k+1) + (1-\alpha)L(i-1,j+1;k+2)\}/(1-\alpha) \\ & \text{if } (a_i, a_j)=(1,0) \\ N(i,j;k) + (\alpha/(1-\alpha))N(i,j+1;k) & \text{if } (a_i, a_j)=(1,1) \end{cases}$$

(by (Su)–(ii) and by (iii) of Lemma 0)

$$= (1-\alpha)^{\#_{01}[a_i\cdots a_j]-1}\mu[a_i\cdots a_j].$$

Further if $(a_i, a_j)=(0, 0)$, for example, we have

$$\begin{aligned} & N(i,j;k) + (\alpha/(1-\alpha))N(i-1,j;k+1) \\ & = N(i,j;k) + (\alpha/(1-\alpha))N(i,j+1;k+1) \\ & = N(i,j+1;k+1) + R(i,j+1;k) + (\alpha/(1-\alpha))N(i,j+1;k+1) \\ & = (1/(1-\alpha))N(i,j+1;k+1) + R(i,j+1;k) = R(i,j;k) \\ & = (1-\alpha)^{\#_{01}[0a_{i+1}\cdots a_{j-1}0]-1}\mu[0a_{i+1}\cdots a_{j-1}0]. \qquad \square \end{aligned}$$

REMARK 1. In [7] Liggett investigated the structure of stationary measures for the simple exclusion process with continuous time whose generator $\Omega^{(1)}$ on $\mathscr{B}_{i,j}$-measurable functions f is of the form

$$(\Omega^{(1)}f)(x) = \sum_{\{k,\ell\}\cap\{i,\dots,j\}\neq\emptyset} p(k,\ell)\{f(x^{k\ell})-f(x)\},$$

$$\text{where } x^{k\ell} = (\cdots x_{k-1}x_\ell x_{k+1}\cdots x_{\ell-1}x_k x_{\ell+1}\cdots)$$

$$\text{for } x \equiv (\cdots x_{k-1}x_k x_{k+1}\cdots x_{\ell-1}x_\ell x_{\ell+1}\cdots).$$

If we consider another exclusion process defined by the following (bounded) operator

$$(\Omega^{(2)}f)(x) = \sum[\textstyle\prod_{\ell=i-1}^{j}\alpha^{x_\ell^{(1)}}\cdot(1-\alpha)^{x_\ell^{(2)}}]\{f(y_i\cdots y_j)-f(x_i\cdots x_j)\},$$

where the summation is taken over all configurations $y_i\cdots y_j$ with ${}_{i-1}[x_{i-1}\cdots x_{j+1}]_{j+1}\rhd_i[y_i\cdots y_j]_j$, it is seen easily that μ satisfying the condition (Su) of Theorem 1 is a stationary measure for the process, that is, $\int_{\mathscr{X}}\Omega^{(2)}fd\mu=0$ for all

$f\in\mathscr{D}(\Omega^{(2)})$. Note that $\Omega^{(2)}$ permits the transition of particles at infinitely many sites likewise in (MP).

§ 2. Gibbs states as stationary measures

In this section we give stationary measures satisfying the condition (Su) of Theorem 1 and show that they are Gibbs states with nearest neighbor interaction on $\mathbf{Z}$.

Let α $(0<\alpha<1)$ be as in §1. Take β $(0<\beta<\alpha)$ and γ (>0) such that

$$(1)\qquad 1-\alpha=(1-\beta)(1-\beta\gamma)\quad\text{i.e.}\quad \alpha=\beta(1+\gamma-\beta\gamma).$$

When β varies from α to 0, γ varies from 0 to ∞. Define a nonnegative function $\tilde{\pi}_\gamma$ on $\mathscr{C}$ as follows:

$$\begin{cases}\tilde{\pi}_\gamma(\phi)=0\\ \tilde{\pi}_\gamma({}_i[a_i\cdots a_{j-1}0]_j)=(\gamma/(1+\gamma))\times(1-\beta)^{\#_{00}}\times(1/\gamma)^{\#_{10}}\\ \qquad\qquad\times((1-\beta\gamma)/\gamma)^{\#_{11}}\times(\beta\gamma/\alpha)^{j-i}\\ \tilde{\pi}_\gamma({}_i[a_i\cdots a_{j-1}1]_j)=(1/(1+\gamma))\times(1-\beta)^{\#_{00}}\times(1/\gamma)^{\#_{10}}\\ \qquad\qquad\times((1-\beta\gamma)/\gamma)^{\#_{11}}\times(\beta\gamma/\alpha)^{j-i},\end{cases}$$

where $\#_{uv}\equiv\#_{uv}[a_i\cdots a_{j-1}a_j]=\#\{\ell:\ a_\ell a_{\ell+1}=uv,\ i\leqq\ell\leqq j-1\}$. Since $\tilde{\pi}_\gamma$ satisfies the assumption of Lemma 0, it is extended uniquely to a translation invariant probability measure π_γ on $(\mathscr{X},\mathscr{B})$. We remark that

$$\pi_\gamma[0]=\gamma/(1+\gamma),\quad \pi_\gamma[1]=1/(1+\gamma)\quad\text{and}\quad \pi_\gamma[0]/\pi_\gamma[1]=\gamma.$$

Set

$$\Omega_\gamma=\{x\in\mathscr{X}:\lim_{n\to\infty}n^{-1}\textstyle\sum_{\ell=0}^{n}x_\ell=\lim_{n\to\infty}n^{-1}\sum_{\ell=-1}^{-n}x_\ell=(1+\gamma)^{-1}\}.$$

Then we have

THEOREM 2. (i) *π_γ is a stationary measure for* (MP) *satisfying the condition* (Su) *of Theorem* 1.

(ii) *π_γ is a renewal measure on $(\mathscr{X},\mathscr{B})$ corresponding to a renewal process whose probability density function (p.d.f.) of the interarrival time is given by*

$$f(n)=\begin{cases}(1-\beta\gamma)(\beta/\alpha) & n=1\\ \gamma^{-1}(1-\beta)^{n-2}(\beta\gamma/\alpha)^n & n=2,3,\dots.\end{cases}$$

(ii)′ *π_γ is a Gibbs state with nearest neighbor interaction on $\mathbf{Z}$ with the chemical potential $J_1=\{-2\log(1-\beta)-\log\gamma\}\cdot kT$ and the interaction potential*

$J_2=\{\log(1-\beta)+\log(1-\beta\gamma)\}\cdot kT$, *where* k *is the Boltzmann constant and* T *is the absolute temperature.*

(iii) $$\pi_\gamma(\Omega_{\gamma'})=\begin{cases}1 & \text{if } \gamma'=\gamma\\ 0 & \text{otherwise.}\end{cases}$$

(iv) *In the weak topology* π_γ *is an extreme point of the compact convex set* $\mathscr{T}$ *consisting of all translation invariant probability measures on* $(\mathscr{X}, \mathscr{B})$.

For the reference of readers we state below the definition of the terms used in the theorem.

DEFINITION 1. A translation invariant probability measure μ on $(\mathscr{X}, \mathscr{B})$ is said to be a renewal measure if there exist p.d.f.'s f_0 and f on $\mathbf{N}$ satisfying

(2) $$\mu[a_1\cdots a_{j-1}1]=f_0(\eta_1)\cdot\prod_{\ell=1}^{k-1}f(\eta_{\ell+1}-\eta_\ell)$$

for every $[a_1\cdots a_{j-1}1]\in\mathscr{C}_{1,j}$, $j\geqq 1$. Here we used the notation

$$k=\sum_{\ell=1}^{j}a_\ell \quad (a_j=1) \quad \text{and} \quad \eta_\ell=\min\{t\colon \sum_{s=1}^{t}a_s=\ell\},\quad 1\leqq\ell\leqq k.$$

Note that if we regard ℓ with $a_\ell=1$ in the left hand side of (2) as a renewal epoch of some renewal process, then the right hand side of (2) states that the p.d.f. of its interarrival time is given by f (cf. Chap. XI of [5]).

For $i, j\in\mathbf{Z}$, $i\leqq j$, let $\mathscr{B}^c_{i,j}$ be the σ-field generated by $\mathscr{C}_{k,\ell}$, $k\leqq\ell<i$ and $j<k\leqq\ell$.

DEFINITION 2. A probability measure μ on $(\mathscr{X}, \mathscr{B})$ is called a Gibbs state with nearest neighbor interaction on $\mathbf{Z}$ with the chemical potential J_1 and the interaction potential J_2 if its conditional probability $\mu\{[a_i\cdots a_j]|\mathscr{B}^c_{i,j}\}(x)$ of $[a_i\cdots a_j]\in\mathscr{C}_{i,j}$ given $\mathscr{B}^c_{i,j}$ is equal to

$$\Xi_{i,j}(x)^{-1}\exp\left[(1/kT)\{J_1\sum_{\ell=i}^{j}a_\ell-J_2(x_{i-1}a_i+a_jx_{j+1}+\sum_{\ell=i}^{j-1}a_\ell a_{\ell+1})\}\right],$$

where $\Xi_{i,j}(x)$ is a normalizing factor which depends on i, j and $x\equiv(x_i)_{i\in\mathbf{Z}}$.

PROOF OF THEOREM 2. (i) is clear from the definition of π_γ. For (ii) and (iii) let f_0 be a function on $\mathbf{N}$ defined by

$$f_0(n)=\begin{cases}(1+\gamma)^{-1} & n=1\\ (1+\gamma)^{-1}(1-\beta)^{n-2}(\beta\gamma/\alpha)^{n-1} & n=2,3,\dots,\end{cases}$$

and f be in the theorem. Since $\sum_{n=1}^{\infty}f_0(n)=\sum_{n=1}^{\infty}f(n)=1$ by (1), f_0 and f are p.d.f.'s on $\mathbf{N}$. Note that

$$\sum_{n=1}^{\infty}nf(n)=1+\gamma=(\pi_\gamma[1])^{-1}=(f_0(1))^{-1}.$$

Since

$$(3)\qquad \pi_\gamma[a_i\cdots a_{j-1}1b_{j+1}\cdots b_k] = \pi_\gamma[a_i\cdots a_{j-1}1]\pi_\gamma[1b_{j+1}\cdots b_k]/\pi_\gamma[1]$$

by the definition of π_γ we can see that π_γ is a renewal measure with the above f_0 and f. Therefore if we define random variables η_i, $i\in\mathbf{N}$, on $(\mathscr{X}, \mathscr{B}, \pi_\gamma)$ by

$$\eta_i(x) = \min\{k\colon \textstyle\sum_{\ell=1}^k x_\ell = i\} \qquad \text{for} \quad x = (\cdots x_{-1}x_0x_1\cdots),$$

then η_1 and $\eta_{i+1}-\eta_i$, $i\in\mathbf{N}$, are mutually independent and their p.d.f.'s are given by f_0 and f respectively. Moreover the law of large numbers implies

$$\lim_{n\to\infty} n^{-1}\{\eta_1 + \textstyle\sum_{i=1}^{n-1}(\eta_{i+1}-\eta_i)\} = 1+\gamma \quad \pi_\gamma\text{-a.a.},$$

and hence

$$\lim_{n\to\infty} n^{-1}\textstyle\sum_{i=0}^n x_i = (1+\gamma)^{-1} \quad \pi_\gamma\text{-a.a.},$$

which proves (ii) and (iii).

