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Infinite-dimensional algebraic and splittable Lie algebras

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About forty years ago the theory of algebraic Lie algebras of endomorphisms of a finite-dimensional vector space had been developed mainly by C. Chevalley in his works [2, 3, 4, 5] and the theory of splittable Lie algebras of endomorphisms of a finite-dimensional vector space had been developed by the present author in his paper [13]. On the other hand, recently the classical structure theorems of finite-dimensional Lie algebras were extended to a certain kind of locally finite Lie algebras by I. Stewart in his works [1, 11, 12].

In this paper, in connexion with the extended structure theorems we shall generalize the theories of algebraic and splittable Lie algebras to a kind of locally finite Lie algebras of endomorphisms of a not necessarily finite-dimensional vector space.

Let V be a not necessarily finite-dimensional vector space over an algebraically closed field $\mathfrak{k}$ of characteristic 0. For an algebraic endomorphism f of V we consider the Chevalley-Jordan decomposition $f=f_s+f_n$ and the rational decomposition $f_s=\sum \xi_\mu f_{s\mu}$, where $\{\xi_\mu\}$ is a basis of $\mathfrak{k}$ over the prime field. For a Lie algebra L of endomorphisms of V of finite rank we call L splittable (resp. algebraic) if with any element f of L f_s (resp. each $f_{s\mu}$) belongs to L. We shall observe the splittable hull $\hat{L}$ and the algebraic hull $\tilde{L}$ of L and show that $L^2=\hat{L}^2=\tilde{L}^2$ (Theorem 4.6). By making use of a known result on Lie algebras consisting of nilpotent endomorphisms of a finite-dimensional vector space, we shall show that L is splittable (resp. algebraic) if and only if L has a splittable (resp. an algebraic) system of generators (Theorem 6.4). We shall also show that L^2 is always algebraic (Theorem 6.7). Finally we shall generalize several known structure theorems of splittable (resp. algebraic) Lie algebras in [3, 7, 13] to ideally finite splittable (resp. algebraic) Lie algebras of endomorphisms of V (Theorems 7.2, 7.9 and 7.10).

§ 1. Preliminaries

Let L be a not necessarily finite-dimensional Lie algebra over a field $\mathfrak{k}$.

We write $H \leq L$ when H is a subalgebra of L and $H \lhd L$ when H is an ideal of L. We denote by $\zeta(L)$ the center of L.

Let λ be an ordinal. A subalgebra H of L is a λ-step ascendant subalgebra of L, denoted by $H \lhd^\lambda L$, if there exists a series $\{H_\alpha | \alpha \leq \lambda\}$ of subalgebras of L such that

(1) $H_0 = H$ and $H_\lambda = L$,

(2) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \lambda$,

(3) $H_\beta = \bigcup_{\alpha<\beta} H_\alpha$ for any limit ordinal $\beta \leq \lambda$.

H is an ascendant subalgebra of L, denoted by $H \operatorname{asc} L$, if $H \lhd^\lambda L$ for some oridnal λ. $\{H_\alpha | \alpha \leq \lambda\}$ is called an ascending series from H to L. Especially when $\lambda = n < \omega$, H is respectively an n-step subideal and a subideal of L, denoted by $H \operatorname{si} L$.

For a totally ordered set Σ, H is a serial subalgebra (of type Σ) of L, denoted by $H \operatorname{ser} L$, if there exists a collection $\{\Lambda_\sigma, V_\sigma | \sigma \in \Sigma\}$ of subalgebras of L such that

(1) $H \leq \Lambda_\sigma$ and $H \leq V_\sigma$ for all $\sigma \in \Sigma$,

(2) $\Lambda_\tau \leq V_\sigma \leq \Lambda_\sigma$ if $\tau < \sigma$,

(3) $L \backslash H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(4) $V_\sigma \lhd \Lambda_\sigma$ for any $\sigma \in \Sigma$.

Then an ascendant subalgebra of L is a serial subalgebra of L.

A class of Lie algebras is a collection of Lie algebras over $\mathfrak{k}$ together with their isomorphic copies and the 0-dimensional Lie algebra. We denote by $\mathfrak{F}$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$, $\mathrm{L}\mathfrak{F}$, $\mathrm{L}\mathfrak{N}$ and $\mathrm{LE}\mathfrak{A}$ the classes of finite-dimensional, abelian, nilpotent, soluble, locally finite, locally nilpotent and locally soluble Lie algebras over $\mathfrak{k}$ respectively.

