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Maximal ordered fields of rank n

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In this paper, the field of real numbers and the field of rational numbers will be denoted by **R** and **Q** respectively.

We denote an ordered field by (F, σ) or simply F, wheıe σ is an ordering of a field F. For ordered fields (F, σ) and (K, τ), we say that K/F is an extension of ordered fields if K/F is an extension of fields and τ is an extension of σ. Let F be an ordered field. Then the following statements are equivalent:

(1) F is isomorphic to **R**.

(2) F is a complete archimedean ordered field.

(3) F is archimedean and for any archimedean ordered field K, there is an order preserving isomorphism of K with a subfield of F.

(4) F is archimedean and for any proper extension of ordered fields K/F, K is not archimedean.

In §2, we define the *rank* of an ordered field. An ordered field F is archimedean if and only if F is of rank 0, and so **R** is characterized as a maximal ordered field of rank 0. For the case of rank n, $n \geqq 1$, we consider the following three conditions:

(1) F is a complete ordered field of rank n.

(2) F is of rank n and for any ordered field K of rank n, there is an order preserving isomorphism of K with a subfield of F.

(3) F is of rank n and for any proper extension of ordered fields K/F, rank $K > n$.

If $n \geqq 1$, the above three conditions are not equivalent. In fact (2) and (3) are equivalent, and (1) follows from (3) but not conversely. We say that F is a *maximal* ordered field of rank n if F satisfies the condition (3). The purpose of this paper is to show that there is a maximal ordered field of rank n and it is unique up to isomorphism.

§1. The rank of an ordered field

Let F be an ordered field and v be a valuation of F. The valuation ring, the group of units and the maximal ideal of v will be denoted by A, U and M respectively. Then the following statements are equivalent (cf. [5], Theorem 2.3):

(1) If $0 < a$ and $v(a) < v(b)$, then $b < a$.

(2) A is convex.

(3) M is convex.

(4) $|m|<1$ for every $m \in M$.

We say that v is compatible (with respect to the ordering of F) if any one (and hence all) of the conditions holds for v and F. In this case, there exists an ordering of $\bar{F}=A/M$ cannonically induced by the ordering of F, namely $\bar{F}^+=f(F^+ \cap U)$ where f is the cannonical surjection $A \to A/M=\bar{F}$ and $\bar{F}^+$, F^+ are the positive cones of $\bar{F}$ and F respectively. Let k be a subfield of F. Then $A(F, k):=\{a \in F;\ |a|<b$ for some $b \in k\}$ is the smallest convex valuation ring which contains k (cf. [5], Theorem 2.6). $A(F, \mathbf{Q})$ is the smallest convex valuation ring and any convex valuation ring is a localization of $A(F, \mathbf{Q})$.

DEFINITION 1.1. rank $F:=$rank $A(F, \mathbf{Q})$.

An ordered field F is called archimedean if for any positive elements a, b of F, there is a natural number n so that $na>b$. It is equivalent to saying that for any element a of F, there is a rational number r so that $a<r$. Hence, an ordered field is of rank 0 if and only if it is archimedean. Let K/F be an extension of ordered fields. Then the valuation ring $A(K, \mathbf{Q})$ is an extension of $A(F, \mathbf{Q})$. Let A be a convex valuation ring of F. Then $B:=\{k \in K;\ |k|<a$ for some $a \in A\}$ is a convex valuation ring of K and $B \cap F=A$. So rank $K \geqq$ rank F. Suppose F is of finite rank. For any convex valuation ring B of K, $B \cap F$ is a convex valuation ring of F. Hence the following statement holds; rank $K>$rank F if and only if there exist convex valuation rings $B_1 \neq B_2$ of K such that $B_1 \subset B_2$ and $B_1 \cap F= B_2 \cap F$.

PROPOSITION 1.2. *Let F be an ordered field of finite rank n. Let K/F be an extension of ordered fields of finite transcendental degree m. Then $n \leqq$ rank $K \leqq n+m$.*

PROOF. It is sufficient to show the following two statements.