Since the ratio

$$\frac{\exp(kT)^{-1}\{J_1\sum_{\ell=i}^j a_\ell - J_2(x_{i-1}a_i + a_jx_{j+1} + \sum_{\ell=i}^{j-1} a_\ell a_{\ell+1})\}}{\pi_\gamma[x_{i-1}a_i\cdots a_jx_{j+1}]}$$
$$= \begin{cases} \{\gamma/(1+\gamma)\}^{-1}\{(1-\beta)\beta\gamma/\alpha\}^{-(j-i+2)} & \text{if} \quad (x_{i-1}, x_{j+1}) = (0, 0) \\ (1+\gamma)(1-\beta)\{(1-\beta)\beta\gamma/\alpha\}^{-(j-i+2)} & \text{if} \quad x_{i-1} \neq x_{j+1} \\ \gamma(1+\gamma)(1-\beta)^2\{(1-\beta)\beta\gamma/\alpha\}^{-(j-i+2)} & \text{if} \quad (x_{i-1}, x_{j+1}) = (1, 1) \end{cases}$$

is independent of $[a_i\cdots a_j]\in\mathscr{C}_{i,j}$ given x_{i-1} and x_{j+1}, we can easily show (ii)′.

By the renewal theory (cf. page 360 of [5]) we have

$$\begin{aligned}&\lim_{\ell\to\infty}\textstyle\sum_{m=1}^{\ell}\prod_{\xi_1+\cdots+\xi_m=\ell} f(\xi.)\\ &\quad= \lim_{\ell\to\infty}\textstyle\sum_{m\geqq 1}(f^{*m})(\ell) = [\sum_{n=1}^{\infty} nf(n)]^{-1} = (1+\gamma)^{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}&\lim_{\ell\to\infty}\pi_\gamma({}_0[a_0\cdots a_{i-1}1]_i \cap {}_{i+\ell}[1b_{i+1}\cdots b_j]_{j+\ell})\\ &\quad= \lim_{\ell\to\infty}\sum_{c_{i+1}\cdots c_{i+\ell-1}\in\{0,1\}^{\ell-1}} \pi_\gamma([a_0\cdots a_{i-1}1c_{i+1}\cdots c_{i+\ell-1}1b_{i+\ell+1}\cdots b_{j+\ell})]\\ &\quad= \lim_{\ell\to\infty}\pi_\gamma[a_0\cdots a_{i-1}1]\cdot\{f_0(1)\sum_{m=1}^{\ell}\prod_{\xi_1+\cdots+\xi_m=\ell} f(\xi.)\}\cdot\pi_\gamma[1b_{i+1}\cdots b_j]/\pi_\gamma[1]^2\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (3) and the definition of } \pi_\gamma)\\ &\quad= \pi_\gamma[a_0\cdots a_{i-1}1]\pi_\gamma[1b_{i+1}\cdots b_j],\end{aligned}$$

which implies π_γ is mixing with respect to the translation:

$$\lim_{\ell\to\infty}\pi_\gamma(A\cap(B+\ell))=\pi_\gamma(A)\pi_\gamma(B),\quad A,B\in\mathscr{B}.$$

Hence we get (iv). □

§3. Convex combination of the Gibbs states

The purpose of this section is to prove the next theorem which determines the structure of the compact convex set $\mathscr{S}$ of all probability measures satisfying the condition (Su) in Theorem 1.

THEOREM 3. *Let π_γ, $0<\gamma<\infty$, be as in §2 and $\pi_0=\delta_{\mathbf{1}}$, $\mathbf{1}\equiv(\cdots 111\cdots)$, and $\pi_\infty=\delta_{\mathbf{0}}$, $\mathbf{0}\equiv(\cdots 000\cdots)$, where δ_x is a Dirac measure concentrated at $x\in\mathscr{X}$. Then the set* ext $\mathscr{S}$ *of extreme points of $\mathscr{S}$ is $\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}$.*

REMARK 2. It follows from Theorem 3 and Choquet's theorem ([10]) that every $\mu\in\mathscr{S}$ can be represented as $\mu=\int_{[0,\infty]}\pi_\gamma m_\mu(d\gamma)$ for some probability measure m_μ on $[0,\infty]$.

We divide the proof of Theorem 3 into several steps. Set

$$M_1=\{x\in\mathscr{X}\colon \textstyle\sum_{-1}^{-\infty}x_i=\sum_{-1}^{-\infty}(1-x_i)=\sum_0^\infty x_i=\sum_0^\infty(1-x_i)=\infty\},$$
$$M_2=\mathscr{X}\smallsetminus(M_1\cup\{\mathbf{0},\mathbf{1}\}).$$

It is easy to check that

(4) $\mu(M_2)=0$ for every translation invariant probability measure μ on $\mathscr{X}$.

LEMMA 1. *Suppose $\mu\in\mathscr{S}$ and $\mu[a_i\cdots a_j]=0$ for some $[a_i\cdots a_j]\in\mathscr{C}$ with $\#_{01}[a_i\cdots a_j]>0$. Then $\mu=\rho\pi_0+(1-\rho)\pi_\infty$ for some $\rho\in(0,1)$.*

PROOF. Since $\mu(M_2)=0$ by (4), it suffices to show that $\mu(M_1)=0$. By the translation invariance of μ we can assume that $j=-1$ and $i=-n$ $(n=j-i+1)$. For $h,k\in\mathbf{Z}$, $h<0<k$, set

$$M_{h,k}=\{x\in M_1\colon \max\{d<0\colon \textstyle\sum_{\ell=d}^{-1}(1-x_\ell)=n+1\}=h,$$
$$\min\{d\geqq 0\colon \textstyle\sum_{\ell=0}^{d}x_\ell=1+\sum_{\ell=-n}^{-1}a_\ell\}=k\}.$$

Then $M_{h,k}$ are mutually disjoint and $M_1=\bigcup_{h<0<k}M_{h,k}$. Let us say $[0b_{i+1}\cdots b_{j-1}1]\in\mathscr{C}_{i,j}$ is linked to $[b'_ib'_{i+1}\cdots b'_{j-1}b'_j]\in\mathscr{C}_{i,j}$ if the latter is obtained from the former by replacing some of $b_\ell b_{\ell+1}=01$'s $(i\leqq\ell\leqq j-1)$ with 10's. Note that if $\mu[0b_{i+1}\cdots b_{j-1}1]>0$ and is linked to $[b'_i\cdots b'_j]$ then $\mu[b'_i\cdots b'_j]>0$ (in fact, the r.h.s. of (Eq)′ for $[b'_i\cdots b'_j]$ contains $\mu[0b_{i+1}\cdots b_{j-1}1]$ multiplied by $\alpha^m(1-\alpha)^n$). Assume $\mu(M_{h,k})>0$ for some h and k. Then there is a basic cylinder $[0b_{h+1}\cdots b_{k-1}1]\in\mathscr{C}_{h,k}$ such that $\mu[0b_{h+1}\cdots b_{k-1}1]>0$. By considering a linked chain from ${}_h[0b_{h+1}\cdots b_{-1}b_0b_1\cdots b_{k-1}1]_k$ to ${}_h[011\cdots 11a_{-n}\cdots a_{-1}00\cdots 001]_k$ via ${}_h[011\cdots$

$1100\cdots 00b_0b_1\cdots b_{k-1}1]_k$, we have $\mu({}_h[011\cdots 11a_{-n}\cdots a_{-1}00\cdots 001]_k)>0$, which implies $\mu[a_{-n}\cdots a_{-1}]>0$. This is a contradiction. Thus $\mu(M_{h,k})=0$ for all h and k, and hence $\mu(M_1)=0$. □

We remark that if $\mu\in\mathscr{S}$ then

$$(5)\qquad \mu[a_ia_{i+1}\cdots a_j] = \mu[a_j\cdots a_{i+1}a_i] \quad\text{for every}\quad [a_i\cdots a_j].$$

(By the translation invariance of μ it is not necessary for us to specify the coordinates of cylinders in (5).) Indeed we have

$$\mu[0a_{i+1}\cdots a_{j-1}1]\cdot(1-\alpha)^s = \mu[0\cdots 01\cdots 1],$$
$$\mu[1b_{i+1}\cdots b_{j-1}0]\cdot(1-\alpha)^t = \mu[1\cdots 10\cdots 0]$$

by (Su)-(ii), where $s=\#_{01}[0a_{i+1}\cdots a_{j-1}1]-1$ and $t=\#_{01}[1b_{i+1}\cdots b_{j-1}0]$; and moreover

$$\begin{aligned}\mu[000111] &= \mu[000] - \mu[0000] - \mu[00010] - \mu[000110]\\ &= \mu[000] - \mu[0000] - \mu[01000] - \mu[011000]\\ &= \mu[111000].\end{aligned}$$

LEMMA 2. *Suppose $\mu\in\operatorname{ext}\mathscr{S}$ and $\mu\neq\pi_0, \pi_\infty$. Then*

$$\mu[00a_i\cdots a_j]/\mu[0a_i\cdots a_j] \equiv q = \mu[00]/\mu[0],$$
$$\mu[a_i\cdots a_j11]/\mu[a_i\cdots a_j1] \equiv q' = \mu[11]/\mu[1]$$

for all $a_i\cdots a_j$; and $0<q, q'<1$.

