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On self H-equivalences of homotopy associative H-spaces

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§ 1. Introduction

For an H-space X, we call a homotopy equivalence $f: X \to X$ a self H-equivalence with respect to a multiplication m on X if f is a self H-map of (X, m), i.e., $fm \sim m(f \times f)$ (homotopic); and we mean by $[f] \in \mathrm{HE}(X)$ that f is such one with respect to any multiplication on X.

In this note, we prove the following theorem similar to Theorem 4.1 of [10] for $U(n)$, $SU(n)$ and $Sp(n)$:

THEOREM 1.1. *Let G be the exceptional Lie group G_2, F_4, E_6, E_7 or E_8. Then, any $f: G \to G$ with $[f] \in \mathrm{HE}(G)$ induces the identity isomorphisms on $H^*(G; \boldsymbol{Z})/\mathrm{Tor}$, $H^*(G; \boldsymbol{Z}_p)$ and $H^*(G; \boldsymbol{Z}_{(p)})$ for any prime $p>1$, 3, 3, 3 or 5, respectively; and $f_{(p)} \sim \mathrm{id}: G_{(p)} \to G_{(p)}$ if $p>1$, 24, 24, 36 or 60, respectively, for the localization $-_{(p)}$ at a prime p.*

To show this, we assume that

(1.2) an H-space X has the homotopy type of a 1-connected finite CW-complex and a homotopy associative multiplication $m: X \times X \to X$, and the type of X is $N = (n_1, \ldots, n_l)$ for odd integers n_i with $3 \leqq n_1 \leqq \cdots \leqq n_l$.

Then, we recall the following

(1.3) *$H^*(X; \boldsymbol{Z})/\mathrm{Tor} = \Lambda(x_1, \ldots, x_l)$ by primitive elements x_i with respect to m of* $\deg x_i = n_i$.

For the p-localization $-_{(p)}$, consider the natural homotopy equivalence $u = ((\mathrm{pr}_j)_{(p)}): (\prod_j Y_j)_{(p)} \simeq \prod_j (Y_j)_{(p)}$ and its homotopy inverse u^{-1}. Then:

(1.4) *$X_{(p)}$ is a homotopy associative H-space by $m_{(p)} u^{-1}: X_{(p)} \times X_{(p)} \simeq (X \times X)_{(p)} \to X_{(p)}$.*

Also, for the n_i-sphere S^{n_i} (n_i: odd) and $p \geqq 5$, the following is due to Adams [1]:

(1.5) *$S_i = S^{n_i}_{(p)}$ is an H-space with a homotopy associative and homotopy commutative multiplication m_i; hence so is $S_N = \prod_{i=1}^{l} S_i$ with $(\prod_i m_i)T: S_N \times S_N \approx \prod_i (S_i \times S_i) \to S_N$.*

Now, we have the following results:

THEOREM 1.6. *For any prime $p>n_l+1$, there is a p-equivalence e: $S^N=\prod_{i=1}^{l} S^{n_i}\to X$ such that the homotopy equivalence $\bar{e}=e_{(p)}u^{-1}$: $S_N=\prod_i S_i\to S_{(p)}^N\to X_{(p)}$ is an H-map with respect to $(\prod_i m_i)T$ and $m_{(p)}u^{-1}$ in (1.4–5). Especially $m_{(p)}u^{-1}$ is also homotopy commutative.*

We note that the latter half was proved by McGibbon [8] when X is a loop space and m is the loop multiplication.

THEOREM 1.7. *Assume $n_i<n_{i+1}$ for $i<l$, and let f: $X\to X$ be a self H-map of (X, m). Then:*

(i) *There are integers η_i $(1\leqq i\leqq l)$ with $f^*x_i=\eta_i x_i$ in $H^*(X; \boldsymbol{Z})$/Tor (see (1.3)).*

(ii) *$f_{(p)}\sim\bar{e}(\prod_i \eta_i)\bar{e}^{-1}$: $X_{(p)}\to X_{(p)}$ for any prime $p>n_l+1$ by $\bar{e}$ in Theorem 1.6 and the product map $\prod_i \eta_i$: $S_N=\prod_i S_i\to S_N$ of η_i: $S_i\to S_i$ of degree η_i.*

(iii) *If $[f]\in \mathrm{HE}(X)$, then the integers η_i $(1\leqq i\leqq l)$ in (i) satisfy $\eta_i=\pm 1$ and $\eta_k=\prod_i \eta_i^{\varepsilon_i}$ for any k and $\varepsilon_i\in\{0, 1, 2\}$ such that the p-component of $\pi_n(S^{n_k})$ $(n=\sum_i \varepsilon_i n_i)$ is non-trivial for some $p>n_l+1$.*

We prove Theorem 1.1 by showing $\eta_i=1$ for $[f]\in \mathrm{HE}(G)$ from the equalities in Theorem 1.7 (iii) when $n=n_k+2p-3$ (see §4). We prove Theorem 1.6 by using the result due to Kumpel [7] and Harper [5] that X is p-equivalent to S^N for $p>n_l/2$ (see §2), and Theorem 1.7 by using Theorem 1.6 in a way similar to [10] (see §3).

