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On strongly increasing entire solutions of even order semilinear elliptic equations

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Introduction

This paper is concerned with entire solutions of even order semilinear elliptic equations of the form

$$\text{(A)} \qquad \Delta^m u = f(|x|, u, \Delta u, \dots, \Delta^{m-1} u), \quad x \in \boldsymbol{R}^n,$$

where $m \geqq 1$, $n \geqq 2$, Δ denotes the n-dimensional Laplacian, $|x|$ is the Euclidean length of x, and f is a given nonnegative continuous function defined on $[0, \infty)^{m+1}$ or on $[0, \infty) \times \boldsymbol{R}^m$. By an entire solution of (A) we mean a function $u \in C^{2m}(\boldsymbol{R}^n)$ which satisfies (A) pointwise in $\boldsymbol{R}^n$. Important special cases of (A) are

$$\text{(B)} \qquad \Delta^m u = p(|x|) u^\gamma, \quad x \in \boldsymbol{R}^n,$$

$$\text{(C)} \qquad \Delta^m u = p(|x|) e^u, \quad x \in \boldsymbol{R}^n,$$

where $\gamma > 1$ and p: $[0, \infty) \to (0, \infty)$ is continuous.

The problem of existence and nonexistence of entire solutions of (A) in the case $m = 1$ has been the subject of intensive investigations in the past three decades, and numerous results have been obtained. Among a vast literature on the subject, we refer the reader to the recent papers [2–5, 10–12, 14–19, 21, 23, 24] which are concerned mainly with second order equations of the forms (B) and (C).

It seems to the author, however, that very little is known about entire solutions for the higher order case of (A) ($m \geqq 2$). As far as the author is aware, Walter [26, 27] and Walter and Rhee [28] were the only references in this area until the appearance of Kusano, Naito and Swanson [7–9] and Kusano and Swanson [13], in which a systematic study of the existence and asymptotic behavior of radial entire solutions of (A) with $m \geqq 2$ has been attempted. In particular it is shown in [9] that equation (A) ($m \geqq 2$) may have a variety of radial entire solutions with different types of asymptotic behavior as $|x| \to \infty$.

The main objective of this paper is to further the theory developed in [7–9] by establishing the existence of a new class of entire solutions for (A). More specifically, we give conditions under which (A) has a radial entire solution $u(x)$ with the asymptotic property

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}\log|x|} = \infty \qquad \text{for} \quad n = 2,$$

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}} = \infty \qquad \text{for} \quad n \geqq 3.$$

Such a solution is said to be a *strongly increasing* entire solution of (A). We note here that the problem under study is essentially an ODE initial value problem. In fact, a radial function $u=y(|x|)$ is an entire solution of (A) if and only if $y(t)$ is of class $C^{2m}[0, \infty)$ and satisfies the ordinary differential equation

(D) $$L^m y = f(t, y, Ly, \ldots, L^{m-1}y), \quad t > 0,$$

as well as the singular initial condition

(E) $$(L^i y)(0) = \alpha_i, \quad (L^i y)'(0) = 0, \quad 0 \leqq i \leqq m-1,$$

for some real constants α_i, where L denotes the polar form of the Laplacian Δ:

$$L = t^{1-n}\frac{d}{dt}t^{n-1}\frac{d}{dt}, \qquad t = |x|.$$

We are thus led to the analysis of the initial value problem (D)–(E).

In Section 1 we derive basic results concerning local solutions of the problem (D)–(E). In Section 2 we construct the desired strongly increasing entire solutions of (A) by using the results of Section 1 and known criteria [7, 8] for the existence of entire solutions $u(x)$ of (A) satisfying

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}\log|x|} = c(u) \in (0, \infty) \qquad \text{for} \quad n = 2,$$

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}} = c(u) \in (0, \infty) \qquad \text{for} \quad n \geqq 3.$$

Some strong nonlinearity hypotheses on f are needed for this purpose. A related question of interest is to characterize the values $c(u)$ in the above limits, that is, to determine the set of positive numbers which can be the limits as $|x|\to\infty$ of $u(x)/|x|^{2m-2}\log|x|$ $(n=2)$ or of $u(x)/|x|^{2m-2}$ $(n\geqq 3)$ for some radial entire solutions $u(x)$ of (A). A fairly complete answer to this question is also given in Section 2. Section 3 concerns the second order case of (A). It is shown that the results for the radial equation $\Delta u = f(|x|, u)$, $x \in \boldsymbol{R}^n$, can be extended, via the supersolution-subsolution method, to the non-radial equations of the form $\Delta u = g(x, u)$, $x \in \boldsymbol{R}^n$. The final section (Section 4) is devoted to the discussion of nonexistence of entire solutions of equation (A) and related differential inequalities.

1. Properties of local solutions

As was stated in the Introduction, the problem of finding a radial entire solution $u=y(|x|)$ of (A) is equivalent to the singular initial value problem

$$L^m y = f(t, y, Ly, \dots, L^{m-1}y), \quad t > 0, \tag{1.1}$$

$$(L^i y)(0) = \alpha_i, \quad (L^i y)'(0) = 0, \quad 0 \leqq i \leqq m-1, \tag{1.2}$$

where α_i are constants and L is the operator defined by

$$L = t^{1-n}\frac{d}{dt}t^{n-1}\frac{d}{dt}.$$

In this section we present some basic results concerning local solutions of the problem (1.1)–(1.2), which play a crucial role in the construction of strongly increasing entire solutions of (A). We begin with the existence of local solutions to the problem. In the next lemmas, the order relation $\boldsymbol{v} \leqq \boldsymbol{w}$ for vectors $\boldsymbol{v}=(v_i)$ and $\boldsymbol{w}=(w_i)$ is defined as $v_i \leqq w_i$ for all i. The symbols $<$, $\geqq$ and $>$ are also used analogously.

LEMMA 1.1. (i) *Let* $f\colon [0, \infty)^{m+1} \to [0, \infty)$ *be continuous. Then the problem* (1.1)–(1.2) *has a local solution for any given* $(\alpha_0, \alpha_1, \dots, \alpha_{m-1}) \in [0, \infty)^m$.

(ii) *Let* $f\colon [0, \infty) \times \boldsymbol{R}^m \to [0, \infty)$ *be continuous. Then the problem* (1.1)–(1.2) *has a local solution for any given* $(\alpha_0, \alpha_1, \dots, \alpha_{m-1}) \in \boldsymbol{R}^m$.

PROOF. We prove only the statement (i); the proof of (ii) is similar. We transform the problem (1.1)–(1.2) into the system of m second order equations

$$L\boldsymbol{y} = \boldsymbol{f}(t, \boldsymbol{y}), \quad t > 0, \tag{1.3}$$

$$\boldsymbol{y}(0) = \boldsymbol{\alpha}, \quad \boldsymbol{y}'(0) = \boldsymbol{0}, \tag{1.4}$$

where $\boldsymbol{y}=(y, Ly, \dots, L^{m-1}y)$, $\boldsymbol{f}(t, \boldsymbol{y})=(Ly, \dots, L^{m-1}y, f(t, y, \dots, L^{m-1}y))$, $\boldsymbol{\alpha}=(\alpha_0, \alpha_1, \dots, \alpha_{m-1})$ and $\boldsymbol{0}=(0, 0, \dots, 0)$. It is easy to see that the problem (1.3)–(1.4) is equivalent to the system of integral equations

$$\boldsymbol{y}(t) = \boldsymbol{\alpha} + \int_0^t s\log(t/s)\cdot\boldsymbol{f}(s, \boldsymbol{y}(s))ds, \quad t \geqq 0, \qquad \text{for} \quad n = 2, \tag{1.5}$$

$$\boldsymbol{y}(t) = \boldsymbol{\alpha} + \frac{1}{n-2}\int_0^t s(1-(s/t)^{n-2})\boldsymbol{f}(s, \boldsymbol{y}(s))ds, \quad t \geqq 0, \qquad \text{for} \quad n \geqq 3. \tag{1.6}$$

In what follows we let a constant $T>0$ be fixed, put $\boldsymbol{1}=(1, 1, \dots, 1)$ and choose $M>0$ so that

$$\boldsymbol{0} \leqq \boldsymbol{f}(t, \boldsymbol{y}) \leqq M\cdot\boldsymbol{1} \qquad \text{for} \quad 0 \leqq t \leqq T, \quad \boldsymbol{\alpha} \leqq \boldsymbol{y} \leqq \boldsymbol{\alpha} + \boldsymbol{1}.$$

Case I: $n=2$. Let $(C[0, \delta])^m$, $\delta=\min\{T, 1/M, e\}$, be the Banach space of all continuous m-vector functions on $[0, \delta]$. Define the set $\mathscr{Y}\subset(C[0, \delta])^m$ and the mapping $\mathscr{F}: \mathscr{Y}\to(C[0, \delta])^m$ by

$$\mathscr{Y} = \{\boldsymbol{y}\in(C[0, \delta])^m: \boldsymbol{\alpha}\leqq\boldsymbol{y}(t)\leqq\boldsymbol{\alpha}+\mathbf{1} \text{ for } t\in[0, \delta]\}$$

and

$$\mathscr{F}\boldsymbol{y}(t) = \boldsymbol{\alpha} + \int_0^t s\log(t/s)\cdot\boldsymbol{f}(s, \boldsymbol{y}(s))ds, \quad t\in[0, \delta].$$

Using the inequality

$$s\log(t/s) \leqq t/e \qquad \text{for} \quad 0\leqq s\leqq t, \tag{1.7}$$

we see that if $\boldsymbol{y}\in\mathscr{Y}$, then

$$\boldsymbol{\alpha} \leqq \mathscr{F}\boldsymbol{y}(t) \leqq \boldsymbol{\alpha} + (t/e)\int_0^t M\cdot\mathbf{1}ds \leqq \boldsymbol{\alpha} + M\delta\cdot\mathbf{1} \leqq \boldsymbol{\alpha} + \mathbf{1}, \quad t\in[0, \delta],$$

which implies that $\mathscr{F}\boldsymbol{y}\in\mathscr{Y}$. Therefore $\mathscr{F}$ maps $\mathscr{Y}$ into itself. It is easily verified that $\mathscr{F}$ is continuous and that $\mathscr{F}\mathscr{Y}$ is relatively compact by the Ascoli-Arzelà theorem. From the Schauder fixed point theorem it follows that $\mathscr{F}$ has a fixed point $\boldsymbol{y}\in\mathscr{Y}$. This $\boldsymbol{y}=\boldsymbol{y}(t)$ is a solution of (1.5), and hence the first component $y(t)$ of $\boldsymbol{y}(t)$ is a solution of the problem (1.1)–(1.2) on $[0, \delta]$.

Case II: $n\geqq3$. Put $\delta=\min\{T, 1/M, 1\}$ and define $(C[0, \delta])^m$ and $\mathscr{Y}$ as in Case I. Then, it can be shown that the mapping $\mathscr{F}$ defined by

$$\mathscr{F}\boldsymbol{y}(t) = \boldsymbol{\alpha} + \frac{1}{n-2}\int_0^t s(1-(s/t)^{n-2})\boldsymbol{f}(s, \boldsymbol{y}(s))ds, \quad t\in[0, \delta],$$

is continuous and maps $\mathscr{Y}$ into a compact subset of $\mathscr{Y}$. A fixed point $\boldsymbol{y}\in\mathscr{Y}$ of $\mathscr{F}$ then gives a solution of (1.6), the first component of which is a solution of the problem (1.1)–(1.2) on $[0, \delta]$. This completes the proof.

The next result concerns the continuous dependence on initial values of solutions of the problem (1.1)–(1.2). It may be regarded as a variant of the well-known theorem of Kamke for regular ODE initial value problems (see Coppel [1, p. 17]).

LEMMA 1.2. *Let f be as in Lemma* 1.1. *Let* $(\alpha_0^\nu, \alpha_1^\nu,\dots, \alpha_{m-1}^\nu)$, $\nu=1, 2,\dots$, *be a sequence of points of* $[0, \infty)^m$ *or* $\boldsymbol{R}^m$ *converging to* $(\alpha_0, \alpha_1,\dots, \alpha_{m-1})$ *as* $\nu\to\infty$, *and let $y_\nu(t)$ be any noncontinuable solution of the equation* (1.1) *satisfying*

$$(L^i y)(0) = \alpha_i^\nu, \quad (L^i y)'(0) = 0, \quad 0\leqq i\leqq m-1, \quad \nu\geqq 1.$$

If the solution $y(t)$ of the problem (1.1)–(1.2) *is defined on* $[0, T]$ *and is unique, then $y_\nu(t)$ is defined on* $[0, T]$ *for all sufficiently large ν and*

$$(1.8)\qquad \begin{aligned} &\lim_{\nu\to\infty} L^i y_\nu(t) = L^i y(t), \\ &\lim_{\nu\to\infty} t^{n-1}(L^i y_\nu)'(t) = t^{n-1}(L^i y)'(t), \quad 0 \leqq i \leqq m-1, \end{aligned}$$

uniformly on $[0, T]$.

PROOF. The proof is given only for the case that $n=2$ and f is defined on $[0, \infty)^{m+1}$, since the other cases can be treated similarly. We employ the same vector notation as in the proof of Lemma 1.1; in particular, $\boldsymbol{\alpha}_\nu=(\alpha_0^\nu, \alpha_1^\nu,\dots, \alpha_{m-1}^\nu)$, $\boldsymbol{y}_\nu(t)=(y_\nu(t), Ly_\nu(t),\dots, L^{m-1}y_\nu(t))$. Let M be a constant satisfying

$$\mathbf{0} \leqq \boldsymbol{f}(t, \boldsymbol{y}) \leqq M\cdot\mathbf{1}, \quad 0 \leqq t \leqq T, \quad \mathbf{0} \leqq \boldsymbol{y} \leqq \boldsymbol{\alpha} + \mathbf{1},$$

and put $\delta=\min\{T, 1/(3M), e\}$. Choose an integer $\nu_0>0$ so that $\mathbf{0}\leqq\boldsymbol{\alpha}_\nu\leqq\boldsymbol{\alpha}+M\delta\cdot\mathbf{1}$ for $\nu\geqq\nu_0$. Then $\boldsymbol{y}_\nu(t)$ exists on $[0, \delta]$ and $\mathbf{0}\leqq\boldsymbol{y}_\nu(t)<\boldsymbol{\alpha}+3M\delta\cdot\mathbf{1}$ for $t\in[0, \delta]$ if $\nu\geqq\nu_0$. In fact, if this is not true, then there exist $t^*\in(0, \delta]$, $\nu\geqq\nu_0$ and $j\in\{0, 1,\dots, m-1\}$ such that

$$\begin{aligned} &L^i y_\nu(t) < \alpha_i + 3M\delta, \quad 0 \leqq t < t^*, \quad 0 \leqq i \leqq m-1, \\ &L^j y_\nu(t^*) = \alpha_j + 3M\delta. \end{aligned}$$

Using (1.7), we have

$$\begin{aligned} \alpha_j + 3M\delta = L^j y_\nu(t^*) &= \alpha_j^\nu + \int_0^{t^*} s\log(t^*/s)\cdot f_j(s, \boldsymbol{y}_\nu(s))ds \\ &\leqq \alpha_j + M\delta + (t^*/e)\int_0^{t^*} M ds \leqq \alpha_j + 2M\delta, \end{aligned}$$

where f_j denotes the j-th component of $\boldsymbol{f}(t, \boldsymbol{y})$. This contradiction shows that $\boldsymbol{y}_\nu(t)$ is defined on $[0, \delta]$ and satisfies $\mathbf{0}\leqq\boldsymbol{y}_\nu(t)<\boldsymbol{\alpha}+3M\delta\cdot\mathbf{1}$ there if $\nu\geqq\nu_0$. Moreover, the sequences $\{\boldsymbol{y}_\nu(t)\}$ and $\{\boldsymbol{y}'_\nu(t)\}$ are equicontinuous on $[0, \delta]$, since

$$\boldsymbol{y}'_\nu(t) = \int_0^t (s/t)\boldsymbol{f}(s, \boldsymbol{y}_\nu(s))ds$$

and

$$\boldsymbol{y}''_\nu(t) = -\frac{1}{t^2}\int_0^t s\boldsymbol{f}(s, \boldsymbol{y}_\nu(s))ds + \boldsymbol{f}(t, \boldsymbol{y}_\nu(t))$$

for $t\in[0, \delta]$ and $\nu\geqq\nu_0$. Let $\{\boldsymbol{y}_\mu(t)\}$ be any subsequence of $\{\boldsymbol{y}_\nu(t)\}$. By the Ascoli-Arzelà theorem and a well-known C^1-convergence theorem, there exist a subsequence $\{\boldsymbol{y}_{\mu(k)}(t)\}$ of $\{\boldsymbol{y}_\mu(t)\}$ and a C^1 m-vector function $\boldsymbol{z}(t)$ on $[0, \delta]$ such that

$$\boldsymbol{y}_{\mu(k)}(t) \to \boldsymbol{z}(t) \quad \text{and} \quad t\boldsymbol{y}'_{\mu(k)}(t) \to t\boldsymbol{z}'(t) \quad \text{as} \quad k\to\infty$$

uniformly on $[0, \delta]$. Letting $k\to\infty$ in the equation

$$\boldsymbol{y}_{\mu(k)}(t) = \boldsymbol{\alpha}_{\mu(k)} + \int_0^t s \log (t/s) \cdot \boldsymbol{f}(s, \boldsymbol{y}_{\mu(k)}(s))ds, \quad t \in [0, \delta],$$

we obtain

$$z(t) = \boldsymbol{\alpha} + \int_0^t s \log (t/s) \cdot \boldsymbol{f}(s, z(s))ds, \quad t \in [0, \delta],$$

which shows that $z(t)$ is a solution of (1.5) on $[0, \delta]$. It follows that $z(t) \equiv \boldsymbol{y}(t)$ on $[0, \delta]$ by uniqueness. Since $\{\boldsymbol{y}_\mu(t)\}$ is an arbitrary subsequence of $\{\boldsymbol{y}_\nu(t)\}$, we conclude that the whole sequences $\{\boldsymbol{y}_\nu(t)\}$, $\{\boldsymbol{y}'_\nu(t)\}$ converge uniformly on $[0, \delta]$ and their components satisfy (1.8) $(n=2)$ there.