(1) For any algebraic extension of ordered fields K/F, rank $K=n$.

(2) Let $F(x)/F$ be a simple transcendental extension of ordered fields. Then rank $F(x)=n$ or $n+1$.

(1): Since K is algebraic over F, any two extensions of a valuation ring of F are incomparable. Let A be any convex valuation ring of F. Let B_1 and B_2 be extensions of A to K. From the fact that B_1 and B_2 are overrings of $A(K, \mathbf{Q})$, $B_1 \subset B_2$ or $B_1 \supset B_2$. Hence $B_1=B_2$ and we have rank $K=n$.

(2): Let G and G' be the value groups of $A(F, \mathbf{Q})$ and $A(K, \mathbf{Q})$ respectively. Then rational rank $G'/G \leqq 1$, and we have rank $G' \leqq n+1$ ([1], Chapter 6, §10, Corollary 2). Q. E. D.

Let K/F be an algebraic extension of ordered fields of finite rank. Then by Proposition 1.2, rank $F=$rank K, and this shows $A(K, F)=K$. For a simple transcendental extension $F(x)$ of an ordered field F of rank n, it is well known that there is a unique extension of the ordering on F to $F(x)$ for which x is infinitely large (cf. [2]). Let $B=\{b\in F(x);\ |b|\leqq a$ for osme $a\in F\}$. Then B is a convex valuation ring of $F(x)$ which does not contain x, and $B\cap F=F$. So in this case, rank $F(x)=n+1$.

PROPOSITION 1.3. *Let F be an ordered field of finite rank. Let K/F be an extension of ordered fields such that F is dense in K (i.e. for any elements $a<b$ in K, there exists an element c of F such that $a<c<b$). Then* rank $F=$rank K.

PROOF. Suppose rank $F<$rank K. Then there exist convex valuation rings $B_1\neq B_2$ of K such that $B_1\subset B_2$ and $B_1\cap F=B_2\cap F$. There exists a positive element $a\in B_2\setminus B_1$. Put $b=a+1$. Then $b\in B_2\setminus B_1$ and by the assumption, there exists an element c of F such that $a<c<b$. So c is contained in $(B_2\cap F)\setminus(B_1\cap F)$. This contradicts the fact $B_1\cap F=B_2\cap F$. Q. E. D.

$\mathbf{Q}$ is dense in $\mathbf{R}$, so $\mathbf{Q}$ is dense in its real closure. But in general, an ordered field is not necessarily dense in its real closure. We give here such an example. Let $F=\mathbf{R}(x)$ where x is infinitely large and K be a real closure of F. Let y and z be positive elements of K such that $y^2=x$ and $z^4=x$ (y and z are uniquely determined). Suppose that there exists an element $w=af(x)/g(x)$ of F such that $z<w<y$ where a is a positive real number and $f(x)$, $g(x)$ are monic polynomials of $\mathbf{R}[x]$. Then $z^4=x<w^4<y^4=x^2$ and so $xg(x)^4<a^4f(x)^4<x^2g(x)^4$. The inequality $xg(x)^4<a^4f(x)^4$ implies $\deg g(x)<\deg f(x)$ and the inequality $a^4f(x)^4<x^2g(x)^4$ implies $\deg f(x)\leqq\deg g(x)$. This contradiction shows that F is not dense in its real closure.

Let A be a convex valuation ring of F. By Zorn's Lemma, there exists a maximal subfield k contained in A.