Let Δ be any one of the relations $\lhd$ and ser. For a class $\mathfrak{X}$ of Lie algebras, $\mathrm{L}(\Delta)\mathfrak{X}$ is the collection of Lie algebras L such that any finite subset of L lies inside a subalgebra H of L satisfying $H \Delta L$ and belonging to $\mathfrak{X}$. Lie algebras belonging to $\mathrm{L}(\lhd)\mathfrak{F}$ and $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ are respectively called ideally finite and serially finite.

From now on let the basic field $\mathfrak{k}$ be of characteristic 0, unless otherwise specified.

It is known [14] that the serially finite Lie algebras coincide with the neoclassical Lie algebras in the sense of [1]. Hence we have the following results by [1, 11, 12].

Radicals. For a locally finite Lie algebra L, we denote by $\rho(L)$ and $\sigma(L)$ the largest locally nilpotent and the largest locally soluble ideals of L respectively.

(1.1) *Let L be locally finite and let $H \operatorname{ser} L$. Then $\rho(H) = H \cap \rho(L)$ and $\sigma(H) = H \cap \sigma(L)$.*

(1.2) *Let L be ideally finite and let $\{F_\lambda | \lambda \in \Lambda\}$ be the collection of finite-dimensional ideals of L. Then $\rho(L) = \sum_{\lambda \in \Lambda} \rho(F_\lambda)$.*

Semisimplicity. A locally finite Lie algebra L is called semisimple if $\sigma(L) = 0$.

(1.3) *Let L be serially finite. L is semisimple if and only if L is a direct sum of finite-dimensional non-abelian simple ideals. Then such a direct sum decomposition is unique.*

(1.4) *Let L be serially finite. If L is semisimple, every ideal of L is a direct summand of L and is semisimple.*

Levi subalgebras. A subalgebra Λ of a locally finite Lie algebra L is called a Levi subalgebra of L if $L=\sigma(L)+\Lambda$ with $\sigma(L)\cap\Lambda=0$.

(1.5) *Every serially finite Lie algebra has a Levi subalgebra.*

Borel subalgebras. For a locally finite Lie algebra L, a maximal locally soluble subalgebra of L is called a Borel subalgebra of L.

(1.6) *For an ideally finite Lie algebra L, any Borel subalgebra of L contains $\sigma(L)$.*

(1.7) *Let L be ideally finite. A subalgebra B of L is a Borel subalgebra of L if and only if, for the decomposition $\Lambda=\oplus_\mu \Lambda_\mu$ in* (1.3) *of a Levi subalgebra Λ of L, $B=\sigma(L)+(\oplus_\mu B_\mu)$ where each B_μ is a Borel subalgebra of Λ_μ.*

Cartan subalgebras. A subalgebra C of L is called a Cartan subalgebra of L if C is locally nilpotent and C equals the idealizer of C in L.

(1.8) *Every ideally finite Lie algebra has a Cartan subalgebra.*

(1.9) *Let C be a Cartan subalgebra of an ideally finite Lie algebra L. Then C is a maximal locally nilpotent subalgebra of L. For an ideal H of L, $(C+H)/H$ is a Cartan subalgebra of L/H.*

(1.10) *Let L be ideally finite. Then a Cartan subalgebra of a Borel subalgebra of L is a Cartan subalgebra of L.*

(1.11) *Let L be a locally soluble, ideally finite Lie algebra. Then a subalgebra C of L is a Cartan subalgebra of L if and only if C is a maximal locally nilpotent subalgebra of L and $L=\rho(L)+C$.*

L-modules. Let L be a Lie algebra over a field $\mathfrak{k}$ of arbitrary characteristic and let V be an L-module. Then the following result can be shown as in [9].

(1.12) *For an L-module V, the following conditions are equivalent*:

(1) *V is a sum of irreducible submodules.*

(2) *V is completely reducible.*

(3) *For any submodule U of V, there exists a submodule U' of V such that $V=U\oplus U'$.*

V is called locally finite if any finite subset of V lies inside a finite-dimensional submodule of V.

§ 2. Semisimple and nil endomorphisms

From now on let $\mathfrak{k}$ be an algebraically closed field of characteristic 0. We identify the prime field of $\mathfrak{k}$ with the field $\boldsymbol{Q}$ of rational numbers and take a basis $\{\xi_\mu | \mu \in M\}$ of $\mathfrak{k}$ over $\boldsymbol{Q}$ containing $\xi_0 = 1$.