§2. Proof of Theorem 1.6

For X in (1.2), the following is due to Browder [3] and [4], Kumpel [7] and Harper [5]:

(2.1) *If $p>n_l/2$, then $H^*(X; \boldsymbol{Z})$ is p-torsion free, and there is a p-equivalence e: $S^N=\prod_{i=1}^{l} S^{n_i}\to X$, and so $e_{(p)}$: $S_{(p)}^N\to X_{(p)}$ is a homotopy equivalence.*

LEMMA 2.2. *If $p>n_l+1$, then e in (2.1) can be so taken that the homotopy equivalence $\bar{e}=e_{(p)}u^{-1}$: $S_N=\prod_i S_i\simeq S_{(p)}^N\to X_{(p)}$ is an H-map with respect to $M=(\prod_i m_i)T$ and $\bar{m}=m_{(p)}u^{-1}$ in (1.4–5).*

PROOF. Take a p-equivalence h: $S^N\to X$ by (2.1), and consider

$$h_i = h\,\mathrm{in}_i\colon S^{n_i}\subset S^N\to X,\quad \bar{h}_i=(h_i)_{(p)}\colon S_i\to X_{(p)}\quad\text{and}$$
$$e = m(\textstyle\prod_i h_i)\colon S^N\to X^l\to X,$$

where m denotes also the iterated multiplication of m. Then, we see that

$$e_{(p)}\sim m_{(p)}(\textstyle\prod_i h_i)_{(p)}\sim \bar{m}(\prod_i \bar{h}_i)u = h'\colon S_{(p)}^N\to X_{(p)},$$
$$h'_* = h_{(p)*}\colon \pi_*(S_{(p)}^N)\to\pi_*(X_{(p)}),$$

and $e_{(p)}$ is a homotopy equivalence since so is $h_{(p)}$. Furthermore, $\pi_*(X_{(p)})\cong \pi_*(S_N)=0$ for $*=n_i+n_j$ by Serre [11], since $p>n_l+1$. Hence, we see that

$$\bar{m}(\bar{h}_i\times\bar{h}_j)\sim\bar{m}(\bar{h}_j\times\bar{h}_i)T\colon S_i\times S_j\to X_{(p)},\quad \bar{m}(\bar{h}_i\times\bar{h}_j)\sim\bar{h}_i m_i \text{ when } j=i,$$

and so $\bar{m}(\bar{e}\times\bar{e})\sim\bar{e}M$ as desired, since $\bar{e}\sim\bar{m}(\prod_i\bar{h}_i)$ and $\bar{m}$ is homotopy associative.

Thus, Theorem 1.6 follows from Lemma 2.2 and (1.5).

REMARK. We note that $e\colon S^N\to X$ in Lemma 2.2 can be taken to be independent of $p>n_l+1$. In fact, take representatives $e_i\colon S^{n_i}\to X$ $(1\leqq i\leqq l)$ of generators of the free part of $\sum_{i\leqq n_l}\pi_i(X)$, and consider $e=m(\prod_i e_i)\colon S^N\to X^l\to X$. Then, we see that $e_*\colon \pi_i(S^N_{(p)})\cong\pi_i(X_{(p)})$ for $i\leqq n_l+1$ and so $e^*\colon H^i(X;\,\boldsymbol{Z}_p)\cong H^i(S^N;\,\boldsymbol{Z}_p)$ for all i. Hence e is a p-equivalence. Also, by the same way as the above proof we see that $e_{(p)}u^{-1}$ is an H-map with respect to M and $\bar{m}$.

§ 3. Proof of Theorem 1.7

Assume that $n_i<n_{i+1}$ $(i<l)$ and $f\colon (X, m)\to(X, m)$ is an H-map. Then, it is clear that

(3.1) $f^*x_i=\eta_i x_i$ *in* $H^*(X;\,\boldsymbol{Z})/\mathrm{Tor}$ *of* (1.3) *for some* $\eta_i\in\boldsymbol{Z}$ $(1\leqq i\leqq l)$.

(3.2) $\bar{f}=\bar{e}^{-1}f_{(p)}\bar{e}\colon S_N\simeq X_{(p)}\to X_{(p)}\simeq S_N$ *for* $p>n_l+1$ *and* $\bar{e}$ *in Theorem* 1.6 *is a self H-map of* $(S_N=\prod_i S_i,\ M=(\prod_i m_i)T)$, *since so is* $f_{(p)}$ *of* $(X_{(p)},\ \bar{m}=m_{(p)}u^{-1})$.