LEMMA 1.4. *Let F be an ordered field and (A, M) be a convex valuation ring of F. Let v be a corresponding valuation and k be a subfield of A. Then the following statements hold*:

(1) *The convex valuation ring $A(F, k)$ is contained in A; in particular M is a prime ideal of $A(F, k)$.*

(2) *$A(\bar{F}, k)=A(F, k)/M$, where $\bar{F}=A/M$ and the ordering of $\bar{F}$ is canonically induced by the ordering of F; in particular the residue field of $A(F, \mathbf{Q})$ is an archimedean ordered field.*

PROOF. It is easy to show the assertion (1) (cf. [5], Theorem 2.6). For the assertion (2), let a be any element of $A(F, k)$. There is an element b of k so that $a<b$. Since the ordering of $\bar{F}$ is the induced one, $\bar{a}\leqq\bar{b}$ in $\bar{F}$. This shows that $\bar{a}$ is an element of $A(\bar{F}, k)$. The converse inclusion is proved similarly. Moreover if $A=A(F, \mathbf{Q})$ and $k=\mathbf{Q}$, then $A(\bar{F}, \mathbf{Q})=\bar{F}$ and so $\bar{F}$ is archimedean. Q. E. D.

PROPOSITION 1.5. *Let F be an ordered field of finite rank and (A, M) be a convex valuation ring of F. Let k be a maximal subfield of A. Then the following statements hold*:

(1) $A=A(F, k)$.

(2) *k is algebraically closed in F and the residue field $\bar{F}=A/M$ is algebraic over k.*

(3) $\operatorname{rank} F=\operatorname{rank} A+\operatorname{rank} \bar{F}$.

(4) $\operatorname{rank} k=\operatorname{rank} \bar{F}$.

PROOF. First we show (2). Since A is integrally closed, it is clear that k is algebraically closed in F. Let u be any element of A which is not contained in k. If $k[u] \cap M=\{0\}$, then $k(u) \subseteq A$ and it contradicts the maximality of k. Thus we have the assertion (2). Next we show (1). Since $\bar{F}$ is algebraic over k, $A(\bar{F}, k)=\bar{F}$. On the other hand, by Lemma 1.4, (2), $A(\bar{F}, k)=A(F, k)/M$. Hence $A(F, k)/M=\bar{F}$ and we have $A=A(F, k)$. By Lemma 1.4, $A(\bar{F}, \mathbf{Q})=A(F, \mathbf{Q})/M$ and this fact shows (3). It is clear that $\operatorname{rank} k=\operatorname{rank} \bar{F}$ by Proposition 1.2.

Q. E. D.

REMARK 1.6. Let $F=\mathbf{Q}(\pi)$, an ordered subfield of $\mathbf{R}$, and $K=F(x)$, where x is infinitely large. Let $F_1=\mathbf{Q}(\pi+x^{-1})$ be an ordered subfield of K. Then $FF_1=K$ and $F \cup F_1 \subset A(K, \mathbf{Q})$. We can readily see that both F and F_1 are maximal subfields contained in $A(K, \mathbf{Q})$. So a maximal subfield of a convex valuation ring is not necessarily unique.

PROPOSITION 1.7. *Let F be a real closed field and A be a convex valuation ring of F. Let k be a maximal subfield of A. Then the following statements hold*:

(1) *k is real closed.*

(2) *The cannonical injection $k \rightarrow A/M$ is an order preserving isomorphism.*

(3) *the value group is divisible.*

PROOF. By Proposition 1.5, (2), k is algebraically closed in F and A/M is algebraic over k. Hence k is real closed and $k=A/M$. For any positive element $a \in F$, and any positive integer n, there exists $b \in F$ such that $b^n=a$. This implies that the value group is divisible. Q. E. D.

§ 2. Maximal ordered fields of rank *n*

In [2], a cut (C, D) of an ordered field was defined as follows. If C and D are subsets of an ordered field F, we write $C<D$ if $c<d$ for all $c \in C$ and $d \in D$. We write $C<a$ or $a<D$ instead of $C<\{a\}$ or $\{a\}<D$, respectively. A pair (C, D) of subsets of F is called a cut in F if $F=C \cup D$ and $C<D$. We regard (F, ϕ) and (ϕ, F) as cuts of F.

DEFINITION 2.1. Let (C, D) be a cut of an ordered field F.

(1) We say that (C, D) is *proper* if $C \neq \phi$, $D \neq \phi$ and neither $\max C$ nor $\min D$ exists.