To complete the proof it suffices to show that the maximal interval on which $\{\boldsymbol{y}_\nu(t)\}$ satisfies (1.8) is $[0, T]$. But this can be done exactly as in Coppel [1, p. 18].

It is important to observe that not all local solutions of the problem (1.1)–(1.2) can be continued to $t=\infty$. In fact, the solutions with sufficiently large initial values α_i are shown to blow up in a finite "time". To see this it is convenient to consider differential equations with "quasi-derivatives", of which (1.1) is a special case.

Let continuous functions q_i: $(0, \infty) \to (0, \infty)$, $0 \leqq i \leqq N$, $N \geqq 2$, be given, define the quasi-derivatives D_i, $0 \leqq i \leqq N$, by

$$D_0 y(t) = \frac{y(t)}{q_0(t)}, \quad D_i y(t) = \frac{1}{q_i(t)} \frac{d}{dt} D_{i-1} y(t), \qquad 1 \leqq i \leqq N,$$

and consider the equation

$$D_N y = g(t, D_0 y, D_1 y, \dots, D_{N-1} y), \tag{1.9}$$

where g is a nonnegative continuous function on $[0, \infty)^{N+1}$ or on $[0, \infty) \times \boldsymbol{R}^N$. Motivated by Kiguradze and Kvinikadze [6], we say that equation (1.9) has the *blow-up property* if for any $t_0 > 0$ there exists a constant $\eta(t_0) > 0$ such that any solution $y(t)$ of (1.9) satisfying

$$D_i y(t_0) \geqq 0, \quad 0 \leqq i \leqq N-2, \quad D_{N-1} y(t_0) \geqq \eta(t_0)$$

blows up in a finite time in the sense that

$$\lim_{t \to T_y^-} D_{N-1} y(t) = \infty$$

for some finite $T_y > t_0$. In case g is defined on $[0, \infty) \times \boldsymbol{R}^N$, the blow-up property of (1.9) implies that for any $t_0 > 0$ and $(\delta_0, \delta_1, \dots, \delta_{N-2}) \in \boldsymbol{R}^{N-1}$ there exists a constant $\eta(t_0; \delta_0, \delta_1, \dots, \delta_{N-2}) > 0$ such that any solution $y(t)$ of (1.9) satisfying

$$D_i y(t_0) \geqq \delta_i, \quad 0 \leqq i \leqq N-2, \quad D_{N-1} y(t_0) \geqq \eta(t_0; \delta_0, \dots, \delta_{N-2})$$

blows up in a finite time.

A sufficient condition for (1.9) to have the blow-up property is given in the next lemma which is a generalization of a result of Kiguradze and Kvinikadze [6, Theorem 1.1] for the equation $y^{(N)}=g(t, y, y',\dots, y^{(N-1)})$. The following notation is used:

$$I_0 = 1,$$
$$I_i(t, s; q_1,\dots, q_i) = \int_s^t q_i(r)I_{i-1}(t, r; q_1,\dots, q_{i-1})dr, \quad 1 \leqq i \leqq N.$$

LEMMA 1.3. *Let $g\colon [0, \infty)^{N+1}\to[0, \infty)$ be continuous. Suppose that there is a nonnegative continuous function $g_*(t, u_0, u_1,\dots, u_{N-1})$ on $[0, \infty)^{N+1}$ which is nonincreasing in t and nondecreasing in each u_i, $0\leqq i\leqq N-1$, and satisfies $g_*(t, u_0,\dots, u_{N-1})>0$ for $\sum_{i=0}^{N-1} u_i>0$ and*

$$g(t, u_0, u_1,\dots, u_{N-1}) \geqq g_*(t, u_0, u_1,\dots, u_{N-1})$$

for $(t, u_0,\dots, u_{N-1})\in[0, \infty)^{N+1}$. For $\tau>0$ we define

$$h_\tau(t, u) = g_*\left(t, u, \frac{u}{I_1(t, \tau; q_1)},\dots, \frac{u}{I_{N-1}(t, \tau; q_1,\dots, q_{N-1})}\right)$$

and

$$H_\tau(t, u) = \left(\frac{N}{(N-2)!}\int_0^u (u-\xi)^{N-2}h_\tau(t, \xi)d\xi\right)^{1/N}$$

for $t>\tau$ and $u\geqq 0$. Then, equation (1.9) has the blow-up property if

$$\int^\infty \frac{du}{H_\tau(t, u)} < \infty \qquad \text{for all} \quad t > \tau > 0.$$

PROOF. Let $t_0>0$ be fixed, choose τ, t_1, t_2, such that $\tau<t_0<t_1<t_2$, and put

$$m_1 = \min_{\tau\leqq t\leqq t_2} q_1(t), \quad m_i = \min_{\tau\leqq t\leqq t_2} \frac{q_i(t)}{q_1(t)}, \quad 2 \leqq i \leqq N.$$

Let $\delta>0$ and $\eta>0$ be such that

$$\int_\delta^\infty \frac{du}{H_\tau(t_2, u)} < m_1(m_2\cdots m_N)^{1/N}(t_2-t_1) \tag{1.10}$$

and

$$\eta \geqq \frac{\delta}{I_{N-1}(t_1, t_0; q_1,\dots, q_{N-1})}. \tag{1.11}$$

It is sufficient to show that any solution $y(t)$ of the equation

$$D_N y = g_*(t, D_0 y, D_1 y,\dots, D_{N-1} y) \tag{1.12}$$

satisfying

$$D_iy(t_0) = 0, \quad 0 \leqq i \leqq N-2, \quad D_{N-1}y(t_0) \geqq \eta \tag{1.13}$$

blows up at some finite $T_y \leqq t_2$. Suppose to the contrary that this $y(t)$ exists on $[t_0, t_2]$. In view of (1.13) we see that

$$D_iy(t) \geqq 0, \quad t \in [t_0, t_2], \quad 0 \leqq i \leqq N-1,$$

and

$$\begin{aligned} D_0y(t) &= \int_{t_0}^{t} I_i(t, r; q_1, \dots, q_i)q_{i+1}(r)D_{i+1}y(r)dr \\ &\leqq I_{i+1}(t, \tau; q_1, \dots, q_{i+1})D_{i+1}y(t), \quad t \in [t_0, t_2], \end{aligned}$$

for $0 \leqq i \leqq N-2$. Therefore, we have

$$D_iy(t) \geqq \frac{D_0y(t)}{I_i(t, \tau; q_1, \dots, q_i)}, \quad t \in [t_0, t_2], \quad 1 \leqq i \leqq N-1,$$

which combined with (1.12) shows that

$$D_Ny(t) \geqq h_\tau(t, D_0y(t)), \quad t \in [t_0, t_2].$$

We rewrite the above inequality as

$$(D_{N-1}y)'(t) \geqq q_N(t)h_\tau(t, D_0y(t)), \quad t \in [t_0, t_2],$$

multiply both sides by $D_1y(t) = (D_0y)'(t)/q_1(t)$, and integrate it from t_0 to t. We then find

$$D_1y(t)D_{N-1}y(t) \geqq \int_{t_0}^{t} \frac{q_N(s)}{q_1(s)} h_\tau(s, D_0y(s))(D_0y)'(s)ds \geqq m_N \int_0^{D_0y(t)} h_\tau(t, u)du,$$

or equivalently

$$D_1y(t)(D_{N-2}y)'(t) \geqq m_Nq_{N-1}(t)\int_0^{D_0y(t)} h_\tau(t, u)du, \quad t \in [t_0, t_2].$$

Multiplying the above by $D_1y(t)$ and integrating on $[t_0, t]$, we get

$$(D_1y(t))^2D_{N-2}y(t) \geqq m_Nm_{N-1}\int_0^{D_0y(t)} (D_0y(t)-u)h_\tau(t, u)du, \quad t \in [t_0, t_2].$$

Continuing this process, we have

$$(D_1y(t))^{N-2}D_2y(t) \geqq \frac{m_Nm_{N-1}\cdots m_3}{(N-3)!}\int_0^{D_0y(t)} (D_0y(t)-u)^{N-3}h_\tau(t, u)du, \quad t \in [t_0, t_2],$$

which leads to

$$(D_1y(t))^N \geqq \frac{Nm_Nm_{N-1}\cdots m_2}{(N-2)!}\int_0^{D_0y(t)} (D_0y(t)-u)^{N-2}h_\tau(t, u)du, \quad t\in[t_0, t_2].$$

It follows that

$$(D_0y)'(t) \geqq m_1(m_2\cdots m_N)^{1/N}H_\tau(t, D_0y(t)), \quad t\in[t_0, t_2]. \tag{1.14}$$

On the other hand, integration of $D_{N-1}y(t)\geqq\eta$ yields

$$D_0y(t) \geqq \eta I_{N-1}(t, t_0; q_1,\dots, q_{N-1}), \quad t\in[t_0, t_2];$$

consequently, $D_0y(t_1)\geqq\delta$ by (1.11). From this inequality, (1.10) and (1.14) we obtain

$$m_1(m_2\cdots m_N)^{1/N}(t_2-t_1) \leqq \int_{t_1}^{t_2}\frac{(D_0y)'(t)}{H_\tau(t_2, D_0y(t))}dt$$
$$= \int_{D_0y(t_1)}^{D_0y(t_2)}\frac{du}{H_\tau(t_2, u)} \leqq \int_\delta^\infty \frac{du}{H_\tau(t_2, u)} < m_1(m_2\cdots m_N)^{1/N}(t_2-t_1).$$

This is a contradiction, and hence $y(t)$ must blow up at some finite $T_y\leqq t_2$. This completes the proof.

Remark 1.1. The proof of Lemma 1.3 shows that the blow-up property is determined only by the nonlinear structure of the function $g(t, u_0,\dots, u_{N-1})$ (not by the functions $q_i(t)$ defining the quasi-derivatives).

Example 1.1. The following equations have the blow-up property:

$$D_Ny = \textstyle\sum_{i=0}^{N-1} p_i(t)(D_iy)^{\gamma_i},$$
$$D_Ny = p(t)\exp\left(\textstyle\sum_{i=0}^{N-1} p_i(t)(D_iy)^{\delta_i}\right),$$

where $\gamma_i>1$, $\delta_i>0$ are the ratios of odd integers and p, p_i: $[0, \infty)\to(0, \infty)$ are continuous functions, $0\leqq i\leqq N-1$.

Lemma 1.4. *Let $f(t, u_0, u_1,\dots, u_{m-1})$ be nonnegative and continuous on $[0, \infty)^{m+1}$ or on $[0, \infty)\times \boldsymbol{R}^m$ and nondecreasing in each u_i, $0\leqq i\leqq m-1$, and satisfy $\lim_{u_0\to\infty}f(t, u_0, 0,\dots, 0)=\infty$ for each $t\geqq0$. Suppose that equation* (1.1) *has the blow-up property. Then, any solution of the problem* (1.1)–(1.2) *with $\alpha_i\geqq0$, $0\leqq i\leqq m-1$, blows up in a finite time provided α_0 is sufficiently large.*

Proof. By the blow-up property of (1.1) there exists a constant $\eta>0$ such that any solution $y(t)$ of (1.1) satisfying

$$L^iy(1) \geqq 0, \quad 0 \leqq i \leqq m-1, \quad (L^iy)'(1) \geqq 0, \quad 0 \leqq i \leqq m-2,$$
$$(L^{m-1}y)'(1) \geqq \eta,$$

blows up in a finite time. Choose $\alpha_0>0$ so that

$$\int_0^1 t^{n-1}f(t, \alpha_0, 0,\dots, 0)dt \geqq \eta,$$

which is possible because $\lim_{\lambda\to\infty}\int_0^1 t^{n-1}f(t, \lambda, 0,\dots, 0)dt=\infty$, and let $y(t)$ be a solution of (1.1)–(1.2) with this choice of α_0 and $\alpha_i\geqq 0$, $1\leqq i\leqq m-1$. It is clear that $y(t)\geqq\alpha_0$, $L^i y(t)\geqq 0$, $1\leqq i\leqq m-1$, and $(L^i y)'(t)\geqq 0$, $0\leqq i\leqq m-1$, throughout the interval of existence of $y(t)$. If $y(t)$ blows up before $t=1$, there is nothing to prove. So, we assume that $y(t)$ exists on [0, 1]. Then, integrating (1.1), we have

$$(L^{m-1}y)'(1) = \int_0^1 t^{n-1}f(t, y(t), Ly(t),\dots, L^{m-1}y(t))dt$$
$$\geqq \int_0^1 t^{n-1}f(t, \alpha_0, 0,\dots, 0)dt \geqq \eta,$$

which implies the blow-up of $y(t)$ at some finite point $T_y>1$. Thus the proof of Lemma 1.4 is complete.

EXAMPLE 1.2. Consider the equations

$$\Delta^m u = \textstyle\sum_{i=0}^{m-1} p_i(|x|)(\Delta^i u)^{\gamma_i}, \qquad x\in \boldsymbol{R}^n, \tag{1.15}$$

$$\Delta^m u = p(|x|)\exp\left(\textstyle\sum_{i=0}^{m-1} p_i(|x|)(\Delta^i u)^{\delta_i}\right), \quad x\in \boldsymbol{R}^n, \tag{1.16}$$

where $\gamma_i>1$, $\delta_i\geqq 1$ are the ratios of odd integers and p, p_i: $[0, \infty)\to(0, \infty)$ are continuous, $0\leqq i\leqq m-1$. From the above results it follows that (1.15) [resp. (1.16)] has a unique local radial solution $u(x)$ satisfying the initial condition $\Delta^i u(0)=\alpha_i$, $0\leqq i\leqq m-1$, for every $(\alpha_0, \alpha_1,\dots, \alpha_{m-1})\in[0, \infty)^m$ [resp. $(\alpha_0, \alpha_1,\dots, \alpha_{m-1})\in \boldsymbol{R}^m$], that $u(x)$ depends continuously on α_i, and that $u(x)$ blows up at a finite value of $|x|$ if $\alpha_i\geqq 0$, $0\leqq i\leqq m-1$, and $\alpha_0>0$ is sufficiently large.

2. Existence of strongly increasing entire solutions

2.1. The purpose of this section is to study the existence of an entire solution $u(x)$ of (A) with the asymptotic property

$$\lim_{|x|\to\infty}\frac{u(x)}{|x|^{2m-2}\log|x|} = \infty \qquad \text{for} \quad n = 2, \tag{2.1}$$

$$\lim_{|x|\to\infty}\frac{u(x)}{|x|^{2m-2}} = \infty \qquad \text{for} \quad n \geqq 3. \tag{2.2}$$

We begin by introducing some notation which will be frequently used in the discussions that follow. Let Φ and Ψ denote the integral operators

$$\Phi h(t) = \int_0^t s^{1-n} \int_0^s r^{n-1} h(r) dr ds, \quad t \geqq 0, \quad n \geqq 2,$$

and

$$\Psi h(t) = \int_t^\infty s^{1-n} \int_0^s r^{n-1} h(r) dr ds, \quad t \geqq 0, \quad n \geqq 3,$$

respectively. It is easily verified that Φ maps $C[0, \infty)$ into $C^2[0, \infty)$ and satifises

$$L\Phi h(t) = h(t), \quad t \geqq 0, \quad \text{or} \quad \Delta\Phi h(|x|) = h(|x|), \quad x \in \boldsymbol{R}^n, \quad \text{if} \quad n \geqq 2,$$

and that Ψ maps the subset of $C[0, \infty)$ consisting of all h such that $\int_0^\infty t|h(t)|dt < \infty$ into $C^2[0, \infty)$ and satisfies

$$L\Psi h(t) = -h(t), \quad t \geqq 0, \quad \text{or} \quad \Delta\Psi h(|x|) = -h(|x|), \quad x \in \boldsymbol{R}^n, \quad \text{if} \quad n \geqq 3.$$

It is sometimes convenient to rewrite Φh and Ψh as

$$\Phi h(t) = \int_0^t s \log(t/s) \cdot h(s) ds, \quad t \geqq 0, \qquad \text{for} \quad n = 2, \tag{2.3}$$

$$\Phi h(t) = \frac{1}{n-2} \int_0^t s(1-(s/t)^{n-2}) h(s) ds, \quad t \geqq 0, \qquad \text{for} \quad n \geqq 3, \tag{2.4}$$

and

$$\Psi h(t) = \frac{1}{n-2}\left(\int_0^t (s/t)^{n-2} sh(s) ds + \int_t^\infty sh(s) ds\right), \quad t \geqq 0, \qquad \text{for} \quad n \geqq 3$$

(note that (2.3) and (2.4) are used in the proofs of Lemmas 1.1. and 1.2). We also use the abbreviations:

$$\begin{aligned} &\rho(n, i) = \textstyle\prod_{j=1}^i [2(m-j)(n+2(m-j-1))], && 1 \leqq i \leqq m-1, \\ &\rho(i) = \rho(2, i) = [2^i(m-1)(m-2)\cdots(m-i)]^2, && 1 \leqq i \leqq m-1, \\ &\rho(n, 0) = \rho(0) = 1. \end{aligned} \tag{2.5}$$

Hypotheses on f will be selected from the following list.