Let V be a vector space over $\mathfrak{k}$ which is not necessarily finite-dimensional. The set End V of endomorphisms of V is a Lie algebra with commutator product, which we denote by [End V]. Let $f \in \mathrm{End}\, V$. Then V is an $\langle f \rangle$-module. For $\alpha \in \mathfrak{k}$, put

$$V_\alpha = \{v \in V | v(f-\alpha)^n = 0 \text{ for some } n\}.$$

LEMMA 2.1. *If V is locally finite as an $\langle f \rangle$-module, then $V = \oplus_\alpha V_\alpha$.*

PROOF. For any finite-dimensional submodule U of V, it is known that $U = \oplus_\alpha U_\alpha$ where each α is an eigenvalue of $f|_U$. Denoting by A the set of eigenvalues of f, we have $V = \sum_{\alpha \in A} V_\alpha$. It follows that $V = \oplus_{\alpha \in A} V_\alpha$.

LEMMA 2.2. *Let W be an f-invariant subspace of V. Then for $\alpha \in \mathfrak{k}$*

a) $W_\alpha = W \cap V_\alpha$.

b) *If $V = \oplus_\alpha V_\alpha$, then $(V/W)_\alpha = (V_\alpha + W)/W$.*

PROOF. a) is evident and b) follows from

$$V/W = \textstyle\sum_\alpha (V_\alpha + W)/W \subseteq \sum_\alpha (V/W)_\alpha = \oplus_\alpha (V/W)_\alpha \subseteq V/W.$$

f is called semisimple if V has a basis consisting of eigenvectors of f. f is called nil if for any $v \in V$ there exists an integer $n = n(v) > 0$ such that $vf^n = 0$. We call f rationally semisimple if f is semisimple and all eigenvalues of f belong to $\boldsymbol{Q}$.

It is immediate that if f is semisimple then V is a locally finite $\langle f \rangle$-module.

LEMMA 2.3. *Let f be semisimple. Then for any eigenvalue α of f, V_α consists of eigenvectors of f corresponding to α.*

PROOF. Let A be the set of eigenvalues of f and for $\alpha \in A$ let $\tilde{V}_\alpha$ be the eigenspace of f corresponding to α. Then $\tilde{V}_\alpha \subseteq V_\alpha$ and therefore $V = \sum_{\alpha \in A} \tilde{V}_\alpha \subseteq \oplus_{\alpha \in A} V_\alpha = V$ by Lemma 2.1. Hence $\tilde{V}_\alpha = V_\alpha$.

LEMMA 2.4. *Let W be an f-invariant subsapce of V and denote by $\bar{f}$ the endomorphism of V/W induced by f. If f is semisimple (resp. rationally semisimple, nil, nilpotent), then so are $f|_W$ and $\bar{f}$.*

PROOF. Let f be semisimple. Then by Lemmas 2.1 and 2.2,

$$V = \oplus_\alpha V_\alpha, \quad W = \oplus_\alpha (W \cap V_\alpha), \quad V/W = \oplus_\alpha (V_\alpha + W)/W.$$

By Lemma 2.3 $W \cap V_\alpha$ and $(V_\alpha + W)/W$ respectively consists of eigenvectors of $f|_W$ and $\bar{f}$ corresponding to α. Therefore $f|_W$ and $\bar{f}$ are semisimple. The case that f is rationally semisimple is similarly shown and the other cases are evident.

LEMMA 2.5. *Let* $f, g \in \operatorname{End} V$ *and assume that* $fg=gf$. *If* f *and* g *are semisimple (resp. rationally semisimple, nil, nilpotent), then so is* $f+g$.

PROOF. Let f and g be semisimple. Then by Lemmas 2.1 and 2.3, $V=\oplus_\alpha V_\alpha$ where each V_α consists of eigenvectors of f corresponding to α. Since $fg=gf$, V_α is g-invariant and by Lemma 2.4 $g|_{V_\alpha}$ is semisimple. Hence by Lemmas 2.1 and 2.3, $V_\alpha=\oplus_\beta V_{\alpha\beta}$ where each $V_{\alpha\beta}$ consists of eigenvectors of $g|_{V_\alpha}$ corresponding to β. It follows that any element of $V_{\alpha\beta}$ is an eigenvector of $f+g$ corresponding to $\alpha+\beta$. Hence $f+g$ is semisimple. The case that f and g are rationally semisimple is similarly shown and the other cases are evident.