PROOF. Let $\tilde\lambda$ be a nonnegative translation invariant function on $\mathscr{C}$ defined by

$$\begin{cases}\tilde\lambda(\phi) = 0, \quad \tilde\lambda[0a_i\cdots a_j] = \mu[00a_i\cdots a_j]\\ \tilde\lambda[1] = (1-\alpha)\mu[01] + \alpha\mu[001]\\ \tilde\lambda[\overbrace{11\cdots 11}^{n}] = (1-\alpha)\mu[10\overbrace{1\cdots 11}^{n-1}] \quad (n>1)\\ \tilde\lambda[1\cdots 10a_i\cdots a_j] = \mu[1\cdots 100a_i\cdots a_j].\end{cases}$$

Since $\tilde\lambda$ satisfies (iii) of Lemma 0 by (Su), we can extend $\tilde\lambda$ to a finite measure λ on $(\mathscr{X}, \mathscr{B})$ uniquely. λ has the property that

$$\lambda[a_i\cdots a_j] < \mu[a_i\cdots a_j], \quad \phi\neq[a_i\cdots a_j]\in\mathscr{C}.$$

Indeed if $\min\{a_i, a_j\}=0$ this is obvious from Lemma 1 and (5). If $a_i=a_j=1$ we have, for example,

$$\begin{aligned}\lambda[11\cdots 11] &= (1-\alpha)\mu[101\cdots 11]\\ &= (1-\alpha)\{\mu[0101\cdots 11]+\mu[01101\cdots 11]+\mu[011101\cdots 11]+\cdots\} \quad \text{(by (4))}\\ &= \mu[0011\cdots 11] + \mu[00111\cdots 11] + \mu[001111\cdots 11] + \cdots \quad \text{(by (Su)–(ii))}\\ &< \mu[011\cdots 11] + \mu[0111\cdots 11] + \mu[01111\cdots 11] + \cdots \quad \text{(by Lemma 1)}\\ &= \mu[11\cdots 11] \quad \text{(by (4))}.\end{aligned}$$

It then follows that

$$0 < \lambda(\mathscr{X}) < \mu(\mathscr{X}) = 1 \quad \text{and} \quad \lambda(A) \leqq \mu(A), \quad A \in \mathscr{B},$$

which allows us to define two probability measures μ_1 and μ_2 on $(\mathscr{X}, \mathscr{B})$ by

$$\mu_1 = (1-\lambda(\mathscr{X}))^{-1}(\mu-\lambda) \quad \text{and} \quad \mu_2 = \lambda(\mathscr{X})^{-1}\lambda$$

respectively. Then $\mu_i \in \mathscr{S}$, $i=1, 2$, by the definition of λ, and

$$\mu = (1-\lambda(\mathscr{X}))\mu_1 + \lambda(\mathscr{X})\mu_2, \quad \mu_i \in \mathscr{S}, \quad i = 1, 2, \quad 0 < \lambda(\mathscr{X}) < 1,$$

which implies $\mu_1=\mu_2$ since $\mu \in \operatorname{ext}\mathscr{S}$. Then a direct computation gives us $\lambda=(\lambda(\mathscr{X})/\mu(\mathscr{X}))\mu$. Thus $q=\lambda(\mathscr{X})/\mu(\mathscr{X})=\mu[00]/\mu[0]$. The second equation is shown similarly. □

PROOF OF THEOREM 3. It is clear that $\pi_0, \pi_\infty \in \operatorname{ext}\mathscr{S}$. Suppose $\mu \in \operatorname{ext}\mathscr{S}$ and $\mu \neq \pi_0, \pi_\infty$. Let q and q' be those in Lemma 2. By (Eq)′ and (Su)

$$\begin{aligned}\mu[01] &= \textstyle\sum_{[b_0b_1b_2b_3]\rhd[01]} \alpha^m(1-\alpha)^n\mu[b_0b_1b_2b_3]\\ &= \alpha\mu[001] + (1-\alpha)\mu[01] + \alpha^2\mu[0101] + \alpha\mu[011]\\ &= \{\alpha q+(1-\alpha)+(\alpha^2/(1-\alpha))qq'+\alpha q'\}\mu[01].\end{aligned}$$

Since $\mu[01]\neq 0$ by Lemma 1, we have

$$1 = q + q' + (\alpha/(1-\alpha))qq'. \tag{6}$$

Set $\beta=(1-q)\alpha$. By (1) and (6)

$$\gamma = q/\{(1-q)(1-\alpha+\alpha q)\} = (1-q')/(1-q).$$

It then follows from (4), (5) and Lemma 2 that

$$\begin{aligned}\mu[0] &= \mu[01] + \mu[001] + \mu[0001] + \cdots\\ &= (1+q+q^2+\cdots)\mu[01] = (1-q)^{-1}\mu[01] = (\alpha/\beta)\mu[01],\\ \mu[1] &= \mu[10] + \mu[110] + \mu[1110] + \cdots\\ &= \mu[01] + \mu[011] + \mu[0111] + \cdots = (1-q')^{-1}\mu[01] = (\alpha/\beta\gamma)\mu[01].\end{aligned}$$

Since $\mu[0]+\mu[1]=1$, we get

$$\mu[01] = (\beta\gamma/\alpha)(1+\gamma)^{-1}, \quad \mu[0] = \gamma/(1+\gamma) \quad \text{and} \quad \mu[1] = 1/(1+\gamma).$$

For $[0a_{i+1}\cdots a_{j-1}1] \in \mathscr{C}_{i,j}$ let

$$k = \textstyle\sum_{\ell=i}^{j} a_\ell \quad \text{and} \quad t = \#_{01}[0a_{i+1}\cdots a_{j-1}1].$$

Then

$$\begin{aligned}(1-\alpha)^{t-1}\pi_\gamma[0a_{i+1}\cdots a_{j-1}1] &= \pi_\gamma({}_i[0\cdots0\overbrace{1\cdots1}^{k}]_j) \quad \text{(by (Su)–(ii))} \\ &= (1/(1+\gamma))(1-\beta)^{j-i-k}((1-\beta\gamma)/\gamma)^{k-1}(\beta\gamma/\alpha)^{j-i}\end{aligned}$$

and

$$\begin{aligned}(1-\alpha)^{t-1}\mu[0a_{i+1}\cdots a_{j-1}1] &= \mu({}_i[0\cdots0\overbrace{1\cdots1}^{k}]_j) \quad \text{(by (Su)–(ii))} \\ &= q^{j-i-k}q'^{k-1}\mu[01] = q^{j-i-k}q'^{k-1}(\beta\gamma/\alpha)(1+\gamma)^{-1}.\end{aligned}$$

By (1) and (6) it follows that

$$\mu[0a_{i+1}\cdots a_{j-1}1] = \pi_\gamma[0a_{i+1}\cdots a_{j-1}1] \qquad \text{for all} \quad a_{i+1}\cdots a_{j-1}.$$

These equations imply $\mu(A)=\pi_\gamma(A)$, $A\in\mathscr{C}$, and hence $\mu=\pi_\gamma$. It was seen in (iv) of Theorem 2 that π_γ is an extreme point of $\mathscr{S}$. Thus we have $\operatorname{ext}\mathscr{S}=\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}$. □

REMARK 3. The theorem can be also proved by using (i) and (ii) of Theorem 2 instead of (iv). In fact it follows from (i) that $\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}\subset\mathscr{S}$ and from (iii) that π_γ's are mutually singular. It is known from the above argument that $\operatorname{ext}\mathscr{S}\subset\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}$. Therefore $\operatorname{ext}\mathscr{S}=\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}$.

§4. The structure of stationary measures

In this section we investigate when the condition (Su) is also a necessary condition. Let $\mathscr{I}$ be the compact convex set consisting of all stationary measures for (MP). Write Θ_n for the probability measure on $\mathscr{X}$ which has a point mass at $\theta_n\equiv(\cdots x_{n-1}x_nx_{n+1}\cdots)$ where $x_i=1$ for $i\leqq n$ and $x_i=0$ for $i>n$. It is clear that $\Theta_n\in\mathscr{I}$.

Now we have the following main theorem, which determines completely the structure of stationary measures for (MP) with $0<\alpha\leqq1/2$.

THEOREM 4. *Suppose* $0<\alpha\leqq1/2$. *Then* $\operatorname{ext}\mathscr{I}=\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}\cup\{\Theta_n\colon n\in\mathbf{Z}\}$.

Since each π_γ satisfies (Su) by (i) of Theorem 2, the next result follows from Theorem 4 and Choquet's theorem.

COROLLARY. *Suppose* $0<\alpha\leqq 1/2$. *Then the condition* (Su) *is a necessary and sufficient condition for a probability measure* μ *on* $\mathscr{X}$ *with* $\mu\{\theta_n\colon n\in\mathbf{Z}\}=0$ *to be a stationary measure for* (MP).

The proof of Theorem 4 can be done by a method of coupled Markov process ([1, 7]). First we consider the translation invariant case:

PROPOSITION 1. *Suppose* $0<\alpha\leqq 1/2$. *Then* $\mathrm{ext}(\mathscr{I}\cap\mathscr{T})=\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}$.

We will define a (coupled) Markov process (CMP) on the state space $(\bar{\mathscr{X}}, \bar{\mathscr{B}})\equiv(\mathscr{X}\times\mathscr{X}, \mathscr{B}\times\mathscr{B})$ below in such a way that each component of (CMP) is the Markov process (MP). It is proceeded by determining transition probabilities $P((x, y), C)$, $(x, y)\in\bar{\mathscr{X}}$, $C\in\bar{\mathscr{B}}$, satisfying

$$(7)\qquad \begin{cases} P((x, y), A\times\mathscr{X}) = P(x, A), & A\in\mathscr{B} \\ P((x, y), \mathscr{X}\times B) = P(y, B), & B\in\mathscr{B}. \end{cases}$$

First we give a local rule of the movement of particles in the configuration $(x, y)\equiv(x_i, y_i)_{i\in\mathbf{Z}}$ under the time evolution as follows:

(i) If $x_{i-1}x_i=y_{i-1}y_i=01$ at time t then at time $t+1$
$x_{i-1}x_i = y_{i-1}y_i = 10$ with probability α,
$x_{i-1}x_i = y_{i-1}y_i = 01$ with probability $1-\alpha$.