(A_1) $f(t, u_0, u_1, \ldots, u_{m-1})$ is continuous and nonnegative on $[0, \infty)^{m+1}$, nondecreasing in each u_i, $1 \leqq i \leqq m-1$, and strictly increasing in u_0. Moreover, $\lambda^{-1} f(t, \lambda u_0, \ldots, \lambda u_{m-1})$ is nondecreasing in $\lambda \in (0, \infty)$ and

$$\lim_{\lambda \to +0} \lambda^{-1} f(t, \lambda u_0, \ldots, \lambda u_{m-1}) = 0 \qquad \text{for each} \quad (t, u_0, \ldots, u_{m-1}) \in [0, \infty)^{m+1}.$$

(A_2) $f(t, u_0, u_1, \ldots, u_{m-1})$ is continuous and nonnegative on $[0, \infty) \times \boldsymbol{R}^m$, nondecreasing in each u_i, $1 \leqq i \leqq m-1$, and strictly increasing in u_0. Moreover

$$\lim_{u_0 \to \infty} f(t, u_0, 0, \ldots, 0) = \infty \qquad \text{for each} \quad t \geqq 0.$$

In case $n=2$ the condition

$$\lim_{u_0\to-\infty} f(t, u_0,\dots, u_{m-1}) = 0 \qquad \text{for each} \quad (t, u_1,\dots, u_{m-1})\in[0, \infty)\times \boldsymbol{R}^{m-1}$$

should be added.

(A_3) For each $t_0>0$ the initial value problem for the ordinary differential equation

$$L^m y = f(t, y, Ly,\dots, L^{m-1}y), \quad t > t_0,$$

with given initial values $L^i y(t_0)$, $(L^i y)'(t_0)$, $0\leqq i\leqq m-1$, at t_0 has a unique (local) solution.

(A_4) The singular initial value problem

$$L^m y = f(t, y, Ly,\dots, L^{m-1}y), \quad t > 0,$$
$$L^i y(0) = \alpha_i, \quad (L^i y)'(0) = 0, \quad 0 \leqq i \leqq m-1,$$

has a unique (local) solution for each admissible $(\alpha_0, \alpha_1,\dots, \alpha_{m-1})$.

(A_5) The ordinary differential equation $L^m y=f(t, y, Ly,\dots, L^{m-1}y)$ has the blow-up property.

REMARK 2.1. The hypotheses (A_3) and (A_4) are satisfied if $f(t, u_0,\dots, u_{m-1})$ is locally Lipschitz continuous with respect to u_i, $0\leqq i\leqq m-1$.

Basic to the proof of the main theorems (Theorems 2.1 and 2.2 below) are the following lemmas ensuring the existence of entire solutions $u(x)$ of (A) such that

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}\log|x|} = const\in(0, \infty) \qquad \text{for} \quad n = 2, \tag{2.6}$$

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}} = const\in(0, \infty) \qquad \text{for} \quad n \geqq 3. \tag{2.7}$$

LEMMA 2.1. *Let $n=2$. Suppose that f satisfies either* (A_1) *or* (A_2). *If there is a constant $c>0$ such that*

$$\int_1^\infty tf(t, ct^{2m-2}\log t, ct^{2m-4}\log t,\dots, c\log t)dt < \infty, \tag{2.8}$$

then equation (A) *has a radial entire solution $u(x)$ satisfying* (2.6).

LEMMA 2.2. *Let $n\geqq 3$. Suppose that f satisfies either* (A_1) *or* (A_2). *If there is a constant $c>0$ such that*

$$\int_0^\infty tf(t, ct^{2m-2}, ct^{2m-4},\dots, c)dt < \infty, \tag{2.9}$$

then equation (A) *has a radial entire solution* $u(x)$ *satisfying* (2.7).

In what follows $\mathscr{C}^{m-1}[0, \infty)$ denotes the set of all functions $y(t)$ such that $L^i y(t)$, $0 \leqq i \leqq m-1$, are continuous on $[0, \infty)$. In particular $(L^i y)'(0)=0$, $0 \leqq i \leqq m-2$, for any $y \in \mathscr{C}^{m-1}[0, \infty)$. Clearly, $\mathscr{C}^{m-1}[0, \infty)$ is a Frèchet space with the topology induced by the seminorms

$$\|y\|_T = \sum_{i=0}^{m-1} \sup_{0 \leqq t \leqq T} |L^i y(t)|, \quad T = 1, 2, \ldots.$$

Proof of Lemma 2.1. We use the fact [7, Lemmas 3 and 4] that if we put

$$k(t) = \max\{1, t\} \quad \text{and} \quad \ell(t) = \max\{1, \log t\},$$

then

$$M_i = \sup_{t \geqq 0} \frac{\Phi(k^{2i-2}\ell)(t)}{[k(t)]^{2i}\ell(t)} < \infty, \quad i = 1, 2, \ldots \tag{2.10}$$

and

$$\begin{aligned} &0 \leqq \Phi h(t) \leqq \ell(t) \int_0^\infty k(t)h(t)dt, \\ &0 \leqq \Phi^j h(t) \leqq M_1 \cdots M_{j-1}[k(t)]^{2j-2}\ell(t) \int_0^\infty k(t)h(t)dt, \quad j = 2, 3, \ldots \end{aligned} \tag{2.11}$$

for any nonnegative continuous function $h(t)$ on $[0, \infty)$ such that $\int_0^\infty th(t)dt < \infty$.

Suppose that (A_1) holds. From condition (2.8) and the Lebesgue dominated convergence theorem it follows that

$$\lim_{\lambda \to +0} \lambda^{-1} \int_0^\infty k(t) f(t, \lambda[k(t)]^{2m-2}\ell(t), \lambda[k(t)]^{2m-4}\ell(t), \ldots, \lambda\ell(t))dt = 0,$$

so that there is an $\alpha > 0$ sufficiently small such that

$$\int_0^\infty k(t)\tilde{f}(t, \alpha)dt \leqq \rho(m-1)\alpha,$$

$$M_1 M_2 \cdots M_{m-i-1} \int_0^\infty k(t)\tilde{f}(t, \alpha)dt \leqq \rho(i)\alpha, \quad 0 \leqq i \leqq m-2,$$

where $\rho(i)$ and M_i are defined by (2.5) and (2.10), and

$$\tilde{f}(t, \alpha) = f(t, 3\alpha[k(t)]^{2m-2}\ell(t), 3\rho(1)\alpha[k(t)]^{2m-4}\ell(t), \ldots, 3\rho(m-1)\alpha\ell(t)).$$

Let $\mathscr{Y}$ be the set of all $y \in \mathscr{C}^{m-1}[0, \infty)$ satisfying the inequalities

$$0 \leqq L^i y(t) \leqq 3\rho(i)\alpha[k(t)]^{2m-2i-2}\ell(t), \quad 0 \leqq i \leqq m-1,$$

for $t \geqq 0$ and define the mapping $\mathscr{F}: \mathscr{Y} \to \mathscr{C}^{m-1}[0, \infty)$ by

$$(2.12)\qquad \mathscr{F}y(t) = \alpha(1+t^{2m-2}) + \Phi^m f(\cdot, y, Ly,\dots, L^{m-1}y)(t), \quad t \geqq 0.$$

Noting that

$$L^i(\mathscr{F}y)(t) = \rho(i)\alpha t^{2m-2i-2} + \Phi^{m-i} f(\cdot, y, Ly,\dots, L^{m-1}y)(t), \quad 1 \leqq i \leqq m-1,$$

and using (2.10) and (2.11) (with $h=f$), we see that $\mathscr{F}$ is continuous and maps $\mathscr{Y}$ into a compact subset of $\mathscr{Y}$. Therefore, by the Schauder-Tychonoff fixed point theorem, $\mathscr{F}$ has a fixed element $y \in \mathscr{Y}$; in particular

$$(2.13)\quad y(t) = \alpha(1+t^{2m-2}) + \Phi^m f(\cdot, y, Ly,\dots, L^{m-1}y)(t), \quad t \geqq 0.$$

Differentiation of (2.13) shows that the function $u(x)=y(|x|)$ is a radial entire solution of (A) in $\boldsymbol{R}^2$. That $u(x)$ has the asymptotic property (2.6) is a consequence of the relation

$$\lim_{t\to\infty} \frac{y(t)}{t^{2m-2}\log t} = \frac{1}{\rho(m-1)}\int_0^\infty tf(t, y(t), Ly(t),\dots, L^{m-1}y(t))dt,$$

which follows from L'Hospital's rule.

Suppose next that (A_2) holds. Let $b \in (0, c]$ be fixed and put

$$\tilde{f}(t, \alpha, b) = f(t, \alpha(1+t^{2m-2}) + b[k(t)]^{2m-2}\ell(t), b[k(t)]^{2m-4}\ell(t),\dots, b\ell(t)).$$

Using the condition $\lim_{u_0\to-\infty} f(t, u_0, u_1,\dots, u_{m-1})=0$, we obtain $\lim_{\alpha\to-\infty}\int_0^\infty k(t) \cdot\tilde{f}(t, \alpha, b)dt=0$, and so there is an $\alpha<0$ such that

$$\int_0^\infty k(t)\tilde{f}(t, \alpha, b)dt \leqq b,$$

$$M_1M_2\cdots M_{m-i-1}\int_0^\infty k(t)\tilde{f}(t, \alpha, b)dt \leqq b, \quad 0 \leqq i \leqq m-2.$$

Then, proceeding as in the case of (A_1), it can be shown that the mapping defined by (2.12) has a fixed element in the set $\mathscr{Y}$ consisting of all $y \in \mathscr{C}^{m-1}[0, \infty)$ such that

$$\alpha(1+t^{2m-2}) \leqq y(t) \leqq \alpha(1+t^{2m-2}) + b[k(t)]^{2m-2}\ell(t),$$

$$\rho(i)\alpha t^{2m-2i-2} \leqq L^i y(t) \leqq b[k(t)]^{2m-2i-2}\ell(t), \quad 1 \leqq i \leqq m-1,$$

for $t\geqq 0$. This fixed element y gives rise to a radial entire solution $u(x)=y(|x|)$ of (A) having the property (2.6). This finishes the proof.

Proof of Lemma 2.2. We only consider the case where (A_2) holds, since the case (A_1) was treated by Kusano, Naito and Swanson [8, Theorem 4], who showed that equation (2.13) has solutions for sufficiently small $\alpha>0$. Let b be any constant such that $0<b\leqq c/\rho(n, m-1)$, and put

$$\tilde{f}(t) = f(t, bt^{2m-2}, b\rho(n, 1)t^{2m-4}, \ldots, b\rho(n, m-1)).$$

In view of (2.9) the operator Ψ is applicable to $\tilde{f}$. Define $\mathscr{Y}$ to be the set of all $y \in \mathscr{C}^{m-1}[0, \infty)$ satisfying for $t \geqq 0$

$$b\rho(n, i)t^{2m-2i-2} - \Phi^{m-i-1}\Psi\tilde{f}(t) \leqq L^i y(t) \leqq b\rho(n, i)t^{2m-2i-2}, \quad 0 \leqq i \leqq m-1.$$

It is a matter of simple computation to check that the conditions of the Schauder-Tychonoff fixed point theorem are satisfied by the operator

$$\mathscr{F}y(t) = bt^{2m-2} - \Phi^{m-1}\Psi f(\cdot, y, Ly, \ldots, L^{m-1}y)(t), \quad t \geqq 0,$$

defined on $\mathscr{Y}$. A fixed point $y \in \mathscr{Y}$ of $\mathscr{F}$ is a solution of the integral equation

$$y(t) = bt^{2m-2} - \Phi^{m-1}\Psi f(\cdot, y, Ly, \ldots, L^{m-1}y)(t), \quad t \geqq 0.$$

Since, by L'Hospital's rule, $\Phi^{m-1}\Psi f(\cdot, y, Ly, \ldots, L^{m-1}y)(t)/t^{2m-2} \to 0$ as $t \to \infty$, the function $u(x) = y(|x|)$ gives an entire solution of (A) satisfying (2.7): $\lim_{|x| \to \infty} u(x)/|x|^{2m-2} = b$. This completes the proof.

REMARK 2.2. The proofs of Lemmas 2.1 and 2.2 show that the entire solutions $u(x) = y(|x|)$ obtained for the case (A_1) are positive throughout $\boldsymbol{R}^n$.

We are now in a position to state and prove the main results of this paper.

THEOREM 2.1. *Let* $n = 2$.

(i) *Suppose that f satisfies* $\{(A_1), (A_3), (A_4), (A_5)\}$. *If* (2.8) *holds for all* $c > 0$, *then equation* (A) *has a positive radial entire solution* $u(x)$ *satisfying* (2.1).

(ii) *Suppose that f satisfies* $\{(A_2), (A_3), (A_4), (A_5)\}$. *If* (2.8) *holds for all* $c > 0$, *then equation* (A) *has an eventually positive radial entire solution* $u(x)$ *satisfying* (2.1).

THEOREM 2.2. *Let* $n \geqq 3$.

(i) *Suppose that f satisfies* $\{(A_1), (A_3), (A_4), (A_5)\}$. *If* (2.9) *holds for all* $c > 0$, *then equation* (A) *has a positive radial entire solution* $u(x)$ *satisfying* (2.2).

(ii) *Suppose that f satisfies* $\{(A_2), (A_3), (A_4), (A_5)\}$. *If* (2.9) *holds for all* $c > 0$, *then equation* (A) *has an eventually positive radial entire solution* $u(x)$ *satisfying* (2.2).

Consider the initial value problem for the equation

$$L^m y = f(t, y, Ly, \ldots, L^{m-1}y), \quad t > 0, \tag{1.1}$$

with singular initial conditions of the type

$$(1.2)_\alpha \quad \begin{array}{l} y(0) = \alpha, \quad L^i y(0) = 0, \quad 1 \leqq i \leqq m-2, \quad L^{m-1}y(0) = \rho(n, m-1)\alpha, \\ (L^i y)'(0) = 0, \quad 0 \leqq i \leqq m-1, \end{array}$$

where α is a real constant (in case $m=1$, $(1.2)_\alpha$ means that $y(0)=\alpha$ and $y'(0)=0$). Under hypothesis (A_1) [resp. (A_2)] the problem (1.1)–$(1.2)_\alpha$ has a local solution $y_\alpha(t)$ for any $\alpha>0$ [resp. $\alpha\in \boldsymbol{R}$] (see Lemma 1.1). From Lemma 1.2 it follows that under hypotheses $\{(A_1), (A_3), (A_4)\}$ [resp. $\{(A_2), (A_3), (A_4)\}$] the solution $y_\alpha(t)$ of (1.1)–$(1.2)_\alpha$ is unique and depends continuously on $\alpha>0$ [resp. $\alpha\in \boldsymbol{R}$]. Integrating (1.1) and using $(1.2)_\alpha$, we see that $y_\alpha(t)$ satisfies the integro-differential equation

$$y_\alpha(t) = \alpha(1+t^{2m-2}) + \Phi^m f(\cdot, y_\alpha, Ly_\alpha,\dots, L^{m-1}y_\alpha)(t)$$

on its interval of existence. Note that this is the same as (2.13).

The following simple comparison principle for the solution of (1.1)–$(1.2)_\alpha$ will be used in the proofs of Theorems 2.1 and 2.2.

LEMMA 2.3. *Suppose that f satisfies either* (A_1) *or* (A_2). *Let* $y_\alpha(t)$ *and* $y_\beta(t)$ *denote, respectively, solutions of the problems* (1.1)–$(1.2)_\alpha$ *and* (1.1)–$(1.2)_\beta$ *defined on* $[0, T)$, $T>0$. *If* $\alpha<\beta$, *then*

$$(2.14)\qquad L^i y_\alpha(t) < L^i y_\beta(t),\quad (L^i y_\alpha)'(t) < (L^i y_\beta)'(t),\quad 0\leqq i\leqq m-1,$$

for $t\in(0, T)$.