§3. Chevalley-Jordan and rational decompositions

Let $f\in \operatorname{End} V$. If f is uniquely expressed in the form

$$f=f_s+f_n \tag{1}$$

where f_s is a semisimple element of End V, f_n is a nil element of End V and $f_sf_n=f_nf_s$, then (1) is called the Chevalley-Jordan decomposition of f. f_s and f_n are respectively called the semisimple and the nil parts of f.

It is shown in Proposition 3.1 that f_s is uniquely expressed in the form

$$f_s = \sum_{\mu\in M} \xi_\mu f_{s\mu} \tag{2}$$

where each $f_{s\mu}$ is a rationally semisimple element of End V and $f_{s\mu}f_{s\nu}=f_{s\nu}f_{s\mu}$ for any $\mu, \nu\in M$. Here by $f_s=\sum_{\mu\in M}\xi_\mu f_{s\mu}$ we mean that for each $v\in V$ $vf_{s\mu}=0$ except a finite number of $\mu\in M$, that is, vf_s is a finite sum $v(\sum_{i=1}^n \xi_{\mu_i} f_{s\mu_i})$. We call (2) the rational decomposition of f_s and each $f_{s\mu}$ the rationally semisimple part of f.

PROPOSITION 3.1. *If* f *is semisimple element of* End V, *then* f *has the rational decomposition.*

PROOF. Let A be the set of eigenvalues of f. Then by Lemmas 2.1 and 2.3 $V=\oplus_{\alpha\in A} V_\alpha$ where each V_α consists of eigenvectors of f corresponding to α. For each $\alpha\in A$ we have

$$\alpha = \sum_{\mu\in M} \xi_\mu \alpha_\mu \quad (\alpha_\mu \in \mathbf{Q}).$$

Define $f_\mu \in \operatorname{End} V$ by

$$f_\mu|_{V_\alpha} = \alpha_\mu 1_{V_\alpha} \quad (\alpha\in A).$$

Then f_μ is rationally semisimple and

$$f = \sum_{\mu\in M} \xi_\mu f_\mu, \quad f_\mu f_\nu = f_\nu f_\mu \quad (\mu, \nu\in M).$$

To show the uniqueness of the decomposition, assume furthermore that

$$f = \sum_{\mu \in M} \xi_\mu \bar{f}_\mu, \quad \bar{f}_\mu \bar{f}_\nu = \bar{f}_\nu \bar{f}_\mu \quad (\mu, \nu \in M)$$

where each $\bar{f}_\mu$ is rationally semisimple. For any $\alpha \in A$, fixe a nonzero element v of V_α. Since $\bar{f}_\mu$ commutes with f, $\bar{f}_\mu$ keeps V_α invariant and by Lemma 2.4 $\bar{f}_\mu|_{V_\alpha}$ is rationally semisimple. Hence V_α has a basis consisting of common eigenvectors of $\bar{f}_\mu|_{V_\alpha}$ $(\mu \in M)$. Write $v = \sum_j v_j$ as the linear sum of elements of this basis. Then

$$v_j \bar{f}_\mu = \gamma_{\mu j} v_j \quad (\gamma_{\mu j} \in \boldsymbol{Q}).$$

It follows that

$$vf = (\textstyle\sum_j v_j)(\sum_\mu \xi_\mu \bar{f}_\mu) = \sum_j (\sum_\mu \xi_\mu \gamma_{\mu j}) v_j.$$

On the other hand

$$vf = (\textstyle\sum_j v_j)(\sum_\nu \xi_\nu f_\nu) = \sum_j (\sum_\nu \xi_\nu \alpha_\nu) v_j.$$

Hence we have

$$\textstyle\sum_\mu \xi_\mu \gamma_{\mu j} = \sum_\nu \xi_\nu \alpha_\nu \qquad \text{for each} \quad j$$

and therefore $\gamma_{\mu j} = \alpha_\mu$ for each j. It follows that

$$v\bar{f}_\mu = \alpha_\mu v = v f_\mu.$$

Since α and v are arbitrary, we have $\bar{f}_\mu = f_\mu$.

f is said to be algebraic if there exists $q(t) \in \mathfrak{k}[t]$ such that $q(f) = 0$. f is algebraic if V is finite-dimensional. Furthermore f is algebraic if f is of finite rank. If f is algebraic, then V is locally finite as an $\langle f \rangle$-module.

The part a) of the following proposition is due to [11].