(ii) If $x_{i-2}x_{i-1}x_i=011$ and $y_{i-1}y_i=01$ at time t then at time $t+1$
$x_{i-2}x_{i-1}x_i = 101$, $y_{i-1}y_i = 01$ with probability α,
$x_{i-2}x_{i-1}x_i = 011$, $y_{i-1}y_i = 10$ with probability α,
$x_{i-2}x_{i-1}x_i = 011$, $y_{i-1}y_i = 01$ with probability $1-2\alpha$.

(iii) The exchange of the roles of x and y in (ii).

(iv) If $x_{i-1}x_i=01$ (resp. $y_{i-1}y_i=01$) and none of the above three cases at time t then at time $t+1$
$x_{i-1}x_i$(resp. $y_{i-1}y_i$) $= 10$ with probability α,
$x_{i-1}x_i$(resp. $y_{i-1}y_i$) $= 01$ with probability $1-\alpha$.

Then the rule of transition, which determines $P((x, y), \cdot)$ for $0<\alpha\leqq 1/2$, is obtained by applying the local rules (i)–(iv) independently to the configuration (x, y) (cf. §1). It is easy to check that $P((x, y), \cdot)$ satisfies (7).

Denote by $\bar{\mathscr{I}}$ the set of all stationary measures for (CMP) and by $\bar{\mathscr{T}}$ the set of translation invariant probability measures on $\bar{\mathscr{X}}$. Here we say that a probability measure ν on $\bar{\mathscr{X}}$ is translation invariant if $\nu(C)=\nu(C+\ell)$ for all $C\in\bar{\mathscr{B}}$ and $\ell\in\mathbf{Z}$ where

$$C+\ell = \{(x', y')\colon (x'_i, y'_i)=(x_{i+\ell}, y_{i+\ell}),\ i\in\mathbf{Z},\ \text{for some } (x, y)\in C\}.$$

Since every $\nu\in\bar{\mathscr{I}}$ satisfies

$$(\overline{\mathrm{E}}\mathrm{q}) \qquad \int_{\bar{\mathscr{X}}} d\nu(x, y) g(x, y) = \int_{\bar{\mathscr{X}}} d\nu(x, y) \int_{\bar{\mathscr{X}}} P((x, y), d(x', y')) g(x', y')$$

for every bounded measurable function g by definition, it follows from (7) that if $\nu \in \bar{\mathscr{I}}$ and μ_1 and μ_2 are the marginal measures of ν defined by $\mu_1(A) = \nu(A \times \mathscr{X})$ and $\mu_2(B) = \nu(\mathscr{X} \times B)$ then μ_1 and $\mu_2 \in \mathscr{I}$.

Let $\mathscr{W}$ be the path space $\bar{\mathscr{X}}^{\mathbf{T}} = \{(x(t), y(t))_{t \in \mathbf{T}}\}$ equipped with the usual Borel structure, and write $P_{(x,y)}(\cdot)$, $(x, y) \in \bar{\mathscr{X}}$, for a probability measure on $\mathscr{W}$ determined by

$$\begin{aligned} &P_{(x,y)}\{(x(t), y(t))_{y \in \mathbf{T}} \colon (x(\ell), y(\ell)) \in S_\ell,\ \ell = 0, 1, \dots, k\} \\ &= \chi_{S_0}(x, y) \int_{S_1} P((x, y), d(x(1), y(1))) \int_{S_2} P((x(1), y(1)), d(x(2), y(2))) \cdots \\ &\quad \cdots \int_{S_{k-1}} P((x(k-2), y(k-2)), d(x(k-1), y(k-1))) P((x(k-1), y(k-1)), S_k), \end{aligned}$$

$k \in \mathbf{N}$, $S_\ell \in \bar{\mathscr{B}}$, $\ell = 0, \dots, k$. The elements of $\mathscr{W}$ are written sometimes by $w \equiv (w(t))_{t \in \mathbf{T}}$. We denote by $P^t((x, y), \cdot)$, $t \in \mathbf{N}$, the t-th iteration of $P((x, y), \cdot)$, and so,

$$P^k((x, y), C) = P_{(x,y)}\{(w(t))_{t \in \mathbf{T}} \colon w(k) \in C\}, \ C \in \bar{\mathscr{B}}.$$

Definition 3. For $x, y \in \mathscr{X}$ we will write $x \leqq y$ if $x_i \leqq y_i$ for all $i \in \mathbf{Z}$, and for two probability measures μ_1 and μ_2 on $\mathscr{X}$, write $\mu_1 \leqq \mu_2$ if there exists a probability measure ν on $\bar{\mathscr{X}}$ with the first marginal μ_1 and the second marginal μ_2, and such that $\nu\{(x, y) \colon x \leqq y\} = 1$.

The following is clear by the definition: if $x \leqq y$ (resp. $x = y$, $x \geqq y$), then

$$P^t((x, y), \{(x', y') \colon x' \leqq y' (\text{resp. } x' = y',\ x' \geqq y')\}) = 1, \quad t \in \mathbf{N}.$$

Given $(x, y) \in \bar{\mathscr{X}}$, we will say $\{i, i+1, \dots, j-1, j\} \subset \mathbf{Z}$, $i \leqq j$, is a plus cluster associated with (x, y) if $\{i, \dots, j\}$ is a maximal set which has the property that $x_i - y_i = x_j - y_j = +1$ and $x_\ell - y_\ell \geqq 0$ for all ℓ with $i \leqq \ell \leqq j$. We permit the case $i = -\infty$ and/or $j = +\infty$. A minus cluster is defined similarly. Note that the clusters are mutually disjoint. Let $s_{m,n}(x, y)$, m, $n \in \mathbf{Z}$, $m \leqq n$, be the sum of numbers of plus and minus clusters which intersect with $\{m, m+1, \dots, n\}$. We sometimes write $s_{\{m,\dots,n\}}(x, y)$ instead of $s_{m,n}(x, y)$. Set

$$\begin{aligned} &\sigma_{m,n}(x, y) = (n - m + 1)^{-1} s_{m,n}(x, y), \\ &\sigma_\infty(x, y) = \limsup_{n \to \infty} \sigma_{0,n}(x, y), \quad \sigma_{-\infty}(x, y) = \limsup_{n \to \infty} \sigma_{-n,0}(x, y), \\ &\bar{\sigma}(x, y) = \lim_{n \to \infty} \sigma_{0,n}(x, y) \quad (\text{if exists}). \end{aligned}$$

Lemma 3. *For any* $(x, y) \in \bar{\mathscr{X}}$ *and* $t \in \mathbf{N}$,

$$\sigma_\infty(x, y) \geqq \int_{\bar{\mathscr{X}}} P^t((x, y), d(x', y'))\sigma_\infty(x', y').$$

The same statement holds for $\sigma_{-\infty}$.

PROOF. By the definition of $P((x, y), \cdot)$ each particle of (x, y) stays or moves to the left-neighboring site under the one step time evolution; and further the number of sites i with $x_i = y_i$ in (x, y) never decreases. Hence for any fixed $(x, y) \in \bar{\mathscr{X}}$ and $k \in \mathbf{N}$,

$$P_{(x,y)}\{(w(t))_{t\in\mathbf{T}}: s_{\ell,m}(w(k)) \leqq 2k + s_{\ell,m}(x, y)\} = 1, \quad \ell \leqq m,$$

which yields

$$P_{(x,y)}\{(w(t))_{t\in\mathbf{T}}: s_{0,n}(w(k)) \leqq 2k + s_{0,n}(x, y),\ n \in \mathbf{N}\} = 1. \tag{8}$$

Hence the lemma follows by the definition of σ_∞. □

The next lemma is fundamental to the proof of Proposition 1.

LEMMA 4. *Let* σ_* $(0 < \sigma_* < 1)$ *be given. Then there exist* $\bar{t} \in \mathbf{N}$ *and* $\delta_* > 0$ *such that for all* $(x, y) \in \bar{\mathscr{X}}$ *with* $\bar{\sigma}(x, y) \geqq \sigma_*$

$$P^{\bar{t}}((x, y), \{(x', y'): \bar{\sigma}(x, y) - \sigma_\infty(x', y') \geqq \delta_*\}) = 1.$$

PROOF. 1°. Take $L = L(\sigma_*) \in \mathbf{N}$ so that

$$b_* \equiv b_*(\sigma_*, L) = (2^{-1}\sigma_* - 3 \cdot 2^{-L})/2^L > 0.$$

Let $\mathscr{D} = \{B(j)\}_{j\in\mathbf{Z}}$ be a partition of $\mathbf{Z}$ into

$$B(j) = \{j \cdot 2^L, j \cdot 2^L + 1, \dots, (j+1) \cdot 2^L - 1\}.$$

Given $(x, y) \in \bar{\mathscr{X}}$ let C_j^ℓ (resp. C_j^r) be the left-(resp. right-)most cluster which intersects with $B(j)$. Denote by $A_{(x,y),j}^t$ the set of all paths $w \in \mathscr{W}$ such that $w(0) = (x, y)$ and such that during the time from 0 to t the configuration on $C_j^\ell \cap B(j)$ and $C_j^r \cap B(j)$ has been all frozen and further at least one cluster in $B(j)$ has disappeared. Then we can choose $\bar{t} = \bar{t}(L) \in \mathbf{N}$ and $q_* = q_*(L, \alpha) > 0$ such that

$$P_{(x,y)}(A_{(x,y),j}^{\bar{t}}) \geqq q_* \tag{9}$$

for all j and (x, y) satisfying $s_{B(j)}(x, y) \geqq 4$. This is possible because the members of $\bar{\mathscr{X}}|_{B(j)}$ (the restriction of $\bar{\mathscr{X}}$ to $B(j)$) are the same and finitely many for all $j \in \mathbf{Z}$; and $s_{B(\cdot)}(x, y) \geqq 4$ implies that $A_{(x,y),\cdot}^t$ is not empty for all sufficiently large t. Let us fix such $\bar{t}$ and q_* and write A_j instead of $A_{(x,y),j}^{\bar{t}}$ for brevity. We remark that if $j_1 \leqq j_2 - 4\bar{t} \leqq \cdots \leqq j_s - (s-1)4\bar{t}$ then

$$P_{(x,y)}(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_s}) = \prod_{\ell=1}^{s} P_{(x,y)}(A_{j_\ell}). \tag{10}$$