PROOF. We first show that

$$(2.15)\qquad L^{m-1}y_\alpha(t) < L^{m-1}y_\beta(t)$$

for $t\in(0, T)$. Since $L^{m-1}y_\alpha(0)=\rho(n, m-1)\alpha<\rho(n, m-1)\beta=L^{m-1}y_\beta(0)$, (2.15) holds for all sufficiently small $t>0$. Suppose that $L^{m-1}y_\alpha(t)\geqq L^{m-1}y_\beta(t)$ for some $t\in[0, T)$. Then, there is $t_1\in(0, T)$ such that

$$(2.16)\quad L^{m-1}y_\alpha(t) < L^{m-1}y_\beta(t),\quad t\in(0, t_1),\quad L^{m-1}y_\alpha(t_1) = L^{m-1}y_\beta(t_1),$$

whence we have

$$(2.17)\qquad L^i y_\alpha(t) \leqq L^i y_\beta(t),\quad t\in[0, t_1],\quad 0\leqq i\leqq m-1,$$

and

$$(2.18)\qquad (L^{m-1}y_\alpha)'(t_1) \geqq (L^{m-1}y_\beta)'(t_1).$$

From (2.16)–(2.18) and (1.1) it follows that

$$\begin{aligned} 0 &\leqq (L^{m-1}y_\alpha)'(t_1) - (L^{m-1}y_\beta)'(t_1)\\ &= \int_0^{t_1}\left(\frac{s}{t_1}\right)^{n-1}[f(s, y_\alpha(s),\dots, L^{m-1}y_\alpha(s)) - f(s, y_\beta(s),\dots, L^{m-1}y_\beta(s))]ds < 0, \end{aligned}$$

where the monotonicity of $f(t, u_0,\dots, u_{m-1})$, in particular the strict increasing nature with respect to u_0, has been used. This is a contradiction, and so (2.15) must hold for $t \in (0, T)$, and repeated integration of (2.15) shows that (2.14) holds on $(0, T)$ except for the last inequality with $i = m-1$. But this follows immediately from the relation

$$(L^{m-1}y_\varepsilon)'(t) = \int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, y_\varepsilon(s), \ldots, L^{m-1}y_\varepsilon(s))ds, \quad \varepsilon = \alpha \quad \text{or} \quad \beta,$$

and the monotonicity of f. This completes the proof.

REMARK 2.3. Suppose that (A_1) or (A_2) holds. Let $\alpha < \beta$ and let $y_\alpha(t)$ and $y_\beta(t)$ be solutions of (1.1)–$(1.2)_\alpha$ and (1.1)–$(1.2)_\beta$, respectively. If $y_\beta(t)$ exists on $[0, T)$, then $y_\alpha(t)$ also exists on $[0, T)$ and satisfies (2.14) there.

PROOF OF THEOREM 2.1. Our idea of the proof is to find an appropriate value of α for which the solution $y_\alpha(t)$ of the problem (1.1)–$(1.2)_\alpha$ exists on $[0, \infty)$ and has the desired asymptotic behavior as $t \to \infty$.

(i) We define the subsets A, B of $(0, \infty)$ by

$$A = \{\alpha \in (0, \infty): y_\alpha(t) \text{ exists on } [0, \infty) \text{ and } t(L^{m-1}y_\alpha)'(t) \text{ is bounded on } [0, \infty)\},$$
$$B = \{\alpha \in (0, \infty): y_\alpha(t) \text{ blows up in a finite time}\}.$$

According to the proof of Lemma 2.1, for any sufficiently small $\alpha > 0$, there exists a solution $y(t)$ of the integro-differential equation (2.13) on $[0, \infty)$, which clearly is a solution of the problem (1.1)–$(1.2)_\alpha$ on $[0, \infty)$ satisfying $\lim_{t\to\infty} y(t)/t^{2m-2}\log t = const > 0$. This shows that A is not empty and contains an interval of the form $(0, \alpha_0)$. Hypothesis (A_1) implies that $\lim_{u_0\to\infty} f(t, u_0, 0, \ldots, 0) = \infty$ for $t \geqq 0$, and by Lemma 1.4 the solution of (1.1)–$(1.2)_\alpha$ blows up in a finite time provided $\alpha > 0$ is sufficiently large. Hence B contains an interval of the form (α_1, ∞).

In view of Lemma 2.3 we see that if $\alpha \in A$ and $\beta \in B$, then $\alpha < \beta$, so that $0 < \sup A \leqq \inf B < \infty$. We claim that

$$\inf B \notin B \quad \text{and} \quad \sup A \notin A. \tag{2.19}$$

Put $\beta_* = \inf B$ and suppose that $\beta_* \in B$. Since $y_{\beta_*}(t)$ blows up in a finite time, there is T, $0 < T < \infty$, such that $\lim_{t\to T^-} t(L^{m-1}y_{\beta_*})'(t) = \infty$. Because of (A_5) there is a constant $\eta(T) > 0$ such that any solution $y(t)$ of (1.1) satisfying

$$\begin{aligned} &L^i y(T) \geqq 0, \quad 0 \leqq i \leqq m-1, \\ &(L^i y)'(T) \geqq 0, \quad 0 \leqq i \leqq m-2, \quad T(L^{m-1}y)'(T) \geqq \eta(T), \end{aligned} \tag{2.20}$$

blows up at some finite point $T_y > T$. Let $\varepsilon > 0$ be small enough so that $(T-\varepsilon)\cdot(L^{m-1}y_{\beta_*})'(T-\varepsilon) > \eta(T)$. Hypotheses (A_3) and (A_4) enable us to apply Lemma 1.2 to infer that if $\beta < \beta_*$ is sufficiently close to β_*, then $y_\beta(t)$ exists on $[0, T-\varepsilon]$ and satisfies $(T-\varepsilon)(L^{m-1}y_\beta)'(T-\varepsilon) > \eta(T)$. If $y_\beta(t)$ blows up before T, then $\beta \in B$, which contradicts the definition of β_*. Therefore $y_\beta(t)$ exists on $[0, T]$. Since $L^i y_\beta(t)$, $t(L^i y_\beta)'(t)$, $0 \leqq i \leqq m-1$, are nondecreasing, $y = y_\beta(t)$ satisfies (2.20) and so blows up at some finite point to the right of T. This is also a contradiction and completes the proof of $\beta_* \notin B$.

Now put $\alpha^* = \sup A$ and suppose that $\alpha^* \in A$. Then $y_{\alpha^*}(t)$ exists on $[0, \infty)$ and satisfies $\lim_{t\to\infty} t(L^{m-1}y_{\alpha^*})'(t) = \lambda$ for some constant $\lambda > 0$. Let K be a positive constant such that

$$L^i y_{\alpha^*}(1) < K, \quad 0 \leqq i \leqq m-1, \quad (L^i y_{\alpha^*})'(1) < K, \quad 0 \leqq i \leqq m-2,$$

and define

$$k_i(t) = K \sum_{j=0}^{m-i-2} \frac{t^{2j}(1+\log t)}{(2^j \cdot j!)^2} + \frac{t^{2(m-i-1)}(K+2\lambda \log t)}{[2^{m-i-1}(m-i-1)!]^2}, \quad 0 \leqq i \leqq m-2,$$

$$k_{m-1}(t) = K + 2\lambda \log t.$$

From condition (2.8) for all $c > 0$, there exists $t_1 > 1$ such that

$$\int_{t_1}^{\infty} tf(t, k_0(t), k_1(t), \dots, k_{m-1}(t))dt < \lambda. \tag{2.21}$$

By Lemma 1.2, if $\alpha > \alpha^*$ is sufficiently close to α^*, then $y_\alpha(t)$ exists on $[0, t_1]$ and satisfies

$$L^i y_\alpha(1) < K, \quad 0 \leqq i \leqq m-1, \quad (L^i y_\alpha)'(1) < K, \quad 0 \leqq i \leqq m-2,$$
$$t(L^{m-1}y_\alpha)'(t) < \lambda, \quad 0 \leqq t \leqq t_1.$$

It will be shown that such a $y_\alpha(t)$ can be extended to $t = \infty$ and satisfies

$$t(L^{m-1}y_\alpha)'(t) < 2\lambda \qquad \text{for} \quad t \geqq 0. \tag{2.22}$$

In fact, if this is not true, then there is $t_2 > t_1$ such that

$$t(L^{m-1}y_\alpha)'(t) < 2\lambda, \quad 0 \leqq t < t_2, \quad t_2(L^{m-1}y_\alpha)'(t_2) = 2\lambda.$$

Integration of (1.1) on $[t_1, t_2]$ yields

$$\begin{aligned} 2\lambda &= t_2(L^{m-1}y_\alpha)'(t_2) \\ &= t_1(L^{m-1}y_\alpha)'(t_1) + \int_{t_1}^{t_2} tf(t, y_\alpha(t), \dots, L^{m-1}y_\alpha(t))dt \\ &< \lambda + \int_{t_1}^{t_2} tf(t, y_\alpha(t), \dots, L^{m-1}y_\alpha(t))dt. \end{aligned} \tag{2.23}$$

On the other hand, successive integration of the inequality $(L^{m-1}y_\alpha)'(t) \leqq 2\lambda/t$, $1 \leqq t \leqq t_2$, shows that

$$L^i y_\alpha(t) \leqq k_i(t), \quad 0 \leqq i \leqq m-1, \quad 1 \leqq t \leqq t_2,$$

which combined with (2.21) and (2.23) leads to a contradiction:

$$2\lambda < \lambda + \int_{t_1}^{t_2} tf(t, k_0(t), k_1(t),\dots, k_{m-1}(t))dt$$
$$< \lambda + \int_{t_1}^{\infty} tf(t, k_0(t), k_1(t),\dots, k_{m-1}(t))dt < 2\lambda.$$

Therefore, $y_\alpha(t)$ must exists on $[0, \infty)$ and satisfy (2.22). But this shows that $\alpha \in A$, contradicting the definition of α^*. Thus we conclude that $\alpha^* \notin A$.

Now consider the interval $[\sup A, \inf B]$, which is not empty because of (2.19) (it may reduce to one point). It is easily seen that for any $\alpha \in [\sup A, \inf B]$ the solution $y_\alpha(t)$ of the initial value problem (1.1)–$(1.2)_\alpha$ exists throughout $[0, \infty)$ and has the asymptotic property $\lim_{t\to\infty} y_\alpha(t)/t^{2m-2}\log t = \infty$. The function $u(x) = y_\alpha(|x|)$ then gives a positive radial entire solution of equation (A) satisfying (2.1).

(ii) Define

$$A = \{\alpha \in \boldsymbol{R}\colon y_\alpha(t) \text{ exists on } [0, \infty) \text{ and } t(L^{m-1}y_\alpha)'(t) \text{ is bounded on } [0, \infty)\},$$
$$B = \{\alpha \in \boldsymbol{R}\colon y_\alpha(t) \text{ blows up in a finite time}\}.$$

From the proofs of Lemmas 2.1 and 1.4 it follows that A contains an interval $(-\infty, \alpha_0)$, $\alpha_0 < 0$, and B contains an interval (α_1, ∞), $\alpha_1 > 0$. Lemma 2.3 implies that $-\infty < \sup A \leqq \inf B < \infty$. That $\sup A \notin A$ is proved exactly as in (i). It remains to prove that $\beta_* = \inf B \notin B$. Suppose to the contrary that $\beta_* \in B$. Let $\tilde{\alpha} \in A$ be fixed. Obviously $\tilde{\alpha} < \beta_*$. Let $T > 0$ be the time at which $y_{\beta_*}(t)$ blows up. In view of (A_5) there exists a constant $\eta > 0$ such that any solution $y(t)$ of (1.1) satisfying

$$\begin{aligned} &L^i y(T) \geqq L^i y_{\tilde{\alpha}}(T), && 0 \leqq i \leqq m-1,\\ &T(L^i y)'(T) \geqq T(L^i y_{\tilde{\alpha}})'(T), && 0 \leqq i \leqq m-2,\\ &T(L^{m-1}y)'(T) \geqq \eta \end{aligned} \tag{2.24}$$

blows up in a finite time. Take a small $\varepsilon > 0$ such that

$$(T-\varepsilon)(L^{m-1}y_{\beta_*})'(T-\varepsilon) > \eta.$$

If $\alpha \in (\tilde{\alpha}, \beta_*)$ is sufficiently close to β_*, then by Lemma 1.2 $y_\alpha(t)$ exists on $[0, T-\varepsilon]$ and satisfies

$$(T-\varepsilon)(L^{m-1}y_\alpha)'(T-\varepsilon) > \eta.$$

Such a $y_\alpha(t)$ blows up in a finite time, because if it exists on $[0, T]$, then $y = y_\alpha(t)$ satisfies (2.24). But this contradicts the definition of β_*, and hence $\inf B \notin B$.

To each $\alpha \in [\sup A, \inf B]$ there corresponds an eventually positive entire solution $u(x) = y_\alpha(|x|)$ which satisfies (2.1). Thus the proof of Theorem 2.1 is complete.

Proof of Theorem 2.2. (i) We now define A and B by

$$A = \{\alpha \in (0, \infty): y_\alpha(t) \text{ exists on } [0, \infty) \text{ and } L^{m-1}y_\alpha(t) \text{ is bounded on } [0, \infty)\},$$

$$B = \{\alpha \in (0, \infty): y_\alpha(t) \text{ blows up in a finite time}\}.$$

Proceeding exactly as in the proof of Theorem 2.1, we can show that $A \neq \emptyset$, $B \neq \emptyset$, $0 < \sup A \leqq \inf B < \infty$ and $\inf B \notin B$ (the proof of $A \neq \emptyset$ is based on Lemma 2.2). It remains to show that $\sup A \notin A$.

Suppose that $\alpha^* = \sup A \in A$. Clearly the limit $\lambda = \lim_{t\to\infty} L^{m-1}y_{\alpha^*}(t)$ is finite and positive. Let $\mu > 0$ and $\tilde{\alpha} > \alpha^*$ be fixed and define

$$k_0(t) = \tilde{\alpha} + \frac{(\lambda+\mu)t^{2(m-1)}}{2^{m-1}(m-1)!n(n+2)\cdots(n+2(m-2))},$$

$$k_i(t) = \frac{(\lambda+\mu)t^{2(m-i-1)}}{2^{m-i-1}(m-i-1)!n(n+2)\cdots(n+2(m-i+2))}, \quad 1 \leqq i \leqq m-2,$$

$$k_{m-1}(t) = \lambda + \mu.$$

Choose $t_1 > 0$ so large that

$$\int_{t_1}^{\infty}\int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, k_0(s), k_1(s), \dots, k_{m-1}(s))dsdt < \mu, \tag{2.25}$$

which is possible because of (2.9). If $\alpha \in (\alpha^*, \tilde{\alpha})$ is sufficiently close to α^*, then by Lemma 1.2 $y_\alpha(t)$ exists on $[0, t_1]$ and satisfies $L^{m-1}y_\alpha(t) < \lambda$ on $[0, t_1]$. This $y_\alpha(t)$ is shown to exist on $[0, \infty)$ and satisfies $L^{m-1}y_\alpha(t) < \lambda + \mu$ there. In fact, if this is not the case, there exists $t_2 > t_1$ such that

$$L^{m-1}y_\alpha(t) < \lambda + \mu \quad \text{on} \quad [0, t_2) \quad \text{and} \quad L^{m-1}y_\alpha(t_2) = \lambda + \mu$$

(note that if $y_\alpha(t)$ blows up at some finite T, then $L^{m-1}y_\alpha(t) \to \infty$ as $t \to T^-$). Then, integrating the above successively, we have $L^i y_\alpha(t) \leqq k_i(t)$, $t \in [0, t_2]$, $0 \leqq i \leqq m-1$, so that from (1.1) and (2.25), we find

$$\begin{aligned} \lambda + \mu &= L^{m-1}y_\alpha(t_2) \\ &= L^{m-1}y_\alpha(t_1) + \int_{t_1}^{t_2}\int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, y_\alpha(s), \dots, L^{m-1}y_\alpha(s))dsdt \\ &< \lambda + \int_{t_1}^{\infty}\int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, k_0(s), k_1(s), \dots, k_{m-1}(s))dsdt < \lambda + \mu. \end{aligned}$$

This contradiction shows that $y_\alpha(t)$ exists on $[0, \infty)$ and satisfies $L^{m-1}y_\alpha(t) < \lambda + \mu$ there, implying that $\alpha \in A$. But this contradicts the definition of α^* and it follows that $\alpha^* = \sup A \notin A$.