PROPOSITION 3.2. *Let f be an algebraic element of* End V. *Then*

a) *f has the Chevalley-Jordan decomposition $f = f_s + f_n$ with f_n nilpotent. Furthermore there exist polynomials $g, h \in \mathfrak{k}[t]$ without constant terms such that $f_s = g(f)$ and $f_n = h(f)$.*

b) *The rational decomposition $f_s = \sum_{\mu \in M} \xi_\mu f_{s\mu}$ of f_s is a finite sum and there exist polynomials $g_\mu \in k[t]$ $(\mu \in M)$ without constant terms such that $f_{s\mu} = g_\mu(f)$.*

PROOF. Let $q(t)$ be the minimal polynomial of f and let

$$q(t) = (t - \alpha_1)^{m_1} \cdots (t - \alpha_k)^{m_k}$$

where $\alpha_1, \ldots, \alpha_k$ are different from each other. Put $V_i = \operatorname{Ker}(f - \alpha_i)^{m_i}$. Then V_i is f-invariant. Putting

$$q_i(t) = q(t)/(t-\alpha_i)^{m_i} \quad (1 \le i \le k),$$

$q_1(t),\dots, q_k(t)$ are relatively prime. Hence there exist polynomials $p_1(t),\dots, p_k(t)$ over $\mathfrak{k}$ such that

$$\sum_{i=1}^k p_i(t)q_i(t) = 1.$$

For any $v \in V$

$$v = \sum_{i=1}^k vp_i(f)q_i(f), \quad vp_i(f)q_i(f) \in V_i \quad (1 \le i \le k).$$

Hence $V = \sum_{i=1}^k V_i$. Since $V_i \subseteq V_{\alpha_i}$, by Lemma 2.1 $V = \oplus_{i=1}^k V_i$.

a) By the Chinese remainder theorem, there exists a polynomial $g(t)$ over $\mathfrak{k}$ such that

$$g(t) = \begin{cases} \alpha_i & \bmod (t-\alpha_i)^{m_i} \quad (1 \le i \le k) \\ 0 & \bmod t. \end{cases}$$

Put $h(t) = t - g(t)$. Then $g(t)$ and $h(t)$ are polynomials over $\mathfrak{k}$ without constant terms. Put

$$f_s = g(f), \quad f_n = h(f).$$

Then f_s and f_n belong to End V. Each V_i is invariant by f_s and f_n, and

$$f_s|_{V_i} = \alpha_i 1_{V_i},$$
$$(f_n|_{V_i})^{m_i} = ((f-\alpha_i)|_{V_i})^{m_i} = 0.$$

Hence f_s is semisimple and f_n is nilpotent. Obviously $f = f_s + f_n$, $f_s f_n = f_n f_s$.

To show uniqueness of the above decomposition, assume that

$$f = \bar{f}_s + \bar{f}_n, \quad \bar{f}_s\bar{f}_n = \bar{f}_n\bar{f}_s$$

where $\bar{f}_s$ is semisimple and $\bar{f}_n$ is nil. Then $f_s - \bar{f}_s = \bar{f}_n - f_n$. Since f_s is expressed as a polynomial of f, $\bar{f}_s$ commutes with f_s and therefore by Lemma 2.5 $f_s - \bar{f}_s$ is semisimple. Similarly $\bar{f}_n - f_n$ is nil. Hence $f_s - \bar{f}_s = \bar{f}_n - f_n = 0$, that is, $\bar{f}_s = f_s$ and $\bar{f}_n = f_n$.

b) Each α_i is expressed as

$$\alpha_i = \sum \xi_\mu \alpha_{i\mu} \quad (\alpha_{i\mu} \in \boldsymbol{Q}).$$

By the Chinese remainder theorem, for each $\mu \in M$ there exists a polynomial $g_\mu(t)$ over $\mathfrak{k}$ such that

$$g_\mu(t) = \begin{cases} \alpha_{i\mu} & \bmod (t-\alpha_i)^{m_i} \quad (1 \le i \le k) \\ 0 & \bmod t. \end{cases}$$

Putting

$$f_{s\mu} = g_\mu(f),$$

we have $f_{s\mu}|_{V_i} = \alpha_{i\mu} 1_{V_i}$ $(1 \leq i \leq k)$. Hence $f_{s\mu}$ is rationally semisimple and

$$f_s = \sum \xi_\mu f_{s\mu}, \quad f_{s\mu} f_{s\nu} = f_{s\nu} f_{s\mu} \quad (\mu, \nu \in M).$$

By Proposition 2.1, this is the rational decomposition of f_s and each $f_{s\mu}$ is a polynomial of f without constant term. For all $\mu \in M$ except a finite number of elements of M we have $\alpha_{i\mu} = 0$ $(1 \leq i \leq k)$ and therefore $f_{s\mu} = 0$.