(2) We say that (C, D) is *archimedean* if for any $e \in F$, $e > 0$, there exist elements $c \in C$ and $d \in D$ such that $d - c < e$.

(3) We say that an ordered field is *complete* if it has no proper archimedean cuts.

Let (C, D) be a non-proper cut of an ordered field F. If $C \neq \phi$ and $D \neq \phi$ (i.e. $\max C$ or $\min D$ exists), then it is clear that (C, D) is archimedean.

LEMMA 2.2. *Let L/F be an extensions of ordered fields. For any $b \in L \setminus F$, we put $C_b = \{a \in F; a < b\}$ and $D_b = \{a \in F; b < a\}$. If F is dense in L, then (C_b, D_b) is a proper archimedean cut of F.*

PROOF. It is clear that there are no infinitely large elements, so $C_b \neq \phi$ and $D_b \neq \phi$. Suppose $d := \max C_b$ exists. Then we can readily see that $F < (b-d)^{-1}$, a contradiction. Hence $\max C_b$ does not exist. Similarly, $\min D_b$ does not exist, and so (C_b, D_b) is a proper cut. Next suppose that (C_b, D_b) is not archimedean. Then there is an element $e \in F$, $e > 0$, such that $d - c > e$ for any $c \in C_b$ and $d \in D_b$. Let f be an element of F where $b < f < b + e/2$. Then $f \in D_b$, $f - e/2 \in C_b$ and $f - (f - e/2) < e$. This is a contradiction, and thus the assertion is proved.

Q. E. D.

Let F_c be the set of archimedean cuts of F such that lower subsets have no greatest elements. Then we can consider F_c as an ordered field and F is an ordered subfield of F_c. In fact the definitions and the proofs are essentially the same as those used in the case of ordinary Dedekind cuts in the development of the real numbers. It is easy to show that F_c is complete and F is dense in F_c.

PROPOSITION 2.3. *For an order field F, the following statements are equivalent:*

(1) *F is complete.*

(2) *For any extension K/F of ordered fields, if F is dense in K, then $F = K$.*

PROOF. Suppose F is complete. Let K/F be an extension of ordered fields such that F is dense in K. If there is an element $b \in K \setminus F$, we put $C_b = \{a \in F; a < b\}$ and $D_b = \{a \in F; b < a\}$. Since F is dense in K, (C_b, D_b) is a proper archimedean cut of F by Lemma 2.2. This contradicts the fact that F is complete, so $F = K$. Conversely if F is not complete, then F_c/F is a proper extension of ordered fields and F is dense in F_c. This shows the assertion (2)$\Rightarrow$(1). Q. E. D.

PROPOSITION 2.4. *Let K/F be an extension of ordered fields. If F is dense in K, then the following statements are equivalent:*

(1) *K is complete.*

(2) *For any extension L/F of ordered fields, if F is dense in L, then there exists a unique order preserving injection $L \to K$ over F.*

PROOF. (1)$\Rightarrow$(2): Let S be the set of all proper archimedean cuts of F. First we show that the map ψ: $K \setminus F \to S$ defined by $\psi(b) = (C_b, D_b)$ (the cut (C_b, D_b) was defined in the proof of Proposition 2.3) is a bijection. Since F is dense in K, it is clear that ψ is injective. Let (C, D) be any proper archimedean cut of F. We put $C_K = \{b \in K;\ b \leqq c \text{ for some } c \in C\}$ and $D_K = \{b \in K;\ b \geqq d \text{ for some } d \in D\}$. If $C_K \cup D_K = K$, then it is easy to show that (C_K, D_K) is a proper archimedean cut of K. It contradicts the fact that K is complete. Hence $K \setminus C_K \cup D_K$ is not empty. Since F is dense in K, $K \setminus C_K \cup D_K$ consists of one element. We put $K \setminus C_K \cup D_K = \{b\}$. Then $\psi(b) = (C, D)$, and so ψ is surjective. We now define the map f: $L \to K$. If $b \in F$, then we let $f(b) = b$ and if $b \in L \setminus F$, we let $f(b) = \psi^{-1}((C_b, D_b))$; here note that since F is dense in L, the cut (C_b, D_b) of F defined in the proof of Proposition 2.3 is proper archimedean. It is clear that f is an order preserving injection $L \to K$ over F. Next we show the uniqueness of f. Let f_1 and f_2 be order preserving injections $L \to K$ over F. Suppose that there exists an element $b \in L$ such that $f_1(b) \neq f_2(b)$. We may assume $f_1(b) < f_2(b)$. From the fact that F is dense in K, it follows that there is an element $a \in F$ with $f_1(b) < a < f_2(b)$. Since f_1 and f_2 are order preserving injections $L \to K$ over F, we have $b < a < b$, a contradiction. So $f_1 = f_2$.