(ii) Let A and B be the sets defined by

$$A = \{\alpha \in \boldsymbol{R}: y_\alpha(t) \text{ exists on } [0, \infty) \text{ and } L^{m-1}y_\alpha(t) \text{ is bounded on } [0, \infty)\},$$

$B = \{\alpha \in \boldsymbol{R}\colon y_\alpha(t) \text{ blows up in a finite time}\}.$

The proof of Lemma 2.2 ensures, for any $b \in \boldsymbol{R}$, the existence of a solution of (1.1) on $[0, \infty)$ obtained as a solution of the integro-differential equation

$$y(t) = bt^{2m-2} - \Phi^{m-1}\Psi f(\cdot, y, Ly,\dots, L^{m-1}y)(t), \quad t \geqq 0. \tag{2.26}$$

Noting by (2.26) that

$$L^i y(0) = 0, \quad 0 \leqq i \leqq m-2,$$
$$L^{m-1}y(0) = \rho(n, m-1)b - \Psi f(\cdot, y,\dots, L^{m-1}y)(0),$$
$$L^{m-1}y(t) \to \rho(n, m-1)b \quad \text{as} \quad t \to \infty,$$

and applying the same argument as in the proof of Lemma 2.3, we infer that, for all $\alpha < 0$ sufficiently close to $-\infty$, the solutions $y_\alpha(t)$ of (1.1)–(1.2)$_\alpha$ exist on $[0, \infty)$ and have bounded derivatives $L^{m-1}y_\alpha(t)$ on $[0, \infty)$. It follows that $A \neq \emptyset$. The relation $\sup A \notin A$ is proved exactly as in the proof of (i). It can also be shown that $B \neq \emptyset$ and $\inf B \notin B$. This completes the proof.

2.2. In this subsection we will strengthen Theorems 2.1 and 2.2 by showing that under the conditions of these theorems equation (A) possesses a radial entire solution $u(x)$ satisfying

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}\log|x|} = a \quad \text{for} \quad n = 2, \tag{2.27}$$

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}} = a \quad \text{for} \quad n \geqq 3, \tag{2.28}$$

for any prescribed value of $a > 0$. In fact, we can prove the following theorems.

THEOREM 2.3. *Let* $n=2$.

(i) *Under the hypotheses* $\{(A_1), (A_3), (A_4), (A_5)\}$, *if* (2.8) *holds for all* $c>0$, *then, for any given* $a \in (0, \infty]$, *equation* (A) *has a positive radial entire solution* $u(x)$ *satisfying* (2.27).

(ii) *Under the hypotheses* $\{(A_2), (A_3), (A_4), (A_5)\}$, *if* (2.8) *holds for all* $c>0$, *then, for any given* $a \in (0, \infty]$, *equation* (A) *has an eventually positive radial entire solution* $u(x)$ *satisfying* (2.27).

THEOREM 2.4. *Let* $n \geqq 3$.

(i) *Under the hypotheses* $\{(A_1), (A_3), (A_4), (A_5)\}$, *if* (2.9) *holds for all* $c>0$, *then, for any given* $a \in (0, \infty]$, *equation* (A) *has a positive radial entire solution* $u(x)$ *satisfying* (2.28).

(ii) *Under the hypotheses* $\{(A_2), (A_3), (A_4), (A_5)\}$, *if* (2.9) *holds for all* $c>0$, *then, for any given* $a \in (0, \infty]$, *equation* (A) *has an eventually positive* *radial entire solution $u(x)$ satisfying* (2.28).

Since the above results are true for $a=\infty$ by Theorems 2.1 and 2.2, we need only to examine the case where $a>0$ is finite. The following theorems will suffice for our purpose.

THEOREM 2.5. *Let* $n=2$.

(i) *Under* $\{(A_1), (A_3), (A_4), (A_5)\}$, *if* (2.8) *holds for some* $c>0$, *then, for any* $a \in (0, c/\rho(m-1)]$, *equation* (A) *has a positive radial entire solution* $u(x)$ *satisfying* (2.27).

(ii) *Under* $\{(A_2), (A_3), (A_4), (A_5)\}$, *if* (2.8) *holds for some* $c>0$, *then, for any* $a \in (0, c/\rho(m-1)]$, *equation* (A) *has an eventually positive radial entire solution* $u(x)$ *satisfying* (2.27).

THEOREM 2.6. *Let* $n \geqq 3$.

(i) *Under* $\{(A_1), (A_3), (A_4), (A_5)\}$, *if* (2.9) *holds for some* $c>0$, *then, for any* $a \in (0, c/\rho(n, m-1)]$, *equation* (A) *has a positive radial entire solution* $u(x)$ *satisfying* (2.28).

(ii) *Under* $\{(A_2), (A_3), (A_4), (A_5)\}$, *if* (2.9) *holds for some* $c>0$, *then, for any* $a \in (0, c/\rho(n, m-1)]$, *equation* (A) *has an eventually positive radial entire solution* $u(x)$ *satisfying* (2.28).

PROOF OF THEOREM 2.5. (i) Let $y_\alpha(t)$ denote the solution of the problem (1.1)–$(1.2)_\alpha$ and define the sets A and B as in the proof of Theorem 2.1–(i). It is clear that $A=(0, \alpha^*)$ or $A=(0, \alpha^*]$ for some finite $\alpha^*>0$ (the second possibility is not excluded because (2.8) is assumed to hold for *some* $c>0$). For $\alpha \in A$ define $\mathscr{L}(\alpha)$ by

$$\mathscr{L}(\alpha) = \lim_{t\to\infty} t(L^{m-1}y_\alpha)'(t) = \int_0^\infty tf(t, y_\alpha(t), Ly_\alpha(t),\dots, L^{m-1}y_\alpha(t))dt.$$

Then the function $\mathscr{L}$: $A \to (0, \infty)$ is continuous and strictly increasing. The increasing nature of $\mathscr{L}$ follows from Lemma 2.3. To prove the continuity let $\alpha, \alpha_0 \in A$ and $\alpha \to \alpha_0$. Take $\alpha_1 \in A$ so that $\alpha_0 \leqq \alpha_1$ and $\alpha \leqq \alpha_1$ for α sufficiently close to α_0. Note that

$$f(t, y_\alpha(t), Ly_\alpha(t),\dots, L^{m-1}y_\alpha(t)) \leqq f(t, y_{\alpha_1}(t), Ly_{\alpha_1}(t),\dots, L^{m-1}y_{\alpha_1}(t))$$

for $t \geqq 0$, and that $L^i y_\alpha(t) \to L^i y_{\alpha_0}(t)$ as $\alpha \to \alpha_0$ pointwise on $[0, \infty)$ by Lemma 1.2. By $\int_0^\infty tf(t, y_{\alpha_1}(t),\dots, L^{m-1}y_{\alpha_1}(t))dt < \infty$, the Lebesgue dominated convergence theorem shows that

$$\lim_{\alpha\to\alpha_0} \mathscr{L}(\alpha) = \lim_{\alpha\to\alpha_0} \int_0^\infty tf(t, y_\alpha(t),\dots, L^{m-1}y_\alpha(t))dt$$
$$= \int_0^\infty tf(t, y_{\alpha_0}(t),\dots, L^{m-1}y_{\alpha_0}(t))dt = \mathscr{L}(\alpha_0),$$

implying the continuity of $\mathscr{L}$.

Since, for a radial entire solution $u(x)=y(|x|)$ of equation (A), $\lim_{t\to\infty} y(t)/t^{2m-2}\log t=a$ if and only if $\lim_{t\to\infty} t(L^{m-1}y)'(t)=\rho(m-1)a$, the proof will be complete if we show that the range of $\mathscr{L}$ contains the interval $(0, c]$. Since $\lim_{\alpha\to 0}\mathscr{L}(\alpha)=0$ by the proof of Lemma 2.1, it is sufficient to show that $\mathscr{L}(\alpha)\geqq c$ for some $\alpha\in A$.

We first consider the case $A=(0, \alpha^*)$. By the definition of A and B, we see that $y_{\alpha^*}(t)$ exists on $[0, \infty)$ and $\lim_{t\to\infty} t(L^{m-1}y_{\alpha^*})'(t)=\infty$. There is $t_1>0$ such that $t_1(L^{m-1}y_{\alpha^*})'(t_1)>c$, and so $t_1(L^{m-1}y_\alpha)'(t_1)>c$ provided $\alpha\in(0, \alpha^*)$ is sufficiently close to α^*. In view of the increasing nature of $t(L^{m-1}y_\alpha)'(t)$ we obtain $\lim_{t\to\infty} t(L^{m-1}y_\alpha)'(t)\in(c, \infty)$, showing that $\mathscr{L}(\alpha)>c$. Therefore $(0, c]\subset \mathscr{L}(A)$ (as a matter of fact, we have $\mathscr{L}(A)=(0, \infty)$ in this case).

Next we consider the case $A=(0, \alpha^*]$. Suppose that the assertion fails to hold. Then, there exists $c^*\in(0, c)$ such that $\mathscr{L}(\alpha^*)=c^*$. Put $2\delta=c-c^*>0$, let $K>0$ be a constant such that

$$L^i y_{\alpha^*}(1) < K, \quad 0\leqq i\leqq m-1, \quad (L^i y_{\alpha^*})'(1) < K, \quad 0\leqq i\leqq m-2,$$

and define the functions $k_i(t)$ by

$$k_i(t) = K\sum_{j=0}^{m-i-2}\frac{t^{2j}(1+\log t)}{(2^j\cdot j!)^2} + \frac{t^{2m-2i-2}[K+(c^*+\delta)\log t]}{[2^{m-i-1}(m-i-1)!]^2}, \quad 0\leqq i\leqq m-2,$$

$$k_{m-1}(t) = K + (c^*+\delta)\log t.$$

Choose $t_2>1$ large enough so that

$$\int_{t_2}^{\infty} tf(t, k_0(t), k_1(t),\dots, k_{m-1}(t))dt < \delta.$$

By Lemma 1.2, if $\alpha>\alpha^*$ is sufficiently close to α^*, then $y_\alpha(t)$ exists on $[0, t_2]$ and satisfies

$$L^i y_\alpha(1) < K, \quad 0\leqq i\leqq m-1, \quad (L^i y_\alpha)'(1) < K, \quad 0\leqq i\leqq m-2,$$
$$t(L^{m-1}y_\alpha)'(t) < c^* \qquad \text{for} \quad t\in[0, t_2].$$

We claim that this $y_\alpha(t)$ exists on $[0, \infty)$ and satisfies

$$(2.29) \qquad t(L^{m-1}y_\alpha)'(t) < c^*+\delta, \quad t\geqq 0.$$

If we assume to the contrary that there is $t_3>t_2$ such that

$$t(L^{m-1}y_\alpha)'(t) < c^*+\delta, \quad 0\leqq t< t_3, \quad t_3(L^{m-1}y_\alpha)'(t_3) = c^*+\delta,$$

then, we have $L^i y_\alpha(t)\leqq k_i(t)$, $1\leqq t\leqq t_3$, $0\leqq i\leqq m-1$, by integrating the above inequality, and furthermore from (1.1) we find

$$\begin{aligned} c^* + \delta &= t_3(L^{m-1}y_\alpha)'(t_3) \\ &= t_2(L^{m-1}y_\alpha)'(t_2) + \int_{t_2}^{t_3} tf(t, y_\alpha(t),\dots, L^{m-1}y_\alpha(t))dt \\ &< c^* + \int_{t_2}^{\infty} tf(t, k_0(t),\dots, k_{m-1}(t))dt < c^* + \delta. \end{aligned}$$

This contradiction implies the truth of (2.29). This, however, leads to a contradiction that there is an $\alpha > \alpha^* = \sup A$ which belongs to A. It follows therefore that $\mathscr{L}(A)$ contains $(0, c]$.

(ii) It suffices to repeat the same arguments as above by replacing the sets A and B by the ones used in the proof of Theorem 2.1–(ii). Clearly, A is of the form $(-\infty, \alpha^*)$ or $(-\infty, \alpha^*]$, and use of the fact that $\lim_{\alpha\to-\infty} \mathscr{L}(\alpha)=0$, which follows from the proof of Lemma 2.1, easily establishes the desired assertion.

Proof of Theorem 2.6. (i) Define the sets A and B as in the proof of Theorem 2.2–(i). From Lemma 2.2 A is not empty, and it is clear that $A=(0, \alpha^*)$ or $A=(0, \alpha^*]$ for some finite $\alpha^*>0$. Here we define the function $\mathscr{L}(\alpha)$, $\alpha \in A$, to be

$$\mathscr{L}(\alpha) = \lim_{t\to\infty} L^{m-1}y_\alpha(t) = \rho(n, m-1)\alpha + \frac{1}{n-2}\int_0^\infty tf(t, y_\alpha(t),\dots, L^{m-1}y_\alpha(t))dt.$$

Then, arguing as in the proof of Theorem 2.5–(i), $\mathscr{L}(\alpha)$ is shown to be continuous and strictly increasing on A. Since, for a radial entire solution $u(x)=y(|x|)$ of equation (A), $\lim_{t\to\infty} y(t)/t^{2m-2}=a$ if and only if $\lim_{t\to\infty} L^{m-1}y(t)=\rho(n, m-1)a$, it is sufficient to show that the range of $\mathscr{L}$ contains the interval $(0, c]$. From Kusano, Naito and Swanson [8, Theorem 4], it follows that $\lim_{\alpha\to 0} \mathscr{L}(\alpha)=0$. We only consider the case $A=(0, \alpha^*]$, because the case $A=(0, \alpha^*)$ is treated as in the proof of Theorem 2.5–(i).

Suppose that the conclusion of the theorem fails. Then there exists $c^* \in (0, c)$ such that $\mathscr{L}(\alpha^*)=c^*$. Put $2\delta = c - c^* > 0$, and let $\tilde{\alpha} > \alpha^*$ be fixed. Define

$$\begin{aligned} k_0(t) &= \tilde{\alpha} + \frac{(c^*+\delta)t^{2(m-1)}}{2^{m-1}(m-1)!n(n+2)\cdots(n+2(m-2))}, \\ k_i(t) &= \frac{(c^*+\delta)t^{2(m-i-1)}}{2^{m-i-1}(m-i-1)!n(n+2)\cdots(n+2(m-i-2))}, \quad 1 \leqq i \leqq m-2, \\ k_{m-1}(t) &= c^* + \delta. \end{aligned}$$

Choose $t_1 > 0$ so large that

$$\int_{t_1}^{\infty}\int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, k_0(s),\dots, k_{m-1}(s))ds\,dt < \delta.$$

By Lemma 1.2, for any $\alpha \in (\alpha^*, \tilde{\alpha})$ sufficiently close to α^*, $y_\alpha(t)$ exists on $[0, t_1]$

and satisfies $L^{m-1}y_\alpha(t)<c^*$ there. Arguing as in the proof of Theorem 2.2–(i), we conclude that this $y_\alpha(t)$ exists on $[0, \infty)$ and satisfies $L^{m-1}y_\alpha(t)<c^*+\delta$ for $t\geqq 0$. This contradicts the definition of α^*, $\alpha^*=\sup A$.

(ii) This statement has been proved in Lemma 2.2.

EXAMPLE 2.1. Consider the equation

$$\Delta^m u = \sum_{i=0}^{m-1} p_i(|x|)(\Delta^i u)^{\gamma_i}, \quad x\in \mathbf{R}^n, \tag{2.30}$$

where $\gamma_i>1$ and p_i: $[0, \infty)\to(0, \infty)$ is continuous, $0\leqq i\leqq m-1$. The corresponding ordinary differential equation is

$$L^m y = \sum_{i=0}^{m-1} p_i(t)(L^i y)^{\gamma_i}, \quad t > 0,$$

for which the hypotheses $\{(A_1), (A_3), (A_4), (A_5)\}$ are clearly satisfied. Conditions (2.8) and (2.9) reduce, respectively, to

$$\int_1^\infty t^{1+2\gamma_i(m-i-1)} (\log t)^{\gamma_i} p_i(t)dt < \infty, \quad 0\leqq i\leqq m-1, \tag{2.31}$$

and

$$\int_0^\infty t^{1+2\gamma_i(m-i-1)} p_i(t)dt < \infty, \qquad 0\leqq i\leqq m-1. \tag{2.32}$$

Theorems 2.3 and 2.4 imply that:

(i) if $n=2$ and (2.31) holds, then (2.30) has a positive radial entire solution $u(x)$ satisfying (2.27) for any given $a\in(0, \infty]$;

(ii) if $n\geqq 3$ and (2.32) holds, then (2.30) has a positive radial entire solution $u(x)$ satisfying (2.28) for any given $a\in(0, \infty]$.

EXAMPLE 2.2. Next consider the equation

$$\Delta^m u = p(|x|)e^u, \quad x\in \mathbf{R}^n, \tag{2.33}$$

where p: $[0, \infty)\to(0, \infty)$ is continuous. Conditions (2.8) and (2.9) for the corresponding ordinary differential equation $L^m y=p(t)e^y$ read

$$\int_1^\infty tp(t)\exp(ct^{2m-2}\log t)dt < \infty, \tag{2.34}$$

and

$$\int_0^\infty tp(t)\exp(ct^{2m-2})dt < \infty, \tag{2.35}$$

respectively, and we have the following statements from Theorems 2.3–2.6.

(i) Let $n=2$. If (2.34) holds for some $c>0$, then (2.33) has an eventually positive radial entire solution $u(x)$ satisfying (2.27) for any given $a\in(0, c/\rho(m-1)]$.

If (2.34) holds for all $c>0$, then (2.33) has an eventually positive radial entire solution $u(x)$ satisfying (2.27) for any given $a \in (0, \infty]$.