Indeed if $C \in \bar{\mathscr{B}}_{m,n}$ (the σ-field $\bar{\mathscr{B}}_{m,n}$ is defined analogously to $\mathscr{B}_{m,n}$ in §1) then $P((x, y), C)$ is $\bar{\mathscr{B}}_{m-1,n+1}$-measurable and further if $D \in \bar{\mathscr{B}}_{m',n'}$ with $n+2 \leqq m'$ then

$$P((x, y), C \cap D) = P((x, y), C)P((x, y), D).$$

Hence if $C_k \in \bar{\mathscr{B}}_{m,n}$ and $D_k \in \bar{\mathscr{B}}_{m',n'}$, $k=1, 2$, and if $n+4 \leqq m'$ then

$$\begin{aligned} P_{(x,y)}&\{w \in \mathscr{W}\colon w(k) \in C_k \cap D_k,\ k=1, 2\} \\ &= \int_{C_1 \cap D_1} P((x, y), d(x', y'))P((x', y'), C_2 \cap D_2)) \\ &= \int_{C_1 \cap D_1} P((x, y), d(x', y'))P((x', y'), C_2)P((x', y'), D_2) \\ &= \int_{C_1} P((x, y), d(x', y'))P((x', y'), C_2) \cdot \int_{D_1} P((x, y), d(x', y'))P((x', y'), D_2) \\ &= P_{(x,y)}\{w\colon w(k) \in C_k,\ k=1, 2\} \cdot P_{(x,y)}\{w\colon w(k) \in D_k,\ k=1, 2\}. \end{aligned}$$

2°. For $(x, y) \in \bar{\mathscr{X}}$ define

$$b(x, y) = \liminf_{n\to\infty} n^{-1}\#\{j \geqq 0\colon s_{B(j)}(x, y) \geqq 4,\ 0<(j+1)2^L<n\}.$$

If $\bar{\sigma}(x, y) \geqq \sigma_*$ we have $b(x, y) \geqq b_*$. In fact it holds that

(11) $\#\{j \geqq 0\colon s_{B(j)}(x, y) \geqq 4,\ 0<(j+1)2^L<n\} \geqq \{n(\sigma_*/2) - 3(n2^{-L}+1)\}/2^L$

for all sufficiently large n, which implies $b(x, y) \geqq b_*$. Let us fix $(x, y) \in \bar{\mathscr{X}}$ with $\bar{\sigma}(x, y) \geqq \sigma_*$ and put

$$\{j_0<j_1<j_2<\cdots\} \equiv \{j \geqq 0\colon s_{B(j)}(x, y) \geqq 4\}.$$

Then $\{j_{4\bar{l}\ell}\}_{\ell \in \mathbf{N}}$ satisfies

(12) $\liminf_{n\to\infty} n^{-1}\#\{\ell\colon 0<(j_{4\bar{l}\ell}+1)2^L<n\} \geqq b(x, y)/(4\bar{l}) \geqq b_*/(4\bar{l}).$

It is easy from (9), (10) and the law of large numbers to conclude

(13) $$\liminf_{s\to\infty} s^{-1}\#\{\ell\colon w \in A_{j_{4\bar{l}\ell}},\ 1 \leqq \ell \leqq s\} \geqq q_*$$

for $P_{(x,y)}$-a.a. w. By the same consideration for obtaining (8) we have

(14) $s_{0,n}(x, y) + 2\bar{l} \geqq s_{0,n}(w(\bar{l})) + \#\{\ell\colon w \in A_{j_{4\bar{l}\ell}},\ 0<(j_{4\bar{l}\ell}+1)2^L<n\}, \quad n \in \mathbf{N},$

for $P_{(x,y)}$-a.a. w. Combining (12)–(14), we get

$$\bar{\sigma}(x, y) - \sigma_\infty(x(\bar{l}), y(\bar{l})) \geqq q_* b_*/(4\bar{l}) \qquad \text{for } P_{(x,y)}\text{-a.a. } w.$$

Hence the lemma holds with $\delta_* = q_* b_*/(4\bar{l})$. □

Let $c\colon \bar{\mathscr{X}}\to\{0,1\}^{\mathbf{Z}}$ be a measurable map defined by $c(x, y)=(\cdots c_{-1}c_0c_1\cdots)$ where $c_i\equiv c(x, y)_i=1$ if i is the left-most site of some plus or minus cluster associated with (x, y) and $c_i=0$ otherwise.

LEMMA 5. *For any* $\nu\in\bar{\mathscr{I}}\cap\bar{\mathscr{T}}$ $\nu\{(x, y)\colon x\leqq y$ *or* $x\geqq y\}=1$.

PROOF. Since $\nu\in\bar{\mathscr{T}}$, Birkhoff's ergodic theorem states that

$$\begin{aligned}\bar{\sigma}(x, y) &= \lim_{n\to\infty}(n+1)^{-1}\textstyle\sum_{i=0}^{n}c(x, y)_i\\ &= \lim_{n\to\infty}(n+1)^{-1}\textstyle\sum_{i=0}^{n}c((x, y)+i)_0\end{aligned}$$

holds for ν-a.a. (x, y) and that

$$\int_{\bar{\mathscr{X}}}\bar{\sigma}(x, y)d\nu(x, y) = \int_{\bar{\mathscr{X}}}c(x, y)_i d\nu(x, y), \quad i\in\mathbf{Z}. \tag{15}$$

Now suppose $\nu\{(x, y)\colon \bar{\sigma}(x, y)>0\}>0$ and so there is a σ_* $(0<\sigma_*<1)$ satisfying $\nu\{(x, y)\colon \bar{\sigma}(x, y)>\sigma_*\}>0$. Since $\nu\in\bar{\mathscr{I}}$, by ($\bar{\mathrm{E}}$q) for $g=\sigma_\infty$ we have

$$\begin{aligned}0 &= \int_{\bar{\mathscr{X}}}d\nu(x, y)\{\sigma_\infty(x, y)-\int_{\bar{\mathscr{X}}}P^t((x, y), d(x', y'))\sigma_\infty(x', y')\}\\ &= \int_{\bar{\sigma}<\sigma_*}+\int_{\bar{\sigma}\geqq\sigma_*}\end{aligned}$$

for every $t\in\mathbf{N}$. If we choose $\bar{t}$ and δ_* as in Lemma 4, the second term in the r.h.s. of the above equation is not smaller than $\nu\{\bar{\sigma}(x, y)\geqq\sigma_*\}\cdot\delta_*>0$ for $t=\bar{t}$. But this is impossible because the first term in the r.h.s. is nonnegative by Lemma 3. Thus $\nu\{\bar{\sigma}(x, y)>0\}$ must be zero, and hence $\nu\{(x, y)\colon c(x, y)_i=1\}=0$ for every $i\in\mathbf{Z}$, which proves the lemma. □

For the last step of the proof of Proposition 1 we summarize the necessary tools below as a proposition which are borrowed from [7] (the proof given there is also valid for our case).

PROPOSITION 2. (i) *If* $\nu\in\mathrm{ext}\,\bar{\mathscr{I}}$, *then each of* $\nu\{(x, y)\colon x=y\}$, $\nu\{(x, y)\colon x\leqq y\}$ *and* $\nu\{(x, y)\colon x\geqq y\}$ *is either zero or one. The same statement holds for* $\nu\in\mathrm{ext}\,(\bar{\mathscr{I}}\cap\bar{\mathscr{T}})$ *in the translation invariant case.*

(ii) (*a*) *If* $\mu_1, \mu_2\in\mathscr{I}$, *there is a* $\nu\in\bar{\mathscr{I}}$ *with marginals* μ_1 *and* μ_2. (*b*) *If* $\mu_1, \mu_2\in\mathrm{ext}\,\mathscr{I}$, *then* ν *can be taken in* $\mathrm{ext}\,\bar{\mathscr{I}}$. (*c*) *In the translation invariant case, if* $\mu_1, \mu_2\in\mathscr{I}\cap\mathscr{T}$, *then* ν *can be taken in* $\bar{\mathscr{I}}\cap\bar{\mathscr{T}}$. (*d*) *If* $\mu_1, \mu_2\in\mathrm{ext}\,(\mathscr{I}\cap\mathscr{T})$, *then* ν *can be taken in* $\mathrm{ext}\,(\bar{\mathscr{I}}\cap\bar{\mathscr{T}})$.

The proof of Proposition 1 is now completed as follows.

PROOF OF PROPOSITION 1. By (iv) of Theorem 2 $\pi_\gamma\in\mathrm{ext}\,(\mathscr{I}\cap\mathscr{T})$. If $\mu\in$

$\mathrm{ext}(\mathscr{I}\cap\mathscr{T})$ then either $\mu\leqq\pi_\gamma$ or $\mu\geqq\pi_\gamma$. Indeed choose $\nu\in\mathrm{ext}(\bar{\mathscr{I}}\cap\bar{\mathscr{T}})$ by (ii)–(d) of the above proposition so that it has marginals μ and π_γ. Then by (i) and Lemma 5, either $\nu\{x\leqq y\}=1$ or $\nu\{x\geqq y\}=1$, and hence $\mu\leqq\pi_\gamma$ or $\mu\geqq\pi_\gamma$. Further if $0\leqq\gamma'\leqq\gamma\leqq\infty$ and $\mu\leqq\pi_\gamma$ (resp. $\mu\geqq\pi_{\gamma'}$) then $\mu\leqq\pi_{\gamma'}$ (resp. $\mu\geqq\pi_\gamma$) by (iii) of Theorem 2. Therefore for a given $\mu\in\mathrm{ext}(\mathscr{I}\cap\mathscr{T})$ there is a $\gamma_0\in[0,\infty]$ such that $\mu\geqq\pi_\gamma$ for $\gamma>\gamma_0$ and $\mu\leqq\pi_\gamma$ for $\gamma<\gamma_0$. If $\mu\leqq\pi_\gamma$, it holds for all integers i and j $(i\leqq j)$ and for all $(k_i,\dots,k_j)\in\{0,1\}^{j-i+1}$ that

$$\begin{aligned}\mu\{x\in\mathscr{X}\colon x_\ell\geqq k_\ell,\ i\leqq\ell\leqq j\} &= \nu\{(x,y)\in\bar{\mathscr{X}}\colon x_\ell\geqq k_\ell,\ i\leqq\ell\leqq j\}\\ \leqq \nu\{(x,y)\in\bar{\mathscr{X}}\colon y_\ell\geqq k_\ell,\ i\leqq\ell\leqq j\} &= \pi_\gamma\{y\in\mathscr{X}\colon y_\ell\geqq k_\ell,\ i\leqq\ell\leqq j\},\end{aligned}$$

where $\nu\in\bar{\mathscr{I}}\cap\bar{\mathscr{T}}$ is such that the first and second marginals are μ and π_γ respectively and such that $\nu\{x\leqq y\}=1$. Letting $\gamma\uparrow\gamma_0$ we have

$$\mu\{x\in\mathscr{X}\colon x_\ell\geqq k_\ell,\ i\leqq\ell\leqq j\}\leqq\pi_{\gamma_0}\{x\in\mathscr{X}\colon x_\ell\geqq k_\ell,\ i\leqq\ell\leqq j\}$$

by the continuity of π_γ. Since the opposite inequality is verified similarly, it follows that $\mu=\pi_{\gamma_0}$. Thus $\mathrm{ext}(\mathscr{I}\cap\mathscr{T})\subset\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}$. The reverse inclusion is clear by (iv) of Theorem 2. □

Now we will prove Theorem 4. First we improve Lemma 4 for the general case.