LEMMA 3.3. *Let $f, g \in \mathrm{End}\, V$ and let $f, g, f+g$ be algebraic. If $fg = gf$, then*

$$(f+g)_s = f_s + g_s, \quad (f+g)_n = f_n + g_n, \quad (f+g)_{s\mu} = f_{s\mu} + g_{s\mu}$$

for each $\mu \in M$.

PROOF. By Proposition 3.2

$$f + g = (f_s + g_s) + (f_n + g_n), \tag{3}$$

$$f_s + g_s = \sum_{\mu \in M} \xi_\mu (f_{s\mu} + g_{s\mu}). \tag{4}$$

Since $fg = gf$, by Proposition 3.2

$$f_s g_s = g_s f_s, \quad f_n g_n = g_n f_n, \quad f_{s\mu} g_{s\mu} = g_{s\mu} f_{s\mu}.$$

Hence by Lemma 2.5 $f_s + g_s$, $f_n + g_n$ and $f_{s\mu} + g_{s\mu}$ are respectively semisimple, nilpotent and rationally semisimple. Since factors in (3) and (4) respectively commute with each other, by Proposition 3.2 (3) is the Chevalley-Jordan decomposition of $f+g$ and (4) is the rational decomposition of $(f+g)_s$.

LEMMA 3.4. *Let f be an algebraic element of* End V. *Let W be an f-invariant subsapce of V and let $\bar{f}$ be an endomorphism of V/W induced by f. Then*

a) *The Chevalley-Jordan decomposition of f induces the Chevalley-Jordan decompositions of $f|_W$ and $\bar{f}$.*

b) *The rational decomposition of f_s induces the rational decompositions of $f_s|_W$ and $\overline{f_s}$.*

PROOF. a) Let $f = f_s + f_n$ be the Chevalley-Jordan decomposition of f. Then by Proposition 3.2 W is invariant by f_s and f_n. Hence by Lemma 2.4 $f_s|_W, \overline{f_s}$ are semisimple and $f_n|_W, \overline{f_n}$ are nilpotent. Since $f|_W$ and $\bar{f}$ are algebraic,

$$f|_W = f_s|_W + f_n|_W \quad \text{and} \quad \bar{f} = \overline{f_s} + \overline{f_n}$$

are respectively the Chevalley-Jordan decomposition of $f|_W$ and $\bar{f}$.

b) is similarly shown.

LEMMA 3.5. *Let $L \leq [\mathrm{End}\, V]$ and let f be an algebraic element of L. If f is semisimple (resp. nilpotent, rationally semisimple), then so is* $\mathrm{ad}_L f$.

PROOF. By Lemma 2.4 it suffices to show the case that L coincides with $[\mathrm{End}\, V]$. Put $E=[\mathrm{End}\, V]$.

Assume that f is a semisimple element of E. Then V_α consists of eigenvectors of f corresponding to α. Since f has only a finite number of eigenvalues, denote them by $\alpha_1, \dots, \alpha_n$ and put $V_i = V_{\alpha_i}$. Then by Lemma 2.1 $V = \oplus_{i=1}^n V_i$. Hence

$$E = \oplus_{i,j=1}^n \mathrm{Hom}(V_i, V_j),$$

where $\mathrm{Hom}(V_i, V_j)$ is the subspace of E consisting of all endomorphisms g of V such that $V_i g \subseteq V_j$ and $V_k g = 0$ $(k \neq i)$. It follows that

$$[g_{ij}, f] = (\alpha_j - \alpha_i) g_{ij} \quad (g_{ij} \in \mathrm{Hom}(V_i, V_j)).$$

Hence choosing a basis of each $\mathrm{Hom}(V_i, V_j)$, we have a basis of E consisting of eigenvectors of $\mathrm{ad}_E f$. Therefore $\mathrm{ad}_E f$ is semisimple.

This reasoning also shows that if f is rationally semisimple then so is $\mathrm{ad}_E f$.

Finally, let f be nilpotent. Since

$$g(\mathrm{ad}_E f)^m = \sum_{i=0}^m (-1)^i \binom{m}{i} f^i g f^{m-i} \quad (g \in E),$$

$f^r = 0$ implies $(\mathrm{ad}_E f)^{2r-1} = 0$. Therefore $\mathrm{ad}_E f$ is nilpotent.