(2)$\Rightarrow$(1): If K is not complete, then $K_c \neq K$. Since F is dense in K_c, there exists an order preserving injection f: $K_c \to K$ over F. From the uniqueness, the restriction map $f|K$: $K \to K$ is identity and we get a contradiction. Q.E.D.

DEFINITION 2.5. Let K/F be an extension of ordered fields. Suppose F is dense in K. If the statement (2) in Proposition 2.4 holds, we say that K is the *completion* of F.

It is clear that the completion of F is unique up to isomorphism as an ordered overfield of F. By Proposition 2.4, F_c is the completion of F.

REMARK 2.6. In general a complete ordered field is not real closed. Let $\mathbf{R}(x)$ be the ordered field where $\mathbf{R} < x$. Let K be the real closure of $\mathbf{R}(x)$ and F be the completion of $\mathbf{R}(x)$. If F is real closed, then K is regarded as a subfield of F and so $\mathbf{R}(x)$ is dense in K. But we remarked that, after Proposition 1.3, $\mathbf{R}(x)$ is not dense in K and this contradiction shows that F is not real closed.

DEFINITION 2.7. We say that F is a *maximal* ordered field of rank n if rank $F = n$ and for any proper extension K/F of ordered fields, rank $K > n$.

Let K/R be an extension of ordered fields. Suppose that K is archimedean.

Then there is an order preserving isomorphism of K with a subfield of $\mathbf{R}$. If the extension $K/\mathbf{R}$ is proper, then $\mathbf{R}$ is siomorphic with a proper subfield of $\mathbf{R}$, a contradiction. Therefore $K=\mathbf{R}$ and so $\mathbf{R}$ is a maximal ordered field of rank 0. By Proposition 1.2, a maximal ordered field of rank n is real closed.

PROPOSITION 2.8. *A maximal ordered field F of finite rank is complete.*

PROOF. Let K be the completion of F. Since F is dense in K, rank $K=$ rank F by Proposition 1.3, and so $F=K$. Q.E.D.

CORORALLY 2.9. *A maximal ordered field of finite rank has no proper archimedean cuts.*

PROPOSITION 2.10. *For an ordered field F of rank n, the following statements are equivalent:*

(1) *F is a maximal ordered field of rank n.*

(2) *F is real closed and* rank $F(x)>n$ *for any ordering of $F(x)$ which is an extension of the ordering of F, where x is a variable.*

PROOF. (1)$\Rightarrow$(2) is trivial. (2)$\Rightarrow$(1): Let K/F be an extension of ordered fields. We fix an element $y\in K\setminus F$. Since F is real closed, y is transcendental over F. By the assumption, rank $F(y)>n$ and we have rank $K>n$. Q.E.D.