LEMMA 4′. *Let* $0<\sigma_*<1$. *Then there exist* $\bar{l}\in\mathbf{N}$ *and* $\delta_*>0$ *such that for all* (x,y) *with* $\sigma_\infty(x,y)\geqq\sigma_*$

$$P^{\bar{l}}((x,y),\{(x',y')\colon\sigma_\infty(x,y)-\sigma_\infty(x',y')\geqq\delta_*\})=1.$$

The same statement holds for $\sigma_{-\infty}$.

PROOF. Choose L, b_*, $\mathscr{D}$, $B(j)$, $\bar{l}$, q_* and $\delta_*=q_*b_*/(4\bar{l})$ as in the proof of Lemma 4, and fix (x,y) with $\sigma_\infty(x,y)\geqq\sigma_*$. Write $\{j_0<j_1<\cdots\}\equiv\{j\geqq 0\colon s_{B(j)}(x,y)\geqq 4\}$. Since $P_{(x,y)}$-a.a. w satisfy (13) and (14), it is sufficient for the proof to show that

$$\sigma_\infty(x,y)-\sigma_\infty(w(\bar{l}))\geqq\delta_*$$

under the assumptions $w(0)=(x,y)$ and (13) & (14). Let us fix such w. If $\sigma_{0,n}(x,y)\leqq(2/3)\sigma_*$, then

$$s_{0,n}(w(\bar{l}))\leqq s_{0,n}(x,y)+2\bar{l}\leqq(2/3)(n+1)\sigma_*+2\bar{l}$$

by (14). If $\sigma_{0,n}(x,y)>(2/3)\sigma_*$, then

$$\sharp\{\ell>0\colon 0<(j_{4\bar{l}\ell}+1)2^L<n\}\geqq[\{n(\sigma_*/2)-3(n2^{-L}+1)\}/(2^L4\bar{l})]-2$$

by (11). Hence, given $\varepsilon>0$, for all sufficiently large n witn $\sigma_n(x,y)>(2/3)\sigma_*$

$$\begin{aligned}(n+1)(\sigma_\infty(x, y)+\varepsilon)-s_{0,n}(w(\tilde{\imath})) &> s_{0,n}(x, y)-s_{0,n}(w(\tilde{\imath}))\\ &\geqq \#\{\ell>0: w\in A_{j_{4\tilde{\imath}\ell}}, 0<(j_{4\tilde{\imath}\ell}+1)2^L<n\} - 2\tilde{\imath} \quad \text{(by (14))}\\ &\geqq \#\{\ell>0; 0<(j_{4\tilde{\imath}\ell}+1)2^L<n\}(q_*-\varepsilon) - 2\tilde{\imath} \quad \text{(by (13))}\\ &\geqq [\{\{n(\sigma_*/2)-3(n2^{-L}+1)\}/(2^L 4\tilde{\imath})\}-2](q_*-\varepsilon) - 2\tilde{\imath},\end{aligned}$$

that is,

$$\begin{aligned}s_{0,n}(w(\tilde{\imath})) \leqq (n+1)(\sigma_\infty(x, y)+\varepsilon)\\ - [\{\{n(\sigma_*/2)-3(n2^{-L}+1)\}/(2^L 4\tilde{\imath})\}-2](q_*-\varepsilon) + 2\tilde{\imath}.\end{aligned}$$

Therefore

$$\begin{aligned}&\limsup_{n\to\infty} \sigma_{0,n}(w(\tilde{\imath}))\\ &\quad \leqq \max\{\sigma_\infty(x, y)+\varepsilon-\{(\sigma_*/2)-3\cdot 2^{-L}\}(q_*-\varepsilon)/(2^L 4\tilde{\imath}), (2/3)\sigma^*\}.\end{aligned}$$

As ε is arbitrary,

$$\sigma_\infty(w(\tilde{\imath})) \leqq \sigma_\infty(x, y) - \{(\sigma_*/2)-3\cdot 2^{-L}\}q^*/(2^L 4\tilde{\imath}) \leqq \sigma_\infty(x, y) - \delta_*,$$

which was to be proved. □

Just like $c(x, y)$ let $\tilde{c}$: $\bar{\mathscr{X}}\to\{0, 1\}^{\mathbf{Z}}$ be a map defined by $\tilde{c}(x, y)_i=1$ if and only if i is the right-most site of some plus or minus cluster associated with (x, y).

LEMMA 6. *Suppose* $\nu\{(x, y): \sigma_{-\infty}(x, y)=\sigma_\infty(x, y)=0\}=1$. *Then there exists an increasing sequence* $\{n_\ell\}_{\ell\in\mathbf{N}}$ *of positive integers satisfying* $\lim_{\ell\to\infty}\nu(C_{n_\ell})=0$, *where*

$$C_n = \{(x, y)\in\bar{\mathscr{X}}: c(x, y)_{-n}+c(x, y)_n+\tilde{c}(x, y)_{-n}+\tilde{c}(x, y)_n\geqq 1\}, \quad n\in\mathbf{N}.$$

PROOF. It is enough to show that for any $\varepsilon>0$ and $L\in\mathbf{N}$ there is $\ell\in\mathbf{N}$ such that $\ell>L$ and $\nu(C_\ell)<\varepsilon$. Assume the contrary, that is, for some $\varepsilon>0$ and $L\in\mathbf{N}$ it holds that $\nu(C_\ell)\geqq\varepsilon$ for all $\ell>L$. Then for

$$h_n(x, y) \equiv (2n+1)^{-1}\sum_{|i|\leqq n}\{c(x, y)_i+\tilde{c}(x, y)_i\}$$

we have

$$\liminf_{n\to\infty}\int_{\bar{\mathscr{X}}} h_n(x, y)d\nu(x, y) \geqq \varepsilon/2.$$

On the other hand, the assumption of the lemma implies

$$\nu\{(x, y): \lim_{n\to\infty}(2n+1)^{-1}\#\{i: c(x, y)_i+\tilde{c}(x, y)_i\geqq 1, |i|\leqq n\}=0\} = 1,$$

and hence

$$\lim_{n\to\infty}\int_{\bar{\mathfrak{X}}} h_n(x, y)d\nu(x, y) = 0$$

by the dominated covergence theorem. This is a contradiction. □

LEMMA 7. *For any* $\nu \in \bar{\mathscr{I}}$

$$\nu\{(x, y): s_\infty(x, y) \equiv \lim_{n\to\infty} s_{-n,n}(x, y) \leqq 2\} = 1.$$

PROOF. For $\nu \in \bar{\mathscr{I}} \cap \bar{\mathscr{T}}$ we proved $\nu\{\bar{\sigma}(x, y)=0\}=1$ in the proof of Lemma 5 using Lemma 4. In the same way, for $\nu \in \bar{\mathscr{I}}$ we can prove $\nu\{\sigma_\infty(x, y)=\sigma_{-\infty}(x, y)=0\}=1$ using Lemma 4′. Then by Lemma 6 there is an increasing sequence $\{n(\ell)\}_{\ell\in\mathbf{N}}$ of positive integers satisfying $\lim_{\ell\to\infty}\nu(C_{n(\ell)})=0$. Let us show $\nu(F_i)=0$ for all $i \in \mathbf{Z}$ where

$$F_i = \{(x, y): c(x, y)_i = c(x, y)_{i+1} = 1\}.$$

Assume the contrary, that is, $\nu(F_i)>0$ for some i. Since $\nu \in \bar{\mathscr{I}}$, ($\bar{\text{E}}$q) with $g(x, y)=s_{-n(\ell)+1,n(\ell)-1}(x, y)$ is written as

$$(16) \qquad 0 = \int_{\bar{\mathfrak{X}}} d\nu(x, y)\,\{g(x, y) - \int_{\bar{\mathfrak{X}}} P((x, y), d(x', y'))g(x', y')\}$$
$$= \int_{C_{n(\ell)}} + \int_{C^c_{n(\ell)}}.$$

By the definition of $P((x, y), \cdot)$ the integrand of the second term in the r.h.s. is nonnegative for all $(x, y) \in C^c_{n(\ell)}$. Hence for ℓ with $n(\ell)>|i|+2$

$$(17) \qquad \int_{C^c_{n(\ell)}} \geqq \int_{C^c_{n(\ell)}\cap F_i} \geqq \alpha(1-\alpha)\nu(C^c_{n(\ell)} \cap F_i).$$

The last inequality is obtained by considering that if the particle at $i+1$ (of x or y) jumps to i and the one at i (of y or x resp.) does not, then at least one cluster disappears. More precisely, if $(x, y) \in C^c_{n(\ell)} \cap F_i$ then

$$P((x, y), E_i^{(x,y)}) \geqq \alpha(1-\alpha),$$
$$g(x, y) - g(x', y') \geqq 1 \qquad \text{for} \quad (x', y') \in E_i^{(x,y)},$$

where

$$E_i^{(x,y)} = \{(x', y'): (x, y) \rhd (x', y'),\ x'_i = y'_i = 1\}.$$

Therefore if we let $\ell\to\infty$ in (16), noticing that the first term in the r.h.s. is not smaller than $-2\nu(C_{n(\ell)})$, we have a contradiction. Thus $\nu(F_i)>0$ can not happen. It is not so hard from $\nu\{s_\infty(x, y)\geqq 3\}>0$ to derive $\nu(F_i)>0$ for some $i \in \mathbf{Z}$ (see the proof of Lemma 1). Hence $\nu\{s_\infty(x, y)\leqq 2\}=1$. □

PROOF OF THEOREM 4. By virtue of Lemma 7 the argument given in the proof of Theorem 1.4 of [7] is also applicable to our case. It is enough to show that $\mathrm{ext}\,\mathscr{I}\subset\{\pi_\gamma\colon 0\leqq\gamma\leqq\infty\}\cup\{\Theta_n\colon n\in\mathbf{Z}\}$. Take any $\mu_1\in\mathrm{ext}\,\mathscr{I}$ and put $\mu_2(\cdot)=\mu_1(\cdot+1)$ (a translation of μ_1). It is clear that

$$(18)\qquad |\textstyle\prod_{i=m}^{n-1}[\mu_1\{x_ix_{i+1}=01\}-\mu_2\{x_ix_{i+1}=01\}]|\leqq 1;$$

which corresponds to the assumption of Corollary 5.3 of [7]. By Proposition 2 there is $\nu\in\mathrm{ext}\,\bar{\mathscr{I}}$ with marginals μ_1 and μ_2.