COROLLARY 3.6. *Let L be an ideally finite subalgebra of* $[\mathrm{End}\, V]$ *and let f be an algebraic element of L. If f_s and f_n belong to L, then*

$$(\mathrm{ad}_L f)_s = \mathrm{ad}_L f_s, \quad (\mathrm{ad}_L f)_n = \mathrm{ad}_L f_n.$$

If furthermore $f_{s\mu}$ belongs to L for any $\mu \in M$, then

$$(\mathrm{ad}_L f)_{s\mu} = \mathrm{ad}_L f_{s\mu} \quad (\mu \in M).$$

PROOF. Let $f_s, f_n \in L$. Then for the Chevalley-Jordan decomposition $f = f_s + f_n$ of f we have

$$\mathrm{ad}_L f = \mathrm{ad}_L f_s + \mathrm{ad}_L f_n. \tag{5}$$

Since f_s and f_n are algebraic, by Lemma 3.5 we see that $\mathrm{ad}_L f_s$ and $\mathrm{ad}_L f_n$ are respectively semisimple and nilpotent and are commutative. By our hypothesis that $L \in \mathrm{L}(\lhd)\mathfrak{F}$, $\mathrm{ad}_L f$ is an algebraic element of $\mathrm{End}\, L$. Hence (5) is the Chevalley-

Jordan decomposition of $\mathrm{ad}_L f$.

Furthermore let $f_{s\mu} \in L$ for any $\mu \in M$. Then it is similarly shown that $\mathrm{ad}_L f_s = \sum \xi_\mu \mathrm{ad}_L f_{s\mu}$ is the rational decomposition of $(\mathrm{ad}_L f)_s$.

Let L be a subalgebra of $[\mathrm{End}\, V]$. L is called splittable, provided every element f of L has the Chevalley-Jordan decomposition and f_s, f_n belong to L. We call L algebraic, provided every element f of L has the Chevalley-Jordan decomposition and $f_s, f_n, f_{s\mu}$ belong to L for any $\mu \in M$.

Especially in the case that every element f of L is of finite rank, by Proposition 3.2 f has the Chevalley-Jordan decomposition and the rational decomposition of f_s is a finite sum. Hence L is algebraic if for every element f of L all the rationally semisimple parts belong to L. In the beginning of Section 2 we fixed a basis $\{\xi_\mu | \mu \in M\}$ of $\mathfrak{k}$ over Q, but in this special case the definition of algebraicity does not depend on the choice of such a basis.

We remark that when V is of finite dimension the above definition of algebraicity coincides with the known definition (e.g. [4], Chap. 2, §4, Definition 1).

Evidently if L is algebraic then L is splittable.

Next, let L be a Lie algebra over $\mathfrak{k}$ which is not necessarily linear. An element x of L is called ad-semisimple (resp. ad-rationally semisimple, ad-nil, ad-nilpotent) if $\mathrm{ad}_L x$ is semisimple (resp. rationally semisimple, nil, nilpotent).

Let L be ideally finite. If for every element x of L

$$x = x_s + x_n, \quad x_s, x_n \in L, \quad [x_s, x_n] = 0$$

and $\mathrm{ad}_L x = \mathrm{ad}_L x_s + \mathrm{ad}_L x_n$ is the Chevalley-Jordan decomposition of $\mathrm{ad}_L x$, then L is called ad-splittable. Furthermore if

$$x_s = \sum_{\mu \in M} \xi_\mu x_{s\mu}, \quad x_{s\mu} \in L, \quad [x_{s\mu}, x_{s\nu}] = 0 \quad (\mu, \nu \in M)$$

and $\mathrm{ad}_L x_s = \sum_{\mu \in M} \xi_\mu \mathrm{ad}_L x_{s\mu}$ is the rational decomposition of $\mathrm{ad}_L x_s$, then L is called ad-algebraic.