PROPOSITION 2.11. *Let (F, σ) be an ordered field and v be a compatible valuation of F. Let K be an extension of F and v' be a valuation of K which is an extension of v. Let $\bar{F}$ and $\bar{K}$ be the residue fields of v and v' respectively. We denote by $\bar{\sigma}$ the ordering of $\bar{F}$ induced by σ. Let τ be an ordering of K such that v' is compatible with τ. Suppose that $\bar{\tau}$, the ordering of $\bar{K}$ induced by τ, is an extension of $\bar{\sigma}$ and $v(\dot{F})=2v(\dot{F})$. Then K/F is an extension of ordered fields (i.e. τ is an extension of σ).*

PROOF. Let μ be the restriction of τ to F. It is easy to show that v is compatible with μ. By the assumption, we have $\bar{\mu}=\bar{\sigma}$. Since the value group of v is two divisible, there is a cannonical one to one correspondence between the set of orderings of F such that v is compatible with them and the set of orderings of $\bar{F}$ (cf. [6], §7). So we have $\mu=\sigma$. Q.E.D.

REMARK 2.12. Let F be a field and v a valuation of F with the residue field $\bar{F}$. It was shown in [6], §7 that for any ordering τ of $\bar{F}$, there exists an ordering σ of F, with which v is compatible, such that $\bar{\sigma}=\tau$.

PROPOSITION 2.13. *Let F be a maximal ordered field of rank n. Let M be the maximal ideal of $A(F, \mathbf{Q})$. Then the following statements hold:*

(1) $A(F, \mathbf{Q})/M=\mathbf{R}$.

(2) $v(\dot{F})=\mathbf{R}\times\cdots\times\mathbf{R}$, *a product of* n *copies of* $\mathbf{R}$, *where* v *is the valuation of* $A(F, \mathbf{Q})$.

PROOF. We put $A:=A(F, \mathbf{Q})$. Since A/M is an archimedean ordered field, A/M is a subfield of $\mathbf{R}$. First we show that $A/M=\mathbf{R}$. Suppose on the contrary that there is an element r of $\mathbf{R}$ which is not contained in A/M. By Proposition 1.7, r is transcendental over A/M. By [1], Chapter 6, §10, Proposition 2, there is a valuation v' of a simple transcendental extension $F(x)$ over F, which is an extension of v, such that the value groups of v and v' coincide and the residue field of v' is $(A/M)(r)$. Let τ be an ordering of $F(x)$ such that v' is compatible with τ and $\bar{\tau}$ coincides with the ordering of $(A/M)(r)$ (cf. Remark 2.12). By virtue of Proposition 2.11, $(F(x), \tau)/F$ is an extension of ordered fields and by Proposition 1.5, rank $F(x)=n$. This contradicts the maximality of F. So $A/M=\mathbf{R}$. Since the rank of the valuation ring $A(F, \mathbf{Q})$ equals n, $v(\dot{F})\subset\mathbf{R}\times\cdots\times\mathbf{R}$ (cf. [7], A). Suppose that $v(\dot{F})\neq\mathbf{R}\times\cdots\times\mathbf{R}$. We take $\xi\in\mathbf{R}\times\cdots\times\mathbf{R}\setminus v(\dot{F})$. Since the value group is divisible by Proposition 1.7, $\mathbf{Z}\xi\cap v(\dot{F})=\{0\}$. By [1], Chapter 6, §10, Proposition 1, there is a valuation v' of $F(x)$ such that the residue fields of v and v' coindice and the value group of v' is $\mathbf{Z}\xi+v(\dot{F})$. Similarly to the former case, we can get a contradiction. Q. E. D.

§3. Main Theorem

In this section we show the existence and the uniqueness of a maximal ordered field of rank n for any positive integer n.

Let F be a field and G be an ordered group. We let $F((x))^G$ stand for the formal power series field, with coefficients in F and exponents in G. An element of $F((x))^G$ is $s=\sum s_g x^g$, $g\in G$, where $\operatorname{supp}(s)=\{g\in G;\ s_g\neq 0\}$ is well ordered. Let $o(s)$ be the initial element of $\operatorname{supp}(s)$ and let v be the valuation of $F((x))^G$ which is defined by $v(s)=o(s)$. Then it is clear that the value group and the residue field of v are G and F respectively. We say that v is the cannonical valuation of $F((x))^G$. Suppose that F is an ordered field with the ordering σ. Then there is an ordering τ of $F((x))^G$ such that the cannonical valuation v is compatible with τ and σ is the restriction of τ. An ordering τ is uniquely determined if the value group G is two divisible (cf. Remark 2.12).