Now, by the localization map $J\colon X\to X_{(p)}$, consider

$$H^*(X;\,\boldsymbol{Z})/\mathrm{Tor}\xrightarrow{\ j\ }H^*(X;\,\boldsymbol{Z}_{(p)})\underset{\cong}{\xleftarrow{\ J^*\ }}H^*(X_{(p)};\,\boldsymbol{Z}_{(p)})$$

$$\underset{\cong}{\xrightarrow{\ \bar{e}^*\ }}H^*(S_N;\,\boldsymbol{Z}_{(p)})=\Lambda_{\boldsymbol{Z}_{(p)}}(s_1,\dots,s_l),$$

where j is the natural monomorphism by the first half of (2.1) and s_i's are primitive generators with respect to M of $\deg s_i=n_i$ corresponding to $S_i=S^{n_i}_{(p)}$. Then:

(3.3) $H^*(S_N;\,\boldsymbol{Z}_{(p)})=\Lambda_{\boldsymbol{Z}_{(p)}}(y_1,\dots,y_l)$ *by primitive elements* $y_i=\bar{e}^*J^{*-1}jx_i$ *with respect to* M *of* $\deg y_i=n_i$; *hence* $y_i=a_is_i$ *for some units* $a_i\in\boldsymbol{Z}_{(p)}$,

by (1.3), Theorem 1.6 and the assumption $n_i<n_{i+1}$ $(i<l)$. Furthermore, (3.1) and (3.3) yield that

(3.4) $\bar{f}^*s_i=\eta_is_i$, *because* $\bar{f}^*y_i=\bar{e}^*J^{*-1}jf^*x_i=\eta_iy_i$.

According to (3.2), (2.3.4) of [10] and $[S_i, S_j]=\pi_{n_i}(S^{n_j})\otimes\boldsymbol{Z}_{(p)}=0$ for $i\neq j$, (3.4) implies that

(3.5) $\bar{f} \sim \prod_i \eta_i : S_N = \prod_i S_i \to \prod_i S_i = S_N$ *by considering*

$$\eta_i \in \boldsymbol{Z} \subset \boldsymbol{Z}_{(p)} = [S_i, S_i].$$

Now, consider any $0 \leqq \varepsilon_i \leqq 2$ $(1 \leqq i \leqq l)$ and $1 \leqq k \leqq l$ such that

(3.6) there is an element $\alpha \in \pi_n(S^{n_k})$ of order p for $n = \sum_i \varepsilon_i n_i$, and so $\sum_i \varepsilon_i > 2$.

Take $\delta = (\delta_1, \ldots, \delta_{2l}) \in \{0, 1\}^{2l}$ with $\varepsilon_i = \delta_i + \delta_{l+i}$ and $\sum_i \delta_i \neq 0 \neq \sum_i \delta_{l+i}$, and

(3.7) the multiplication $M(\alpha) = M + \mathrm{in}_k\, \alpha_{(p)} \pi_\delta : S_N \times S_N \to S_N$ ($+$ is induced by M),

where $\pi_\delta : S_N \times S_N \to \wedge_{\delta_j = 1} S_j \simeq S^n_{(p)}$ $(S_{l+i} = S_i)$ is the composition of the collapsing map and the homotopy equivalence, and $\mathrm{in}_k : S_k \subset S_N$. Then:

(3.8) *There is a multiplication* $m(\alpha): X \times X \to X$ *such that if* $f: X \to X$ *is a self H-map with respect to* $m(\alpha)$, *then so is* $\bar{f} = \bar{e}^{-1} f_{(p)} \bar{e} : S_N \to S_N$ *with respect to* $M(\alpha)$.

In fact, let $-_{\bar{p}}$ be the localization at the set $\bar{p}$ of all primes $\neq p$, and consider

$$m' = \bar{e} M(\alpha)(\bar{e} \times \bar{e})^{-1} u : (X \times X)_{(p)} \simeq X_{(p)} \times X_{(p)} \to X_{(p)} \quad \text{and}$$

$$m_{\bar{p}} : (X \times X)_{\bar{p}} \to X_{\bar{p}}.$$

Then, we see that their rationalizations coincide with each other, because α is of finite order and $\bar{e} M(\bar{e} \times \bar{e})^{-1} u \sim \overline{m} u \sim m_{(p)}$ by Theorem 1.6. Hence, we have a multiplication $m(\alpha)$ on X with $m(\alpha)_{(p)} \sim m'$ and $m(\alpha)_{\bar{p}} \sim m_{\bar{p}}$ by Corollary 5.13 of Hilton-Mislin-Roitberg [6], and (3.8) holds.