Let us show

$$(19)\qquad \nu\{x\leqq y\}=1 \quad\text{or}\quad \nu\{x\geqq y\}=1.$$

($\bar{\mathrm{E}}$q) for the number of coupled sites $f_n(x, y)=\sum_{|i|\leqq n}\{1-|x_i-y_i|\}$ is written as

$$(20)\qquad 0=\textstyle\sum_{(1)}\nu([z]_{n+1})\sum_{(2)}P([z]_{n+1},[\tilde{z}]_n)\{f_n([\tilde{z}]_n)-f_n([z]_{n+1})\},$$

where the summations $\sum_{(1)}$ and $\sum_{(2)}$ are taken over all configurations $[z]_{n+1}\equiv[(a_i, b_i)_{|i|\leqq n+1}]$ and $[\tilde{z}]_n\equiv[(\tilde{a}_i, \tilde{b}_i)_{|i|\leqq n}]$ respectively. The variation $f_n([\tilde{z}]_n)-f_n([z]_{n+1})$ of coupled sites is divided into two parts; the increment $f_n^{in}([z]_{n+1},[\tilde{z}]_n)$ caused by the movement of particles staying in the interval $[-n,\dots, n]$ and the variation f_n^{bd} caused by that of particles crossing the boundary ($-n-0$ and $n+0$). Then (20) becomes

$$\begin{aligned}(21)\quad 0=&\textstyle\sum_{(1)}\nu([z]_{n+1})\sum_{(2)}P([z]_{n+1},[\tilde{z}]_n)f_n^{in}([z]_{n+1},[\tilde{z}]_n)\\ &+\alpha[\nu\{x_{-n-1}=y_{-n-1}=0,\ x_{-n}\neq y_{-n}\}+\nu\{x_{-n-1}=y_{-n}\neq y_{-n-1}=x_{-n}\}\\ &\quad+\nu\{x_n\neq y_n,\ x_{n+1}=y_{n+1}=1\}\\ &\quad+\nu\{x_{n-1}=x_n=y_{n+1}=1\neq y_n=x_{n+1}\\ &\qquad\qquad\text{or}\quad y_{n-1}=y_n=x_{n+1}=1\neq x_n=y_{n+1}\}\\ &\quad+(1-\alpha)\nu\{x_{n-1}=y_n=x_{n+1}=0\neq x_n=y_{n+1}\\ &\qquad\qquad\text{or}\quad y_{n-1}=x_n=y_{n+1}=0\neq y_n=x_{n+1}\}\\ &\quad-\nu\{x_{-n-1}\neq y_{-n-1},\ x_{-n}=y_{-n}=1\}-\nu\{x_n=y_n=0,\ x_{n+1}\neq y_{n+1}\}\\ &\quad-\alpha\nu\{x_{n-1}=y_n=x_{n+1}=0\neq x_n=y_{n+1}\\ &\qquad\qquad\text{or}\quad y_{n-1}=x_n=y_{n+1}=0\neq y_n=x_{n+1}\}]\end{aligned}$$

for $n>1$. By the same reason as for (17) the first term in the r.h.s. of (21) is not smaller than $\alpha(1-\alpha)\nu\{x_i=y_{i+1}\neq y_i=x_{i+1}\}$ for $|i|+1\leqq n$. By (18) the Cesàro limit of the second term as $n\to\infty$ is zero (see the proof of Corollary 5.3 of [7]). Hence $\nu\{x_i=y_{i+1}\neq y_i=x_{i+1}\}=0$ for all $i\in\mathbf{Z}$. Assume (19) does not hold. Then Lemma 7 implies that $\nu(B)=1$ or $\nu\{(y, x)\colon (x, y)\in B\}=1$ where $B\equiv\{(x, y)\colon \exists i_0\in\mathbf{Z}$

such that $x_i \leqq y_i$ for all $i<i_0$, $x_i<y_i$ for infinitely many $i<i_0$, $x_j \geqq y_j$ for all $j \geqq i_0$, and $x_j>y_j$ for infinitely many $j \geqq i_0\}$. Since $\nu \in \bar{\mathscr{I}}$, $\nu\{x_i=y_{i+1} \neq y_i=x_{i+1}\}>0$ for some $i \in \mathbf{Z}$. This is a contradiction. Thus we get (19).

Now it is not so hard to follow the route laid by [7] if we notice that

$$\mu_1\{x_{i-1}x_i=01\} = \mu_1\{x_i x_{i+1}=01\}, \quad i \in \mathbf{Z},$$

which follows from (Eq) with $f(x) \equiv x_i$. □

§5. Stochastic properties of the drift of particles

In the previous sections we have been concerned with the structure of stationary measures for the Markov process (MP). In this last section we consider some statistical properties of a particle under the time evolution in the stationary state.

Suppose the configuration at $t=0$ is $x \equiv (\cdots x_{-2}x_{-1}1x_1\cdots) \in \mathscr{X}$ and evolves according to the transition probabilities $P(x, A)$, $x \in \mathscr{X}$, $A \in \mathscr{B}$. Then the particle which was located at the origin drifts to the left. Problems are

i) what the expected value m_t of the drift is

and

ii) what the variance σ_t^2 from the expected value is.

We will consider these problems under the assumption that the distribution μ of the configuration x at $t=0$ is π_γ ($\gamma \in \mathbf{R}$). Recall that Theorem 4 states that if $0<\alpha \leqq 1/2$ then π_γ is an extreme point of the set of stationary measures for (MP).

Let $\mathscr{U}$ be the path space $\mathscr{X}^{\mathbf{T}}$ with the Borel structure $\mathscr{F}$ generated by cylinder sets $\{u \in \mathscr{U}: u(s) \in A_s, s=0,\dots, t\}$, $A_s \in \mathscr{B}$, $t \in \mathbf{T}$. Fix $0<\alpha<1$. For $0<\gamma<\infty$ define a probability measure ξ_γ on $(\mathscr{U}, \mathscr{F})$ by

$$\xi_\gamma(F) = \int_{\mathscr{X}} \pi_\gamma(dx) P_x(F), \quad F \in \mathscr{F},$$

where $P_x(\cdot)$, $x \in \mathscr{X}$, is a measure on $\mathscr{U}$ defined similarly to $P_{(x,y)}(\cdot)$ in §4. Set

$$\mathscr{U}^* = \{u \in \mathscr{U}: u(0)=(\cdots x_{-1}1x_1\cdots)\}$$

and denote by ξ_γ^* the conditional probability measure of ξ_γ with respect to $\mathscr{U}^*$, that is,

$$\xi_\gamma^*(\cdot) = \xi_\gamma(\cdot \cap \mathscr{U}^*)/\xi_\gamma(\mathscr{U}^*) = (1+\gamma)\xi_\gamma(\cdot \cap \mathscr{U}^*). \tag{22}$$

For $u \in \mathscr{U}^*$ let $r_0(0)=0$ and

$$r_n(0) = \max\{i<0: \textstyle\sum_{\ell=i}^{-1} u_\ell(0)=n\}, \quad n \in \mathbf{N},$$

which represents the site where the n-th particle from the origin is seen to the left at $t=0$. For each $n\in\mathbf{N}$ and $t\in\mathbf{T}$ we denote by $r_n(t)$ the random variable on $(\mathscr{U}^*, \xi_\gamma^*)$ which represents the position of the particle at time t which started from $r_n(0)$. Let β be the positive number satisfying (1). Then we obtain

THEOREM 5. (i)

$$\xi_\gamma^*\{u\in\mathscr{U}^*: r_0(s-1)-r_0(s)=e_s,\ s=1,\dots,t\}$$
$$= \prod_{s=1}^t(\beta\gamma)^{e_s}(1-\beta\gamma)^{1-e_s}, \quad (e_1,\dots,e_t)\in\{0,1\}^t, \quad t\in\mathbf{N},$$

and hence

(ii) $$m_t \equiv \int_{\mathscr{U}^*} r_0(t)\xi_\gamma^*(du) = \beta\gamma t,$$

(iii) $$\sigma_t^2 \equiv \int_{\mathscr{U}^*}(r_0(t)-m_t)^2\xi_\gamma^*(du) = \beta\gamma(1-\beta\gamma)t,$$

(iv) (*central limit theorem*)

$$\{r_0(t)-m_t\}/\sigma_t \xrightarrow[d]{} N(0,1) \quad \text{as} \quad t\to\infty,$$

where $N(0, 1)$ *is the normal distribution with mean* 0 *and variance* 1.