PROPOSITION 3.1. *Let* n *be a non-negative integer. Then there exists a cardinal number* $c(n)$ *such that* $|F|<c(n)$ *for any ordered field* F *of rank* n, *where* $|F|$ *is the cardinal number of the set* F.

PROOF. Let F be an ordered field of rank n and let v be the valuation of F with the valuation ring $A(F, \mathbf{Q})$. Let $\bar{F}$ and G be the residue field and the value group of v, respectively. It is well known that $|F|\leqq|S|$ where $S=\bar{F}((x))^G$ (cf.

[7], p. 80, Lemme 1). Since $G \subset \mathbf{R} \times \cdots \times \mathbf{R}$ and $\bar{F} \subset \mathbf{R}$, we have $|F| \leqq |T|$ where $T = \mathbf{R}(x))^H$, $H = \mathbf{R} \times \cdots \times \mathbf{R}$, a product of n copies of $\mathbf{R}$. So the assertion is proved. Q. E. D.

We borrow the following results on a maximal valued field from Krull [3] and Kaplansky [2].

PROPOSITION 3.2.

(1) *For any valued field F, there exists a maximal immediate extension of F. If the residue field has characteristic zero, then it is unique up to isomorphism as valued fields.*

(2) *For any field k and any ordered group G, the formal power series field $k((x))^G$ is a maximal valued field with respect to the canonical valuation.*

Let $G = \mathbf{R} \times \cdots \times \mathbf{R}$ be a product of n copies of $\mathbf{R}$. As we remarked, there is a unique ordering σ in $\mathbf{R}((x))^G$ with which the cannonical valuation is compatible. By Proposition 1.5, $\mathbf{R}((x))^G$ is of rank n.

PROPOSITION 3.3. *In the above situation, $\mathbf{R}((x))^G$ is a maximal ordered field of rank n.*

PROOF. Let $K/\mathbf{R}((x))^G$ be an extension of ordered fields. Suppose rank $K = n$. Let v be the cannonical valuation of $\mathbf{R}((x))^G$ and v' be the valuation of K with the valuation ring $A(K, \mathbf{Q})$. It is clear that the valuation ring of v is $A(\mathbf{R}((x))^G, \mathbf{Q})$. So v' is an extension of v. By Lemma 1.4, the residue field of v' is archimedean and this implies that the residue fields of v and v' coincide. Since rank $K = n$ and $G = \mathbf{R} \times \cdots \times \mathbf{R}$, the value groups of v and v' also coincide. Therefore we see that K is an immediate extension of $\mathbf{R}((x))^G$. Hence $K = \mathbf{R}((x))^G$ by Proposition 3.2. This shows that $\mathbf{R}((x))^G$ is a maximal ordered field of rank n. Q. E. D.

PROPOSITION 3.4. *Let F be a maximal ordered field of finite rank and v be any compatible valuation of F. Then F is a maximal valued field with respect to the valuation v.*

PROOF. Let K be a maximal immediate extension of F. Let $\bar{F}$ be the residue field of F (and K). Let σ be the ordering of F and $\bar{\sigma}$ be the induced ordering of $\bar{F}$ by σ. There is an ordering τ of K so that the valuation of K is compatible with τ and $\bar{\sigma}$ is induced by τ (cf. Remark 2.12). By Proposition 2.11, τ is an extension of σ and any Proposition 1.5, rank $(K, \tau) =$ rank F. So $K = F$. Q. E. D.