(ii) Let $n \geqq 3$. If (2.35) holds for some $c>0$, then (2.33) has an eventually positive radial entire solution $u(x)$ satisfying (2.28) for any given $a \in (0, c/\rho(n, m-1)]$. If (2.35) holds for all $c>0$, then (2.33) has an eventually positive radial entire solution $u(x)$ satisfying (2.28) for any given $a \in (0, \infty]$.

3. Second order equations without radial symmetry

We have so far been concerned with radial entire solutions of equation (A) which is radially symmetric. The results obtained in the preceding section seem to be new even when specialized to the second order case, i.e.,

$$\Delta u = f(|x|, u), \quad x \in \boldsymbol{R}^n. \tag{3.1}$$

The purpose of this section is to proceed further and show that our results for (3.1) can be applied, with the aid of the supersolution-subsolution method, to establish the existence of strongly increasing entire solutions (not necessarily radial) for second order equations without radial symmetry of the form

$$\Delta u = g(x, u), \quad x \in \boldsymbol{R}^n. \tag{3.2}$$

We refer to the paper [25] for closely related results concerning the exterior Dirichlet problem for equations of the form (3.2).

Hypotheses required for (3.2) are as follows:

(B_1) $g(x, u)$ is a positive locally Hölder continuous (with exponent $\theta \in (0, 1)$) function on $\boldsymbol{R}^n \times (0, \infty)$ which is strictly increasing in u for any $x \in \boldsymbol{R}^n$.

(B_2) There exist positive locally Hölder continuous (with exponent θ) functions $g^*(t, u)$ and $g_*(t, u)$ on $[0, \infty) \times (0, \infty)$ such that

$$g_*(|x|, u) \leqq g(x, u) \leqq g^*(|x|, u) \qquad \text{for} \quad (x, u) \in \boldsymbol{R}^n \times (0, \infty).$$

Moreover $g^*(t, u)$ and $g_*(t, u)$ satisfy the hypotheses $\{(A_1), (A_3), (A_4), (A_5)\}$ (with $m=1$ and $f=g^*, g_*$) given in Section 2.

(B_3) $g(x, u)$ is a positive locally Hölder continuous (with exponent θ) function on $\boldsymbol{R}^{n+1}$ which is strictly increasing in u for any $x \in \boldsymbol{R}^n$.

(B_4) There exist positive locally Hölder continuous (with exponent θ) functions $g^*(t, u)$ and $g_*(t, u)$ on $[0, \infty) \times \boldsymbol{R}$ such that

$$g_*(|x|, u) \leqq g(x, u) \leqq g^*(|x|, u) \qquad \text{for} \quad (x, u) \in \boldsymbol{R}^{n+1}.$$

Moreover $g^*(t, u)$ and $g_*(t, u)$ satisfy the hypotheses $\{(A_2), (A_3), (A_4), (A_5)\}$ (with $m=1$ and $f=g^*, g_*$) given in Section 2.

We begin by stating an existence result on which the supersolution-subsolution method is based.

LEMMA 3.1. *Let $g(x, u)$ be locally Hölder continuous (with exponent $\theta \in (0, 1)$) on $\boldsymbol{R}^n \times (0, \infty)$ or $\boldsymbol{R}^{n+1}$. If there exist functions v, $w \in C^{2+\theta}_{loc}(\boldsymbol{R}^n)$ such that*

$$\Delta v(x) \leqq g(x, v(x)), \quad x \in \boldsymbol{R}^n, \tag{3.3}$$

$$\Delta w(x) \geqq g(x, w(x)), \quad x \in \boldsymbol{R}^n, \tag{3.4}$$

$$w(x) \leqq v(x), \quad x \in \boldsymbol{R}^n, \tag{3.5}$$

then equation (3.2) *has an entire solution $u \in C^{2+\theta}_{loc}(\boldsymbol{R}^n)$ satisfying $w(x) \leqq u(x) \leqq v(x)$ throughout $\boldsymbol{R}^n$.*

For the proof of this lemma see Noussair and Swanson [20]. A function $v(x)$ [resp. $w(x)$] satisfying (3.3) [resp. (3.4)] is called a supersolution [resp. subsolution] of equation (3.2).

Our first result concerns the existence of entire solutions $u(x)$ of (3.2) having the prescribed limits $\lim_{|x| \to \infty} u(x)/\log |x|$ for $n=2$, and $\lim_{|x| \to \infty} u(x)$ for $n \geqq 3$.

THEOREM 3.1. *Let $n=2$.*

(i) *Suppose that* (B_1) *and* (B_2) *hold. If*

$$\int_1^\infty t g^*(t, c \log t) dt < \infty \tag{3.6}$$

for some $c>0$, then, for every $a \in (0, c]$, equation (3.2) *has a unique positive entire solution $u(x)$ satisfying*

$$\lim_{|x| \to \infty} \frac{u(x)}{\log |x|} = a. \tag{3.7}$$

(ii) *Suppose that* (B_3) *and* (B_4) *hold. If* (3.6) *holds for some $c>0$, then, for every $a \in (0, c]$, equation* (3.2) *has a unique eventually positive entire solution $u(x)$ satisfying* (3.7).

THEOREM 3.2. *Let $n \geqq 3$.*

(i) *Suppose that* (B_1) *and* (B_2) *hold. If*

$$\int_0^\infty t g^*(t, c) dt < \infty \tag{3.8}$$

for some $c>0$, then, for every $a \in (0, c]$, equation (3.2) *has a unique positive entire solution $u(x)$ satisfying*

(3.9) $$\lim_{|x|\to\infty} u(x) = a.$$

(ii) *Suppose that* (B_3) *and* (B_4) *hold. If* (3.8) *holds for some* $c>0$, *then, for every* $a\in(0, c]$, *equation* (3.2) *has a unique eventually positive entire solution* $u(x)$ *satisfying* (3.9).

Proof of Theorem 3.1. (i) (*Existence*) Let $a\in(0, c]$ be fixed arbitrarily. By (i) of Theorem 2.5 there exist positive functions $y, z\in C^{2+\theta}_{loc}[0, \infty)$ which satisfy

(3.10) $$Ly(t) = g_*(t, y(t)), \quad t > 0; \quad y'(0) = 0,$$

(3.11) $$\lim_{t\to\infty}\frac{y(t)}{\log t} = a$$

and

(3.12) $$Lz(t) = g^*(t, z(t)), \quad t > 0; \quad z'(0) = 0,$$

(3.13) $$\lim_{t\to\infty}\frac{z(t)}{\log t} = a,$$

respectively. It is clear that $v(x)=y(|x|)$ is a supersolution of (3.2) and $w(x)=z(|x|)$ is a subsolution of (3.2). We shall show that $z(t)\leqq y(t)$ for $t\geqq 0$. To prove first that $z(0)\leqq y(0)$, we assume the contrary: $z(0)>y(0)$. Then, the argument of the proof of Lemma 2.3 shows that $z(t)>y(t)$ for $t\geqq 0$. By (3.10) and (3.12) we then have

$$t[z'(t)-y'(t)] = \int_0^t s[g^*(s, z(s))-g_*(s, y(s))]ds, \quad t\geqq 0.$$

The right hand side of the above is clearly bounded away from zero for all $t\geqq 1$, whereas the left hand side tends to zero as $t\to\infty$ because of (3.11) and (3.13). Therefore we must have $z(0)\leqq y(0)$. Suppose now that $z(t_1)>y(t_1)$ for some $t_1>0$. Let $T\in[0, t_1)$ be the first point at which $y(t)=z(t)$, and choose $t_2\in(T, t_1)$ so that $z(t_2)>y(t_2)$ and $z'(t_2)>y'(t_2)$. Then, $z(t)>y(t)$ for $t\geqq t_2$ (see Lemma 1 of [25]). Again from (3.10) and (3.12) we have

$$\begin{aligned} t[z'(t)-y'(t)] &= t_2[z'(t_2)-y'(t_2)] + \int_{t_2}^t s[g^*(s, z(s))-g_*(s, y(s))]ds \\ &\geqq t_2[z'(t_2)-y'(t_2)], \quad t\geqq t_2, \end{aligned}$$

which is a contradiction since $t[z'(t)-y'(t)]\to 0$ as $t\to\infty$. We therefore conclude that $z(t)\leqq y(t)$, $t\geqq 0$, as desired. From Lemma 3.1 it follows that equation (3.2) possesses an entire solution $u(x)$ satisfying $z(|x|)\leqq u(x)\leqq y(|x|)$ in $\boldsymbol{R}^2$. The asymptotic behavior (3.7) of $u(x)$ is a consequence of (3.11) and (3.13).

(*Uniqueness*) Let $u_1(x)$ and $u_2(x)$ be positive entire solutions of (3.2) satisfying the same condition (3.7). We distinguish the two cases: (a) $u_1(x)-u_2(x)$ is of constant sign throughout $\boldsymbol{R}^2$; and (b) $u_1(x)-u_2(x)$ changes sign in $\boldsymbol{R}^2$.

Case (a). We may suppose that $u_1(x) \geqq u_2(x)$ in $\boldsymbol{R}^2$. Since

$$\Delta(u_1(x)-u_2(x)) = g(x, u_1(x)) - g(x, u_2(x)) \geqq 0, \quad x \in \boldsymbol{R}^2,$$

and

$$\lim_{|x|\to 0} \frac{u_1(x)-u_2(x)}{\log|x|} = \lim_{|x|\to\infty} \frac{u_1(x)-u_2(x)}{\log|x|} = 0,$$

we see from a Liouville type theorem (Protter and Weinberger [22, p. 130]) that $u_1(x)-u_2(x)=C$ in $\boldsymbol{R}^2$ for some constant C. This constant C must be zero, because

$$0 = \Delta(u_1(x)-u_2(x)) = g(x, u_2(x)+C) - g(x, u_2(x)), \quad x \in \boldsymbol{R}^2,$$

and $g(x, u)$ is strictly increasing in u. Thus $u_1(x)=u_2(x)$ in $\boldsymbol{R}^2$.

Case (b). Put $\Gamma=\{x \in \boldsymbol{R}^2 \colon u_1(x)-u_2(x)>0\}$ and let $x_0 \in \boldsymbol{R}^2$ be a point at which $u_1(x_0)-u_2(x_0)<0$. Let R_0 denote the distance between x_0 and $\partial\Gamma$, choose a positive constant R with $0<R<R_0$, and consider the function

$$U(x) = \frac{u_1(x)-u_2(x)}{\log(|x-x_0|/R)}, \quad x \in \bar{\Gamma}.$$

Take a point $x_1 \in \Gamma$. For any $\varepsilon>0$ there is $R_\varepsilon>R_0$ such that $|x_1-x_0|<R_\varepsilon$ and

$$|u_1(x)-u_2(x)| \leqq \varepsilon \log(|x-x_0|/R) \qquad \text{for} \quad |x-x_0| = R_\varepsilon.$$

Let D be a connected component of $\Gamma \cap \{x \in \boldsymbol{R}^2 \colon |x-x_0|<R_\varepsilon\}$ containing x_1. Since

$$\begin{aligned}\Delta(U(x)\log(|x-x_0|/R)) &= \Delta(u_1(x)-u_2(x)) \\ &= g(x, u_1(x)) - g(x, u_2(x)) \geqq 0, \quad x \in D,\end{aligned}$$

and

$$\begin{aligned}\Delta(U(x)\log(|x-x_0|/R)) &= \Delta U(x)\cdot\log(|x-x_0|/R) \\ &\quad + 2\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\log(|x-x_0|/R))\frac{\partial U(x)}{\partial x_i}, \quad x \in D,\end{aligned}$$

$U(x)$ satisfies the differential inequality

$$\Delta U(x) + \frac{2}{\log(|x-x_0|/R)}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\log(|x-x_0|/R))\frac{\partial U(x)}{\partial x_i} \geqq 0, \quad x \in D.$$

Note that $\partial D \subset \partial\Gamma \cup \{x \colon |x-x_0|=R_\varepsilon\}$, $U(x)=0$ for $x \in \partial\Gamma$ and $|U(x)| \leqq \varepsilon$ for $|x-x_0| = R_\varepsilon$. Applying the strong maximum principle, we have

$$0 < U(x) = \frac{u_1(x) - u_2(x)}{\log(|x - x_0|/R)} \leqq \varepsilon \qquad \text{for} \quad x \in D;$$

in particular

$$0 < u_1(x_1) - u_2(x_1) \leqq \varepsilon \log(|x_1 - x_0|/R).$$

Since $\varepsilon > 0$ is arbitrary, this leads to a contradiction, and hence case (b) is impossible. This completes the proof of (i) of Theorem 3.1.

To prove (ii) it suffices to construct eventually positive solutions $y(t)$, $z(t)$ of the problems (3.10)–(3.11), (3.12)–(3.13), by means of (ii) of Theorem 2.5 and repeat the same arguments as in (i).

The proof of Theorem 3.2 is essentially same as that of Theorem 3.1, and so will be omitted.

REMARK 3.1. Theorem 3.1 improves Theorem 1 of Kawano, Kusano and Naito [5] and Theorem 1 of Usami [23]. Theorem 3.2 is an extension of the superlinear part of the result of Naito [17].

Application of Theorems 2.1 and 2.2 to the construction of strongly increasing solutions of (3.2) will be given below.

THEOREM 3.3. *In addition to* (B_1) *and* (B_2) *suppose that there exist positive constants* C *and* ε *such that*

$$\frac{g^*(t, u)}{g_*(t, u)} \leqq C \qquad \text{for all} \quad (t, u) \in [0, \infty) \times (0, \infty), \tag{3.14}$$

and

$$u^{-1-\varepsilon} g^*(t, u) \text{ is nondecreasing in } u \text{ for any } t \geqq 0. \tag{3.15}$$

(i) *In case* $n=2$, *if* (3.6) *holds for all* $c>0$, *then equation* (3.2) *has a positive entire solution* $u(x)$ *satisfying*

$$\lim_{|x| \to \infty} \frac{u(x)}{\log|x|} = \infty. \tag{3.16}$$

(ii) *In case* $n \geqq 3$, *if* (3.8) *holds for all* $c>0$, *then equation* (3.2) *has a positive entire solution* $u(x)$ *satisfying*

$$\lim_{|x| \to \infty} u(x) = \infty. \tag{3.17}$$

THEOREM 3.4. *In addition to* (B_3) *and* (B_4) *suppose that there exist positive constants* C, k *and* δ *such that*

$$\frac{g^*(t, u)}{g_*(t, u)} \leqq C \qquad \text{for all} \quad (t, u) \in [0, \infty) \times \boldsymbol{R} \tag{3.18}$$

and

$$\frac{g^*(t, u-\delta)}{g^*(t, u)} \leqq k < 1 \qquad \text{for all} \quad (t, u) \in [0, \infty) \times \boldsymbol{R}. \tag{3.19}$$

(i) *In case* $n=2$, *if* (3.6) *holds for all* $c>0$, *then equation* (3.2) *has an eventually positive entire solution* $u(x)$ *satisfying* (3.16).

(ii) *In case* $n \geqq 3$, *if* (3.8) *holds for all* $c>0$, *then equation* (3.2) *has an eventually positive entire solution* $u(x)$ *satisfying* (3.17).

PROOF OF THEOREM 3.3. By (i) of Theorems 2.1 and 2.2, there exists a positive function $y \in C^{2+\theta}_{loc}[0, \infty)$ such that

$$Ly(t) = g_*(t, y(t)), \quad t > 0; \quad y'(0) = 0 \tag{3.20}$$

and

$$\lim_{t\to\infty} \frac{y(t)}{\log t} = \infty \qquad \text{for} \quad n = 2, \quad \lim_{t\to\infty} y(t) = \infty \qquad \text{for} \quad n \geqq 3. \tag{3.21}$$

If $\lambda \in (0, 1)$, then by (3.14) and (3.15)

$$\begin{aligned}\frac{g^*(t, \lambda u)}{g_*(t, u)} &= \frac{g^*(t, \lambda u)}{(\lambda u)^{1+\varepsilon}} \cdot \frac{u^{1+\varepsilon}}{g_*(t, u)} \cdot \lambda^{1+\varepsilon} \\ &\leqq \frac{g^*(t, u)}{u^{1+\varepsilon}} \cdot \frac{u^{1+\varepsilon}}{g_*(t, u)} \cdot \lambda^{1+\varepsilon} \leqq C\lambda^{1+\varepsilon}\end{aligned}$$

for $(t, u) \in [0, \infty) \times (0, \infty)$, so that there is $\lambda \in (0, 1)$ such that

$$g^*(t, \lambda u) \leqq \lambda g_*(t, u), \quad (t, u) \in [0, \infty) \times (0, \infty).$$

Define $z(t) = \lambda y(t)$. Then $z(t)$ satisfies $z'(0) = 0$,

$$Lz(t) = \lambda g_*(t, y(t)) \geqq g^*(t, \lambda y(t)) = g^*(t, z(t)), \quad t > 0$$

and

$$\lim_{t\to\infty} \frac{z(t)}{\log t} = \infty \qquad \text{for} \quad n = 2, \quad \lim_{t\to\infty} z(t) = \infty \qquad \text{for} \quad n \geqq 3.$$

Since $z(t) \leqq y(t)$, $t \geqq 0$, the functions $v(x) = y(|x|)$ and $w(x) = z(|x|)$ are a supersolution and a subsolution of equation (3.2) satisfying (3.5), and the conclusions of the theorem follow from Lemma 3.1.