The theorem is an immediate consequence of the following lemma:

LEMMA 8. *For all* $t\in\mathbf{T}$, $t\geqq 1$,

$$\xi_\gamma^*\left\{u\in\mathscr{U}^*\ \middle|\ \begin{array}{l} r_0(s-1)-r_0(s)=e_s,\ s=1,\dots,t, \\ r_{\ell-1}(t)-r_\ell(t)=z_\ell,\ \ell=1,\dots,k \end{array}\right\}$$
$$= \{\prod_{s=1}^t(\beta\gamma)^{e_s}(1-\beta\gamma)^{1-e_s}\}\cdot\{\prod_{\ell=1}^k f(z_\ell)\},$$
$$(z_1,\dots,z_k)\in\mathbf{N}^k,\ k\in\mathbf{N},\ (e_1,\dots,e_t)\in\{0,1\}^t,$$

where $f(\cdot)$ *is the p.d.f. defined in Theorem* 2.

PROOF. We first show

$$\xi_\gamma^*\{u\in\mathscr{U}^*: r_0(0)-r_0(1)=e_1,\ r_{\ell-1}(1)-r_\ell(1)=z_\ell,\ \ell=1,\dots,k\}$$
$$= (\beta\gamma)^{e_1}(1-\beta\gamma)^{1-e_1}\prod_{\ell=1}^k f(z_\ell) \tag{23}$$

for every $e_1\in\{0,1\}$ and $(z_1,\dots,z_k)\in\mathbf{N}^k$, $k\in\mathbf{N}$, which is the assertion of the lemma for $t=1$. Suppose $z=z_1+\cdots+z_k$ and

$$\left\{x\in\mathscr{X}\ \middle|\ \begin{array}{l} x_0=1;\ x_i=1 \text{ for } i \text{ with } i=-\sum_{j=1}^\ell z_j,\ \ell=1,\dots,k; \\ \text{and } x_i=0 \text{ for the other } i \text{ with } -z<i<0 \end{array}\right\}$$
$$= {}_{-z}[1a_{-z+1}\cdots a_{-2}a_{-1}1]_0 \equiv A.$$

Note that $A+1={}_{-z-1}[1a_{-z+1}\cdots a_{-2}a_{-1}1]_{-1}$. Then by (22)

$$\begin{aligned}
&\xi_\gamma^*\{u\in\mathscr{U}^*: r_{\ell-1}(1)-r_\ell(1)=z_\ell,\ \ell=1,\dots,k\}\cdot(1+\gamma)^{-1}\\
&\quad=\xi_\gamma\{u\in\mathscr{U}: u_0(0)=u_0(1)=1,\ u(1)\in A\}\\
&\qquad+\xi_\gamma\{u\in\mathscr{U}: u_0(0)=1,\ u_0(1)=0,\ u(1)\in A+1\}\\
&\quad=\sum_{{}_{-z-1}[b_{-z-1}b_{-z}\cdots b_{-1}1b_1]_1\rhd A}\alpha^m(1-\alpha)^n\pi_\gamma[b_{-z-1}\cdots 1b_1]\\
&\qquad+\sum_{{}_{-z-1}[b_{-z-1}b_{-z}\cdots b_{-1}01]_1\rhd A}\alpha^m(1-\alpha)^n\pi_\gamma[b_{-z-1}\cdots b_{-1}01]\\
&\quad\equiv S_1+S_2\\
&\quad=\pi_\gamma(A)\qquad\text{(by (Eq)')}\\
&\quad=f_0(1)\cdot\textstyle\prod_{\ell=1}^k f(z_\ell)\quad\text{(by (2))}.
\end{aligned}$$

Therefore

$$\xi_\gamma^*\{u\in\mathscr{U}^*: r_{\ell-1}(1)-r_\ell(1)=z_\ell,\ \ell=1,\dots,k\}=\textstyle\prod_{\ell=1}^k f(z_\ell). \tag{24}$$

Further it holds that

$$S_1/S_2=(1-\beta\gamma)/\beta\gamma\qquad\text{for all}\quad(z_1,\dots,z_k). \tag{25}$$

In fact if $(\#_{01}+\#_{10})(A)>0$, by the same simplification as in the proof of Theorem 1 (the case that $(a_i, a_j)=(1, 1)$)

$$\begin{aligned}
S_1&=(1-\alpha)^{-\#_{01}(A)+1}N(-z-1,0;k+1),\\
S_2&=(1-\alpha)^{-\#_{01}(A)+1}(\alpha/(1-\alpha))N(-z-1,1;k+1);
\end{aligned}$$

and hence $S_1/S_2=(1-\beta\gamma)/\beta\gamma$. If $(\#_{01}+\#_{10})(A)=0$, that is, if

$A={}_{-z}[11\cdots 11]_0$, then

$$\begin{aligned}
S_1&=(1-\alpha)\pi_\gamma({}_{-z-1}[011\cdots 11]_0)+\pi_\gamma({}_{-z-1}[111\cdots 11]_0),\\
S_2&=(1-\alpha)\alpha\pi_\gamma({}_{-z-1}[011\cdots 101]_1)+\alpha\pi_\gamma({}_{-z-1}[111\cdots 101]_1);
\end{aligned}$$

and so

$$\begin{aligned}
S_1/S_2&=\{(1-\alpha)N(-z-1,0;k+1)+L(-z-1,0;k+2)\}\\
&\quad\times\{\alpha N(-z-1,1;k+1)+\alpha L(-z-1,1;k+2)\}^{-1}\\
&=(1-\beta\gamma)/\beta\gamma.
\end{aligned}$$

Then combining (25) with (24) we get (23). Notice that (23) implies

$$\begin{aligned}
&\xi_\gamma^*\{r_0(0)-r_0(1)=e_1,\ u(1)\in(E+e_1)\}\\
&\qquad=(\beta\gamma)^{e_1}(1-\beta\gamma)^{1-e_1}\xi_\gamma^*\{r_0(0)=0,\ u(0)\in E\}
\end{aligned} \tag{26}$$

for all $E \in \mathscr{B}_{-\infty,0} \equiv \sigma(\cup_{i \leqq j \leqq 0} \mathscr{C}_{i,j})$.

In order to prove the lemma for $t=2$, we set

$$\tilde{\mathscr{F}} = \sigma\{r_n(t),\ n=0, 1,\dots;\ t \in \mathbf{T}\}\ (\subset \mathscr{F})\ ;$$

$$F + e_1 = \{u \in \mathscr{U}\colon \exists u' \in F \text{ s.t. } u_i(t) = u'_{i+e_1}(t) \text{ for all } i \text{ and } t\}, \quad F \in \mathscr{F},$$

(the translation of the set F to the left by e_1) ;

$$\tau G = \{u \in \mathscr{U}\colon \exists u' \in G \text{ s.t. } u(t+1) = u'(t) \text{ for all } t \in \mathbf{T}\}, \quad G \in \mathscr{F};$$

and define

$$\xi^*_{\gamma,e_1}(F) = \xi^*_\gamma(\{r_0(0) - r_0(1) = e_1\} \cap \tau(F + e_1)) / \xi^*_\gamma\{r_0(0) - r_0(1) = e_1\}.$$

Then

$$\xi^*_{\gamma,e_1}\{u(0) \in E\} = \xi^*_\gamma\{u(0) \in E\} \qquad \text{for all} \quad E \in \mathscr{B}_{-\infty,0}$$

by (26), and hence $\xi^*_{\gamma,e_1}(F) = \xi^*_\gamma(F)$, that is,

$$\xi^*_\gamma(\{r_0(0) - r_0(1) = e_1\} \cap \tau(F + e_1)) = (\beta\gamma)^{e_1}(1-\beta\gamma)^{1-e_1}\xi^*_\gamma(\{r_0(0) = 0\} \cap F)$$

for all $F \in \tilde{\mathscr{F}}$. If $F = \{r_0(0) - r_0(1) = e_2,\ r_{\ell-1}(1) - r_\ell(1) = z_\ell,\ 1 \leqq \ell \leqq k\}$ in the above, we have

$$\begin{aligned} &\xi^*_\gamma(\{r_0(0) - r_0(1) = e_1\} \cap \{r_0(1) - r_0(2) = e_2,\ r_{\ell-1}(2) - r_\ell(2) = z_\ell,\ 1 \leqq \ell \leqq k\}) \\ &\quad = \{\textstyle\prod_{s=1}^2 (\beta\gamma)^{e_s}(1-\beta\gamma)^{1-e_s}\} \cdot \{\prod_{\ell=1}^k f(z_\ell)\} \end{aligned}$$

by (23), which is the assertion of the lemma for $t=2$. In the same manner, by defining $\xi^*_{\gamma,e_1e_2\cdots e_{t-1}}(\cdot)$, we can prove the lemma for all $t \in \mathbf{T}$ inductively. □

REMARK 4. By (i) of Theorem 5 it is known that under ξ^*_γ the particle located at the origin at $t=0$ acts as if it is a random walker on $\mathbf{Z}$ which moves to the left with probability $\beta\gamma$ and stays with probability $1-\beta\gamma$. By Lemma 8

$$\begin{aligned} &\xi^*_\gamma\{u \in \mathscr{U}^*\colon r_0(t) - r_1(t) \geqq 2,\ r_0(s-1) - r_0(s) = e_s,\ s = 1,\dots, t\} \\ &\quad = (1 - f(1)) \textstyle\prod_{s=1}^t (\beta\gamma)^{e_s}(1-\beta\gamma)^{1-e_s}, \end{aligned}$$

which implies that the conditional probability that there exists no particle at the left-neighboring site of $r_0(t)$ given $r_0(s-1) - r_0(s) = e_s$, $s = 1,\dots, t$, is $1 - f(1)$ for all $(e_1,\dots, e_t) \in \{0, 1\}^t$, $t \in \mathbf{N}$. Since $\beta\gamma = \alpha(1 - f(1))$ by (1), we can understand that the transition rate $\beta\gamma$ is determined by two elementary probabilities: the probability $1 - f(1)$ that the site $r_0(t) - 1$ is unoccupied and the probability α that a particle at $r_0(t)$ jumps to the left when it is unoccupied.

REMARK 5. If we consider $(r_0(t) - r_1(t),\ r_1(t) - r_2(t),\dots)$, $t \in \mathbf{T}$, as a time evolution on the state space $\mathbf{N}^{\mathbf{N}}$, then Lemma 8 implies that the product measure

$\prod_{\ell=1}^{\infty} f_{\ell}$ $(f_{\ell}=f,\ \ell \in \mathbf{N})$ is a stationary measure for the process. (This is a special case of the so called zero range process.)

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