THEOREM 3.5. *For any ordered field (F, σ) of rank n, there exists an extension K/F of ordered fields such that K is a maximal ordered field of rank n.*

PROOF. Let L be an algebraically closed field which is an extension of F and

$|L|=c(n)$ (the cardinal number $c(n)$ is defined in Proposition 3.1). Let S be the set of ordered fields (T, σ_T) of rank n such that $F\subseteq T\subseteq L$ and $\sigma_T|F=\sigma$. For (T, σ_T) and $(U, \sigma_U)\in S$, we write $(T, \sigma_T)\leqq(U, \sigma_U)$ if $T\subseteq U$ and $\sigma_U|T=\sigma_T$. Then S is an inductive set. Let (K, σ_K) be a maximal pair of S. It is clear that (K, σ_K) is real closed. Since $|K|<c(n)$ by Proposition 3.1, there exists a simple transcendental extension $K(x)$ over K in L. By the maximality of (K, σ_K), rank $K(x)$ $>n$ for any ordering τ of $K(x)$ such that $\tau\,|\,K=\sigma_K$. Hence by Proposition 2.10, K is a maximal ordered field of rank n. Q. E. D.

LEMMA 3.6. *For any maximal ordered field of finite rank, there is an ordered subfield which is isomorphic to* $\mathbf{R}$.

PROOF. Let F be a maximal ordered field of finite rank. Then by Proposition 2.13, $A(F, \mathbf{Q})/M=\mathbf{R}$. Let k be a maximal subfield of $A(F, \mathbf{Q})$. Then by Proposition 1.7 (2), k is isomorphic to $\mathbf{R}$. Q. E. D.

THEOREM 3.7. *A maximal ordered field F of rank n is isomorphic to* $\mathbf{R}((x))^G$ *where* $G=\mathbf{R}\times\cdots\times\mathbf{R}$, *a product of n copies of* $\mathbf{R}$.

PROOF. Let $A=A(F, \mathbf{Q})$ and v be the corresponding valuation. We denote by M and G the maximal ideal and the value group of v respectively. By Proposition 2.13, $A/M=\mathbf{R}$ and $G=\mathbf{R}\times\cdots\times\mathbf{R}$, a product of n copies of $\mathbf{R}$. G is a vector space over $\mathbf{Q}$. Let $\{g_i\}$, $i\in I$, be a basis of G over $\mathbf{Q}$. For each g_i, we fix an element $t_{g_i}\in F$ so that $t_{g_i}>0$ and $v(t_{g_i})=g_i$. Let $r=h/m\in\mathbf{Q}$, $m>0$. Since t^h_{gi} is positive and F is real closed, the equation $x^m-t^h_{gi}=0$ has a unique root $(t^h_{gi})^{1/m}$ in F. We put $t_{rg_i}=(t^h_{gi})^{1/m}$ and for $s\in G$, we also put $t_s=\Pi t_{r_ig_i}$, where $s=\Sigma r_ig_i$, $r_i\in\mathbf{Q}$.

It is clear that $t_st_{s'}=t_{s+s'}$ for any $s, s'\in G$. By Lemma 3.6, there is a subfield of F, which is isomorphic to $\mathbf{R}$. Therefore we can identify $\mathbf{R}$ with a subfield of F. We consider the subfield $K=\mathbf{R}(t^s)$, $s\in G$, of F, and the subfield $E=\mathbf{R}(x)^G$, $s\in G$, of $\mathbf{R}((x))^G$. It is easy to show that there exists an isomorphism f: $K\rightarrow E$ over $\mathbf{R}$ such that $f(t_s)=x^s$. F is a maximal immediate extension of K by Proposition 3.4. So by Proposition 3.2, there is an isomorphism of F with $\mathbf{R}((x))^G$. Since F and $\mathbf{R}((x))^G$ are real closed, this is an order preserving isomorphism. Q. E. D.

CORORRLLY 3.8. *Let F be a maximal ordered field of rank n. For any ordered field K of rank n, there is an order preserving iosmorphism of K with an ordered subfield of F.*

PROOF. A maximal ordered extension of K is isomorphic to $F=\mathbf{R}((x))^G$ and the assertion is clear. Q. E. D.

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