PROOF OF THEOREM 3.4. From (ii) of Theorems 2.1 and 2.2 there exists an eventually positive solution $y(t)$ of the problem (3.20)–(3.21). Conditions (3.18) and (3.19) imply that

$$\frac{g^*(t, u-N\delta)}{g_*(t, u)} = \frac{g^*(t, u-N\delta)}{g^*(t, u)} \frac{g^*(t, u)}{g_*(t, u)} \leqq Ck^N$$

for $(t, u) \in [0, \infty) \times \boldsymbol{R}$ and $N=1, 2, \dots$, whence there is a constant $\mu>0$ that

$$g^*(t, u-\mu) \leqq g_*(t, u), \quad (t, u) \in [0, \infty) \times \boldsymbol{R}. \tag{3.22}$$

If we define $z(t)=y(t)-\mu$, then in view of (3.22) it is easy to see that $w(x)=z(|x|)$ is a subsolution of (3.2). Since $v(x)=y(|x|)$ is a supersolution of (3.2) and $z(t) \leqq y(t)$, $t \geqq 0$, Lemma 3.1 guarantees the existence of an eventually positive entire solution $u(x)$ of (3.2) satisfying (3.16) or (3.17) according as $n=2$ or $n \geqq 3$.

EXAMPLE 3.1. Consider the equation

$$\Delta u = \phi(x)u^{\gamma}, \quad x \in \boldsymbol{R}^n, \tag{3.23}$$

where $\gamma>1$ and ϕ: $\boldsymbol{R}^n \to (0, \infty)$ is locally Hölder continuous (with exponent $\theta \in (0, 1)$).

Hypotheses (B_1) and (B_2) are satisfied for (3.23) with

$$g(x, u) = \phi(x)u^{\gamma}, \quad g^*(t, u) = \phi^*(t)u^{\gamma}, \quad g_*(t, u) = \phi_*(t)u^{\gamma},$$

where

$$\phi^*(t) = \max_{|x|=t} \phi(x), \quad \phi_*(t) = \min_{|x|=t} \phi(x), \quad t \geqq 0. \tag{3.24}$$

Conditions (3.6) and (3.8) reduce, respectively, to

$$\int_1^{\infty} t(\log t)^{\gamma}\phi^*(t)dt < \infty \tag{3.25}$$

and

$$\int_0^{\infty} t\phi^*(t)dt < \infty. \tag{3.26}$$

Since (3.25) and (3.26) are independent of c, from (i) of Theorems 3.1 and 3.2 we see that, for any given $a>0$, equation (3.23) has a positive entire solution $u(x)$ satisfying (3.7) for $n=2$ or (3.9) for $n \geqq 3$ if (3.25) or (3.26) is satisfied.

Noting that (3.14) for (3.23) is equivalent to

$$\frac{\phi^*(t)}{\phi_*(t)} \leqq C \qquad \text{for} \quad t \geqq 0 \tag{3.27}$$

and that (3.15) is satisfied with $\varepsilon=\gamma-1$, we conclude from Theorem 3.3 that if (3.27) holds for some $C>0$, then (3.25) or (3.26) is a sufficient condition for (3.23) with $n=2$ or $n \geqq 3$ to possess a positive entire solution $u(x)$ which satisfies (3.16) or (3.17).

EXAMPLE 3.2. Consider the equation

$$\Delta u = \phi(x)e^{u}, \quad x \in \boldsymbol{R}^n, \tag{3.28}$$

where ϕ: $\boldsymbol{R}^n \to (0, \infty)$ is locally Hölder continuous (with exponent θ). Hypotheses (B_3) and (B_4) hold with

$$g(x, u) = \phi(x)e^u, \quad g^*(t, u) = \phi^*(t)e^u, \quad g_*(t, u) = \phi_*(t)e^u,$$

where $\phi^*(t)$ and $\phi_*(t)$ are defined by (3.24). In this case condition (3.6) reduces to

$$\int_1^\infty t^{1+c}\phi^*(t)dt < \infty, \tag{3.29}$$

conditions (3.8) and (3.18) become, respectively, to (3.26) and (3.27), and condition (3.19) is automatically satisfied.

From (ii) of Theorem 3.1 it follows that if (3.29) holds for some $c > 0$, then, for every $a \in (0, c]$, equation (3.28) ($n = 2$) has a unique eventually positive entire solution $u(x)$ satisfying (3.7). If, for example, there are constants $k > 0$ and $\ell > 2$ such that

$$\phi(x) \leqq \frac{k}{|x|^\ell} \quad \text{for large} \quad |x|, \tag{3.30}$$

then (3.29) holds for any $c < \ell - 2$, so that for every $a \in (0, \ell - 2)$, there exists a unique entire solution $u(x)$ of (3.28) ($n = 2$) satisfying (3.7). Recently McOwen [15] has show that under (3.30) equation (3.28) ($n = 2$) has an entire solution $u_M(x)$ such that

$$u_M(x) = a \log|x| + O(1) \quad \text{as} \quad |x| \to \infty$$

for every $a \in (0, \ell - 2)$. By uniqueness, his solution coincides with the one constructed by our procedure.

From (i) of Theorem 3.4 it follows that if (3.27) and (3.29) hold for all $c > 0$, then equation (3.28) ($n = 2$) has an eventually positive entire solution $u(x)$ with the property (3.16). Any locally Hölder continuous function $\phi(x)$ on $\boldsymbol{R}^2$ such that

$$K_1 \exp(-|x|^2) \leqq \phi(x) \leqq K_2 \exp(-|x|^2) \quad \text{for} \quad |x| \quad \text{large} \tag{3.31}$$

for some positive constants K_1 and K_2 satisfies the above mentioned requirements. We note that, under (3.31), Ni [18] has proved the existence of an entire solution $u_N(x)$ of (3.28) ($n = 2$) such that

$$u_N(x) = \frac{1}{2}|x|^2 + O(1) \quad \text{as} \quad |x| \to \infty,$$

which is strongly increasing in the sense that $\lim_{|x| \to \infty} u_N(x)/\log|x| = \infty$.

4. Nonexistence of entire solutions

The question of nonexistence of entire solutions for higher order elliptic equations was first studied by Walter [26] and then by Walter and Rhee [28]. To the best of author's knowledge, there is no paper other than [26, 28] which is concerned with this question. Our purpose here is to extend the result of Walter [26] to elliptic equations of the form (A). We first discuss the differential inequality

$$\Delta^m u \geqq p(|x|)f(u), \quad x \in \boldsymbol{R}^n, \tag{4.1}$$

and then establish criteria for the nonexistence of entire solutions of equation (A) with the use of the spherical mean of solutions.

The conditions we assume for (4.1) are as follows:

(C_1) f: $(0, \infty) \to (0, \infty)$ is continuous and nondecreasing.

(C_2) f: $\boldsymbol{R} \to (0, \infty)$ is continuous and nondecreasing.

(C_3) p: $[0, \infty) \to (0, \infty)$ is continuous.

We use the notation:

$$f_0(u) = f(u), \quad f_i(u) = \int_0^u f_{i-1}(\xi)d\xi, \quad 1 \leqq i \leqq 2m-1,$$

that is,

$$f_i(u) = \frac{1}{(i-1)!}\int_0^u (u-\xi)^{i-1} f(\xi)d\xi, \quad 1 \leqq i \leqq 2m-1.$$

We start with the two-dimensional case of (4.1).

THEOREM 4.1. *Let* $m \geqq 2$ *and* $n = 2$. *Suppose that either* $\{(C_1), (C_3)\}$ *or* $\{(C_2), (C_3)\}$ *is satisfied. Suppose moreover that there exists a continuous function* p_*: $[0, \infty) \to (0, \infty)$ *such that*

$$p(t) \geqq p_*(t), \quad t \geqq 0, \tag{4.2}$$

and

(4.3) $t^{-\delta} p_*(t)$ *is nonincreasing on* $(0, \infty)$ *for some* $\delta \geqq 0$.

If

$$\int^\infty [f_{2m-1}(u)]^{-1/(2m)} du < \infty \tag{4.4}$$

and

$$\int^\infty [p_*(t)]^{1/(2m)} dt = \infty, \tag{4.5}$$

then, in case $\{(C_1), (C_3)\}$ *holds,* (4.1) *has no positive radial entire solution, and in case* $\{(C_2), (C_3)\}$ *holds,* (4.1) *has no radial entire solution.*

The following simple lemma, which is a variant of L'Hopsital's rule, is needed in the proof of this theorem.

LEMMA 4.1. *If $g(t)$ and $h(t)$ are continuously differentiable on $[0, \infty)$ and satisfy $\lim_{t\to\infty} h(t)=\infty$ and $g'(t)\geqq Mh'(t)$ for some constant $M>0$ and for all sufficiently large t, then*

$$\liminf_{t\to\infty} \frac{g(t)}{h(t)} \geqq M.$$

PROOF OF THEOREM 4.1. Let $\{(C_1), (C_3)\}$ be satisfied and suppose the existence of a positive entire solution $u(x)=y(|x|)$ of (4.1).

Step 1. Let $T>0$ be fixed. Integrating $t^{-1}(t(L^{m-1}y)'(t))'=L^m y(t)$ repeatedly and taking account of the positivity of the right side of (4.1), we find

$$L^{m-i}y(t) = a_i t^{2i-2}\log t + P_i(t) + \Phi_T^i L^m y(t), \quad t \geqq T, \tag{4.6}$$

for $1\leqq i\leqq m$, where $a_i>0$ are constants, $P_i(t)$ are continuous functions such that $P_i(t)=o(t^{2i-2}\log t)$ as $t\to\infty$ and Φ_T^i is the i-th iterate of $\Phi_T\colon C[T, \infty)\to C^2[T, \infty)$ defined by

$$\Phi_T h(t) = \int_T^t s^{-1}\int_T^s rh(r)drds, \quad t \geqq T, \quad h\in C[T, \infty).$$

An important consequence of (4.6) is that $L^i y(t)$, $(L^i y)'(t)$, $0\leqq i\leqq m-1$, are all eventually positive.

Step 2. We assert that $y''(t)>0$ for all large t. Integrate (4.6) with $i=m-1$ to obtain

$$y'(t) = at^{2m-3}\log t + bt^{2m-3} + ct^{-1} + t^{-1}\int_T^t s[P_{m-1}(s)+\Phi_T^{m-1}L^m y(s)]ds$$

for $t\geqq T$ with some constants $a>0$, b and c, and then differentiate the above to obtain

$$\begin{aligned} y''(t) &= (2m-3)at^{2m-4}\log t + [a+(2m-3)b]t^{2m-4} - ct^{-2} \\ &\quad + P_{m-1}(t) - t^{-2}\int_T^t sP_{m-1}(s)ds + \Phi_T^{m-1}L^m y(t) - t^{-2}\int_T^t s\Phi_T^{m-1}L^m y(s)ds \end{aligned} \tag{4.7}$$

for $t\geqq T$. Noting that

$$P_{m-1}(t) - t^{-2}\int_T^t sP_{m-1}(s)ds = o(t^{2m-4}\log t) \quad \text{as} \quad t\to\infty$$

and

$$\Phi_T^{m-1}L^m y(t) - t^{-2}\int_T^t s\Phi_T^{m-1}L^m y(s)ds \geqq \frac{1}{2}[1+(T/t)^2]\Phi_T^{m-1}L^m y(t)$$

for $t \geqq T$, we see from (4.7) that $y''(t) > 0$ for all sufficiently large t.

Step 3. Let $t_0 > 0$ be such that $y'(t) > 0$ and $y''(t) > 0$ for $t \geqq t_0$. We show that

$$\liminf_{t\to\infty} \frac{\int_{t_0}^{t} s^{\alpha} f_i'(y(s))y'(s)\,ds}{t^{\alpha} f_i(y(t))} \geqq \frac{1}{\alpha+1} \tag{4.8}$$

for any $\alpha \geqq 0$ and $1 \leqq i \leqq 2m-1$. We easily see that for $1 \leqq i \leqq 2m-1$

$$\begin{aligned} f_i(y(t)) &= f_i(y(t_0)) + \int_{t_0}^{t} f_i'(y(s))y'(s)ds \\ &\leqq f_i(y(t_0)) + f_i'(y(t))y'(t)(t-t_0), \quad t \geqq t_0, \end{aligned}$$

which implies

$$\frac{f_i(y(t))}{tf_i'(y(t))y'(t)} \leqq \frac{f_i(y(t_0))}{tf_i'(y(t))y'(t)} + \frac{t-t_0}{t}, \quad t \geqq t_0.$$

Therefore we have

$$\limsup_{t\to\infty} \frac{f_i(y(t))}{tf_i'(y(t))y'(t)} \leqq 1. \tag{4.9}$$

Since

$$\frac{\dfrac{d}{dt}\displaystyle\int_{t_0}^{t} s^{\alpha} f_i'(y(s))y'(s)ds}{\dfrac{d}{dt}[t^{\alpha} f_i(y(t))]} = \left[1 + \frac{\alpha f_i(y(t))}{tf_i'(y(t))y'(t)}\right]^{-1},$$

the desired inequality (4.8) follows from Lemma 4.1 and (4.9).

Step 4. Let t_0 be so large that, in addition to $y'(t)$ and $y''(t)$, $L^i y(t)$ and $(L^i y)'(t)$, $0 \leqq i \leqq m-1$, are positive for $t \geqq t_0$. From (4.1)

$$(t(L^{m-1}y)'(t))' \geqq tp_*(t)f(y(t)), \quad t \geqq t_0.$$

We multiply the above by $ty'(t)$ and integrate it from t_0 to t. Integration by parts and use of the monotonicity of $t^{-\delta}p_*(t)$ then show that

$$t(L^{m-1}y)'(t)\cdot ty'(t) \geqq t^{-\delta}p_*(t)\int_{t_0}^{t} s^{2+\delta} f_1'(y(s))y'(s)ds, \quad t \geqq t_0. \tag{4.10}$$

From Step 3 there exist constants $C_1 > 0$ and $t_1 \geqq t_0$ such that

$$\int_{t_0}^{t} s^{2+\delta} f_1'(y(s))y'(s)ds \geqq C_1 t^{2+\delta} f_1(y(t)), \quad t \geqq t_1,$$

which, combined with (4.10), yields

$$(L^{m-1}y)'(t)\cdot t^2 y'(t) \geqq C_1 t^2 p_*(t) f_1(y(t)), \quad t \geqq t_1.$$

Multiply the above by $y'(t)$, integrate it on $[t_1, t]$ and apply the same argument as in Step 3. We then see that there exist constants $C_2>0$ and $t_2 \geqq t_1$ such that

$$L^{m-1}y(t)\cdot(ty'(t))^2 \geqq C_2t^2p_*(t)f_2(y(t)), \quad t \geqq t_2.$$

Continuing in this manner, we are led to the inequality

$$(4.11)\quad (ty'(t))'\cdot(ty'(t))^{2m-2} \geqq C_{2m-2}t^{2m-1}p_*(t)f_{2m-2}(y(t)), \quad t \geqq t_{2m-2},$$

for some constants $C_{2m-2}>0$ and $t_{2m-2}>0$. Again multiplying (4.11) by $ty'(t)$ and integrating on $[t_{2m-2}, t]$, we obtain after some manipulations

$$(4.12)\qquad y'(t) \geqq C[p_*(t)]^{1/(2m)}[f_{2m-1}(y(t))]^{1/(2m)}, \quad t \geqq t^*,$$

where $C>0$ and $t^* \geqq t_{2m-2}$ are constants, whence we find

$$\begin{aligned} C\int_{t^*}^t [p_*(s)]^{1/(2m)}ds &\leqq \int_{t^*}^t y'(s)[f_{2m-1}(y(s))]^{-1/(2m)}ds \\ &= \int_{y(t^*)}^{y(t)} [f_{2m-1}(u)]^{-1/(2m)}du \\ &\leqq \int_{y(t^*)}^{\infty} [f_{2m-1}(u)]^{-1/(2m)}du, \quad t \geqq t^*. \end{aligned}$$

This leads to a contradiction because of (4.4) and (4.5). Consequently, there is no positive radial entire solution of (4.1).

To complete the proof it suffices to observe that, when (C_2) and (C_3) hold, the relation (4.6) is also satisfied by any possible radial entire solution of (4.1).

We now turn to the higher dimensional case of (4.1). The result below is formally weaker than Theorem 4.1 in that a stronger assumption is needed with respect to $f(u)$.