(3.9) *If* $f: X \to X$ *is a self H-equivalence with respect to* m *and also to* $m(\alpha)$ *in* (3.8), *then the integers* η_i*'s given in* (3.1) *satisfy*

$$\eta_i = \pm 1 \ (1 \leqq i \leqq l) \ \textit{and} \ \prod_i \eta_i^{\varepsilon_i} = \eta_k \ \textit{for} \ 0 \leqq \varepsilon_i \leqq 2 \ \textit{and} \ 1 \leqq k \leqq l \ \textit{with} \ (3.6).$$

In fact, $\eta_i = \pm 1$ since f^* is isomorphic. Furthermore, in the same way as (3.2.1) of [10] we see that $(\prod_i \eta_i^{\varepsilon_i}) \cdot \alpha = \eta_k \cdot \alpha$ in $\pi_n(S^{n_k}) \otimes \boldsymbol{Z}_{(p)}$ by (3.7–8) and (3.5). Hence the second equality holds, since α is of order p.

Thus, Theorem 1.7 is proved completely.

§4. Proof of Theorem 1.1

Theorem 1.1 for $G = G_2$ is trivial, because $f \sim \mathrm{id}: G_2 \to G_2$ for any self H-equivalence f with respect to the group multiplication on G_2 by Theorem II of [9].

For $G_l = F_4$ $(l = 4)$, E_l $(l = 6, 7$ or $8)$, we recall the following:

(4.1) *The type* $(n_1,\dots, n_l)$ *of* G_l *is* (3, 11, 15, 23), (3, 9, 11, 15, 17, 23), (3, 11, 15, 19, 23, 27, 35) *or* (3, 15, 23, 27, 35, 39, 47, 59), *respectively.*

Note that the p-component of $\pi_n(S^{n_k})$ is Z_p for $n=n_k+2p-3$ by Serre [11], and $H^*(G_l; Z)$ is p-torsion free if $p>3$, 3, 3 or 5, respectively, by Borel [2]. Then, Theorem 1.1 is proved for G_l by Theorem 1.7 and the following:

(4.2) *Assume that* $\eta_i=\pm 1$ $(1\leqq i\leqq l)$ *satisfy* $\eta_k=\prod_i \eta_i^{\varepsilon_i}$ *for any* $1\leqq k\leqq l$ *and* $0\leqq\varepsilon_i\leqq 2$ *such that* $\sum_i \varepsilon_i n_i=n_k+2p-3$ *for* $(n_1,\dots, n_l)$ *in* (4.1) *and a prime* $p>n_l+1$. *Then,* η_i*'s are all equal to* 1.

We see (4.2) quite arithmetically, because the assumptions contain the following equalities which imply $\eta_i=1$ as desired:

$$(l=4)\quad \eta_1=\eta_1^2\eta_2^2\eta_3^2\quad (p=29),\quad \eta_2=\eta_1^2\eta_3^2\eta_4^2\quad (p=37),$$
$$\eta_3=\eta_1^2\eta_2^2\eta_4^2,\quad \eta_4=\eta_1^2\eta_3^2\eta_4^2\quad (p=31).$$
$$(l=6)\quad \eta_1=\eta_1^2\eta_3^2\eta_4^2=\eta_1^2\eta_2^2\eta_3\eta_6=\eta_1\eta_2^2\eta_3^2\eta_4=\eta_1^2\eta_2\eta_3\eta_4\eta_5$$
$$=\eta_1\eta_2\eta_6^2\quad (p=29),\quad \eta_1=\eta_1\eta_2^2\eta_3\eta_4^2\quad (p=31).$$
$$(l=7)\quad \eta_1=\eta_1\eta_2\eta_3\eta_5\eta_7^2=\eta_1\eta_2\eta_5^2\eta_6\eta_7=\eta_1\eta_2^2\eta_6\eta_7^2=\eta_1\eta_3\eta_4\eta_5\eta_6\eta_7$$
$$=\eta_2^2\eta_3^2\eta_7^2=\eta_1\eta_3^2\eta_6^2\eta_7\quad (p=61),\quad \eta_1=\eta_1\eta_3\eta_5^2\eta_7^2\quad (p=67).$$
$$(l=8)\quad \eta_1=\eta_1^2\eta_3^2\eta_5^2=\eta_1^2\eta_2^2\eta_6\eta_7=\eta_1^2\eta_2\eta_4\eta_5\eta_6=\eta_1\eta_2^2\eta_4^2\eta_5=\eta_1^2\eta_2\eta_4^2\eta_7$$
$$=\eta_1^2\eta_2^2\eta_4\eta_8=\eta_1^2\eta_2\eta_3\eta_6^2\quad (p=61),\quad \eta_1=\eta_1^2\eta_3^2\eta_5\eta_7\quad (p=67).$$

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