THEOREM 4.2. *Let* $m \geqq 2$ *and* $n \geqq 3$. *Suppose that either* $\{(C_1), (C_3)\}$ *or* $\{(C_2), (C_3)\}$ *is satisfied. In addition to the same conditions on* $p(t)$ *and* $f(u)$ *as in Theorem* 4.1, *assume that there exists a constant* $k>0$ *such that*

$$(4.13)\qquad f(u) \geqq k \quad \text{for} \quad u \in (0, \infty)$$

in case $\{(C_1), (C_3)\}$ *holds, or*

$$(4.14)\qquad f(u) \geqq k \quad \text{for} \quad u \in \boldsymbol{R}$$

in case $\{(C_2), (C_3)\}$ *holds, and that*

$$(4.15)\qquad \int^{\infty} tp_*(t)dt = \infty.$$

Then, in the case of $\{(C_1), (C_3)\}$, (4.1) *has no positive radial entire solution, and in the case of* $\{(C_2), (C_3)\}$, (4.1) *has no radial entire solution.*

PROOF. A sketch of the proof is given only for the case where $\{(C_1), (C_3)\}$ is satisfied. Suppose to the contrary that there is a positive radial entire solution $u(x)=y(|x|)$ of (4.1). Integrating (4.1), we have in view of (4.13)

$$L^{m-1}y(t) \geqq c_0 + \Phi(p_*f(y))(t) \geqq c_0 + k\Phi p_*(t), \quad t \geqq 0, \tag{4.16}$$

where c_0 is a constant. Since $\Phi p_*(t)\to\infty$ as $t\to\infty$ by (4.15), (4.16) implies the existence of a constant c_1 such that $L^{m-1}y(t)\geqq c_1>0$ for all large t, and hence, there is a constant $c>0$ such that

$$y(t) \geqq ct^{2m-2} \qquad \text{for all large} \quad t. \tag{4.17}$$

Successive integration of $t^{1-n}(t^{n-1}(L^{m-1}y)'(t))'=L^m y(t)$ and use of (4.17) show that

$$L^{m-i}y(t) = a_i t^{2i-2} + P_i(t) + \Phi_T^i L^m y(t), \quad t \geqq T, \quad 1 \leqq i \leqq m, \tag{4.18}$$

provided $T>0$ is sufficiently large, where $a_i>0$ are constants, $P_i(t)$ are continuous functions such that $P_i(t)=o(t^{2i-2})$ as $t\to\infty$, and Φ_T is the operator defined by

$$\Phi_T h(t) = \int_T^t s^{1-n}\int_T^s r^{n-1}h(r)drds, \quad t \geqq T, \quad h \in C[T, \infty).$$

It can be shown that $y''(t)>0$, $L^i y(t)>0$ and $(L^i y)'(t)>0$, $0\leqq i\leqq m-1$, for t sufficiently large, say $t\geqq t_0$ (see Step 2 of the proof of Theorem 4.1).

We now proceed similarly to Step 4 of the proof of Theorem 4.1 starting with the inequality

$$(t^{n-1}(L^{m-1}y)'(t))' \geqq t^{n-1}p_*(t)f(y(t)), \quad t \geqq t_0. \tag{4.19}$$

Namely, we multiply (4.19) by $t^{n-1}y'(t)$, integrate on $[t_0, t]$ and use the nonincreasing nature of $t^{-\delta}p_*(t)$ to obtain

$$t^{n-1}(L^{m-1}y)'(t)\cdot t^{n-1}y'(t) \geqq t^{-\delta}p_*(t)\int_{t_0}^t s^{2(n-1)+\delta}f_1'(y(s))y'(s)ds, \quad t \geqq t_0.$$

Since (4.8) also holds in this case, it follows from the above that

$$(L^{m-1}y)'(t)\cdot t^{2(n-1)}y'(t) \geqq C_1 t^{2(n-1)}p_*(t)f_1(y(t)), \quad t \geqq t_1,$$

for some constants $C_1>0$ and $t_1\geqq t_0$. Next we integrate the above multiplied by $y'(t)$ and transform the resulting inequality by using the decreasing nature of $t^{-\delta}p_*(t)$ and (4.8) again. Continuing in this manner, we arrive at (4.12), from which a contradiction follows as before.

REMARK 4.1. (i) If condition (4.13) [resp. (4.14)] is deleted from

Theorem 4.2, then we assert that, in case $\{(C_1), (C_3)\}$ [resp. $\{(C_2), (C_3)\}$] holds, (4.1) has no radial entire solution which is bounded below by a positive constant [resp. bounded below] in $\boldsymbol{R}^n$, $n \geqq 3$. In fact, if such an entire solution $u(x) = y(|x|)$ exists, then it satisfies $f(y(t)) \geqq k$, $t \geqq 0$, for some constant $k > 0$.

(ii) Moreover if conditions (4.13) [resp. (4.14)] and (4.15) are deleted from Theorem 4.2, then we assert that, in case $\{(C_1), (C_3)\}$ [resp. $\{(C_2), (C_3)\}$] holds, (4.1) has no radial entire solution $u(x) = y(|x|)$ satisfying (4.17) for any $c > 0$. Especially, (4.1) has no strongly increasing radial entire solution.

Nonexistence criteria given in Theorems 4.1 and 4.2 can be slightly improved if (4.1) is specialized to the inequalities

$$\Delta^m u \geqq p(|x|)u^\gamma, \quad x \in \boldsymbol{R}^n, \tag{4.20}$$

$$\Delta^m u \geqq p(|x|)e^u, \quad x \in \boldsymbol{R}^n, \tag{4.21}$$

where $m \geqq 2$, $\gamma > 1$, and p: $[0, \infty) \to (0, \infty)$ is continuous.

Corollary 4.1. *Suppose that there is a continuous function* p_*: $[0, \infty) \to (0, \infty)$ *satisfying* (4.2) *and* (4.3). *Moreover* (4.15) *is added for* $n \geqq 3$.

(i) *If*

$$\int^\infty [p_*(t)]^{1/(2m)}[q_n(t)]^{(\gamma-1)/(2m)-\varepsilon}dt = \infty \qquad \text{for some} \quad \varepsilon \in \left(0, \frac{\gamma-1}{2m}\right), \tag{4.22}$$

where

$$q_2(t) = t^{2m-2}\log t, \quad q_n(t) = t^{2m-2} \qquad \text{for} \quad n \geqq 3,$$

then (4.20) *for* $n=2$ *has no positive radial entire solution, and* (4.20) *for* $n \geqq 3$ *has no radial entire solution which is bounded below by a positive constant.*

(ii) *If*

$$\int^\infty [p_*(t)]^{1/(2m)} \exp [cq_n(t)]dt = \infty \qquad \text{for all} \quad c > 0, \tag{4.23}$$

then (4.21) *for* $n=2$ *has no radial entire solution, and* (4.21) *for* $n \geqq 3$ *has no radial entire solution which is bounded below.*

Proof. If $u(x) = y(|x|)$ is such an entire solution of (4.20) or (4.21), then $y(t)$ satisfies (4.12) (specialized to the case $f(u) = u^\gamma$ or $f(u) = e^u$) for some $C > 0$ and $t^* > 0$ (see Remark 4.1–(i) for $n \geqq 3$).

(i) Rewriting (4.12) (with $f(u) = u^\gamma$) as

$$y'(t) \geqq C[p_*(t)]^{1/(2m)}[y(t)]^{(\gamma-1)/(2m)-\varepsilon}[y(t)]^{1+\varepsilon}, \quad t \geqq t^*,$$

and using the fact that $y(t) \geqq C_1 q_n(t)$, $t \geqq \bar{t}$, for some $C_1 > 0$ and $\bar{t} \geqq t^*$, we have

$$[y(t)]^{-1-\varepsilon}y'(t) \geqq C_2[p_*(t)]^{1/(2m)}[q_n(t)]^{(\gamma-1)/(2m)-\varepsilon}, \quad t \geqq \tilde{t},$$

where $C_2>0$ is a constant. Integrating the above from $\tilde{t}$ to t and letting $t\to\infty$, we obtain a contradiction to (4.22).

(ii) We rewrite (4.12) (with $f(u)=e^u$) as

$$y'(t) \geqq C[p_*(t)]^{1/(2m)}e^{(1/(2m)-\varepsilon)y(t)}\cdot e^{\varepsilon y(t)}, \quad t \geqq t^*,$$

where $\varepsilon\in(0, 1/(2m))$, and use $y(t)\geqq C_1q_n(t)$ for $t\geqq\tilde{t}$ with some $C_1>0$ and $\tilde{t}\geqq t^*$, to obtain

$$e^{-\varepsilon y(t)}y'(t) \geqq C_2[p_*(t)]^{1/(2m)}\exp\left(\left(\frac{1}{2m}-\varepsilon\right)C_1q_n(t)\right), \quad t \geqq \tilde{t},$$

which upon integration yields a contradiction to (4.23). This completes the proof.

Finally, we derive conditions which guarantee the nonexistence of entire solutions for the equation

$$(4.24) \qquad \Delta^m u = g(x, u), \quad x\in \boldsymbol{R}^n,$$

where $g(x, u)$ is continuous either on $\boldsymbol{R}^n\times(0, \infty)$ or on $\boldsymbol{R}^{n+1}$. In what follows I denotes either $(0, \infty)$ or $\boldsymbol{R}$.

THEOREM 4.3. *Let $m\geqq 2$ and suppose that*

$$g(x, u) \geqq p_*(|x|)f(u) \qquad \textit{for} \quad (x, u)\in \boldsymbol{R}^n\times I,$$

where p_*: $[0, \infty)\to(0, \infty)$ is continuous and f: $I\to(0, \infty)$ is continuous, non-decreasing and convex. Suppose moreover that (4.3), (4.4) *and* (4.5) *are satisfied. Then the following statements hold*:

(i) *If $n=2$ and $I=(0, \infty)$, then equation* (4.24) *has no positive entire solution.*

(ii) *If $n=2$ and $I=\boldsymbol{R}$, then equation* (4.24) *has no entire solution.*

(iii) *If $n\geqq 3$ and $I=(0, \infty)$, and if* (4.15) *and* (4.13) *with some $k>0$ hold, then equation* (4.24) *has no positive entire solution.*

(iv) *If $n\geqq 3$ and $I=\boldsymbol{R}$, and if* (4.15) *and* (4.14) *with some $k>0$ hold, then equation* (4.24) *has no entire solution.*

PROOF. Suppose that $n=2$ and $I=(0, \infty)$. Let $u(x)$ be a positive entire solution of (4.24). Let $\bar{u}(t)$ denote the mean value of $u(x)$ over the circle $|x|=t$, $t\geqq 0$:

$$\bar{u}(t) = \frac{1}{2\pi t}\int_{|x|=t}u(x)ds = \frac{1}{2\pi}\int_0^{2\pi}u(t\cos\theta, t\sin\theta)d\theta.$$

By taking the mean value of the inequality

$$\Delta^m u(x) \geqq p_*(|x|)f(u(x)), \quad x \in \boldsymbol{R}^2,$$

over $|x|=t$ and using Jensen's inequality, we see that $\bar{u}(t)$ satisfies the differential inequality

$$L^m \bar{u}(t) \geqq p_*(t)f(\bar{u}(t)), \quad t > 0,$$

and the initial condition

$$(L^i \bar{u})'(0) = 0, \quad 0 \leqq i \leqq m-1,$$

which means that $\bar{u}(|x|)$ is a positive radial entire solution of an inequality of the form (4.1) in $\boldsymbol{R}^2$. This, however, is impossible in view of Theorem 4.1, proving the truth of (i). The remaining statements are proved similarly.

EXAMPLE 4.1. Consider the equations

(4.25) $$\Delta^m u = \phi(x)u^\gamma, \quad x \in \boldsymbol{R}^n,$$

(4.26) $$\Delta^m u = \phi(x)e^u, \quad x \in \boldsymbol{R}^n,$$

where $m \geqq 2$, $\gamma > 1$ and ϕ: $\boldsymbol{R}^n \to (0, \infty)$ is continuous. Assume the existence of a positive continuous function $p_*(t)$ satisfying $\phi(x) \geqq p_*(|x|)$ for $x \in \boldsymbol{R}^n$ and (4.3). Assume moreover that (4.15) holds when $n \geqq 3$. Combining the proofs of Theorem 4.3 and Corollary 4.1, we have the following statements:

(i) If (4.22) holds, then (4.25) for $n=2$ has no positive entire solution, and (4.25) for $n \geqq 3$ has no positive entire solution which is bounded below by a positive constant.

(ii) If (4.23) holds, then (4.26) for $n=2$ has no entire solution, and (4.26) for $n \geqq 3$ has no entire solution which is bounded below.

REMARK 4.2. From the proof of Theorem 4.3 we see that the same conclusion holds for a more general equation

$$\Delta^m u = g(x, u, \Delta u, \dots, \Delta^{m-1}u), \quad x \in \boldsymbol{R}^n, \quad n \geqq 2,$$

if $g(x, u_0, u_1, \dots, u_{m-1})$ is a continuous function such that

$$g(x, u_0, u_1, \dots, u_{m-1}) \geqq p_*(|x|)f(u_0) \quad \text{on} \quad \boldsymbol{R}^n \times I \times \boldsymbol{R}^{m-1}.$$

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References

[1] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, 1965.

[2] N. Fukagai, On decaying entire solutions of second order sublinear elliptic equations, Hiroshima Math. J. **14** (1984), 551–562.

[3] N. Fukagai, Existence and uniqueness of entire solutions of second order sublinear elliptic equations, Funkcial. Ekvac. **29** (1986), 151–165.

[4] N. Kawano, On bounded entire solutions of semilinear elliptic equations, Hiroshima Math. J. **14** (1984), 125–158.

[5] N. Kawano, T. Kusano and M. Naito, On the elliptic equation $\Delta u=\phi(x)u^{\gamma}$ in $\boldsymbol{R}^2$, Proc. Amer. Math. Soc. **93** (1985), 73–78.

[6] I. T. Kiguradze and G. G. Kvinikadze, On strongly increasing solutions of nonlinear ordinary differential equations, Ann. Mat. Pura Appl. **130** (1982), 67–87.

[7] T. Kusano, M. Naito and C. A. Swanson, Radial entire solutions of even order semilinear elliptic equations in the plane, Proc. Royal Soc. Edinburgh, to appear.

[8] T. Kusano, M. Naito and C. A. Swanson, Entire solutions of a class of even order quasilinear elliptic equations, Math. Z., to appear.

[9] T. Kusano, M. Naito and C. A. Swanson, Radial entire solutions of even order semilinear elliptic equations, preprint.

[10] T. Kusano and S. Oharu, Bounded entire solutions of second order semilinear elliptic equations with application to a parabolic initial value problem, Indiana Univ. Math. J. **34** (1985), 85–95.

[11] T. Kusano and S. Oharu, On entire solutions of second order semilinear elliptic equations, J. Math. Anal. Appl. **113** (1986), 123–135.

[12] T. Kusano and C. A. Swanson, Decaying entire positive solutions of quasilinear elliptic equations, Monatsh. Math. **101** (1986), 39–51.

[13] T. Kusano and C. A. Swanson, Positive entire solutions of semilinear biharmonic equations, Hiroshima Math. J., to appear.

[14] T. Kusano and H. Usami, Positive solutions of a class of second order semilinear elliptic equations in the plane, Math. Ann. **268** (1984), 255–264.

[15] R. C. McOwen, On the equation $\Delta u+Ke^{2u}=f$ and prescribed negative curvature in $\boldsymbol{R}^2$, J. Math. Anal. Appl. **103** (1984), 365–370.

[16] R. C. McOwen, Conformal metrics in $\boldsymbol{R}^2$ with prescribed Gaussian curvature and positive total curvature, Indiana Univ. Math. J. **34** (1985), 97–104.

[17] M. Naito, A note on bounded positive entire solutions of semilinear elliptic equations, Hiroshima Math. J. **14** (1984), 211–214.

[18] W.-M. Ni, On the elliptic equation $\Delta u+K(x)e^{2u}=0$ and conformal metrics with prescribed Gaussian curvatures, Invent. Math. **66** (1982), 343–352.

[19] W.-M. Ni, On the elliptic equation $\Delta u+K(x)u^{(n+2)/(n-2)}=0$, its generalizations, and applications in geometry, Indiana Univ. Math. J. **31** (1982), 493–529.

[20] E. S. Noussair and C. A. Swanson, Positive solutions of quasilinear elliptic equations in exterior domains, J. Math. Anal. Appl. **75** (1980), 121–133.

[21] E. S. Noussair and C. A. Swanson, Positive decaying entire solutions of superlinear elliptic equations, preprint.

[22] M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

[23] H. Usami, On the elliptic equation $\Delta u=\phi(x)e^u$ in the plane, Hiroshima Math. J. **14** (1984), 571–574.
[24] H. Usami, On bounded positive entire solutions of semilinear elliptic equations, Funkcial. Ekvac. **29** (1986), 189–195.
[25] H. Usami, Global existence and asymptotic behavior of solutions of second order nonlinear differential equations, J. Math. Anal. Appl., to appear.
[26] W. Walter, Ganze Lösungen der Differentialgleichung $\Delta^p u=f(u)$, Math. Z. **67** (1957), 32–37.
[27] W. Walter, Zur Existenz ganzer Lösungen der Differentialgleichung $\Delta^p u=e^u$, Arch. Math. **9** (1958), 308–312.
[28] W. Walter and H. Rhee, Entire solutions of $\Delta^p u=f(r, u)$, Proc. Royal Soc. Edinburgh **82A** (1979), 189–192.

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