We here give examples of algebraic Lie algebras in the following

PROPOSITION 3.7. *Let A be an algebra over $\mathfrak{k}$.*

a) *The Lie algebra* $\mathrm{Der}_f A$ *of all derivations of A of finite rank is algebraic.*

b) *If A is finite-dimensional, then the derivation algebra* $\mathrm{Der}\, A$ *is algebraic.*

PROOF. a) Let $\delta \in \mathrm{Der}_f A$. Then A is a locally finite $\langle \delta \rangle$-module. Hence by Lemma 1.1

$$A = \oplus_\alpha A_\alpha .$$

For any eigenvalues α, β of δ, we have $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. If $\alpha = \sum_{\mu \in M} \xi_\mu \alpha_\mu$ $(\alpha_\mu \in Q)$, by Proposition 3.2 b)

$$\delta_{s\mu} | _{A_\alpha} = \alpha_\mu 1_{A_\alpha}.$$

Hence for $x \in A_\alpha$ and $y \in A_\beta$

$$(xy)\delta_{s\mu} = (\alpha+\beta)_\mu(xy) = (\alpha_\mu+\beta_\mu)(xy) = (x\delta_{s\mu})y + x(y\delta_{s\mu}).$$

That is, $\delta_{s\mu} \in \mathrm{Der}\, A$. Since $\delta_{s\mu}$ is expressed as a polynomial of δ without constant term, $\delta_{s\mu} \in \mathrm{Der}_f\, A$.

b) is a special case of a).

§4. Splittable and algebraic hulls

Let $\mathrm{F}(V)$ be the set of endomorphisms of V of finite rank. Then $\mathrm{F}(V)$ is a subalgebra of $[\mathrm{End}\, V]$. $\mathrm{F}(V)=[\mathrm{End}\, V]$ if V is finite-dimensional.

LEMMA 4.1. *Let L be a subalgebra of* $\mathrm{F}(V)$. *For $f_1,\dots,f_m \in L$, put*

$$W = \textstyle\sum_{i=1}^m \mathrm{Im} f_i, \quad U = \bigcap_{i=1}^m \mathrm{Ker} f_i,$$

$$K = \{f \in L |\, Vf \subseteq W,\ Uf=0\}.$$

Then K is a finite-dimensional subalgebra of L containing $f_1,\dots,f_m$. Especially if L is splittable (resp. algebraic), then so is K.

PROOF. Evidently $f_1,\dots,f_m \in K \leq L$. W is of finite dimension and U is of finite codimension. Hence if we take a subspace U' of V complementary to U,

$$\dim K \leq \dim \mathrm{Hom}\,(U', W) < \infty.$$

Especially, let L be splittable (resp. algebraic). Then for $f \in K$ f_s (resp. $f_{s\mu}$ $(\mu \in M)$) belongs to L. By Proposition 3.2 we see that f_s (resp. $f_{s\mu}$ $(\mu \in M)$) belongs to K. Therefore K is splittable (resp. algebraic).

PROPOSITION 4.2. a) $\mathrm{F}(V)$ *is locally finite and algebraic.*

b) *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$. *If L is splittable (resp. algebraic), then L is ad-splittable (resp. ad-algebraic).*

PROOF. a) Applying the first part of Lemma 4.1 to $L=\mathrm{F}(V)$, we see that $\mathrm{F}(V)$ is locally finite. For any element f of $\mathrm{F}(V)$, by Proposition 3.2 b) $f_{s\mu}$ belongs to $\mathrm{F}(V)$ for any $\mu \in M$. Hence $\mathrm{F}(V)$ is algebraic.

b) Let L be splittable (resp. algebraic). Then for any element f of L we have the Chevalley-Jordan decomposition of f (resp. the rational decomposition of f_s)

$$f = f_s + f_n, \quad f_s, f_n \in L$$

$$(\text{resp. } f_s = \textstyle\sum_{\mu \in M} \xi_\mu f_{s\mu}, \quad f_{s\mu} \in L\,(\mu \in M)).$$

Then by Corollary 3.6

$$\mathrm{ad}_L f = \mathrm{ad}_L f_s + \mathrm{ad}_L f_n \quad (\text{resp. } \mathrm{ad}_L f_s = \textstyle\sum_{\mu \in M} \xi_\mu \mathrm{ad}_L f_{s\mu})$$

is the Chevalley-Jordan decomposition of $\mathrm{ad}_L f$ (resp. the rational decomposition of $(\mathrm{ad}_L f)_s$). Hence L is ad-splittable (resp. ad-algebraic).

The first half of Lemma 4.1 and local finiteness of $\mathrm{F}(V)$ in Proposition 4.2 a) are due to [11].

By Proposition 4.2 a) $\mathrm{F}(V)$ is algebraic and therefore splittable. Hence for any subalgebra L of $\mathrm{F}(V)$ there exist the smallest splittable subalgebra and the smallest algebraic subalgebra of $\mathrm{F}(V)$ containing L. We call them the splittable hull and the algebraic hull of L, and denote them by $\hat{L}$ (or $L^\wedge$) and $\tilde{L}$ (or $L^\sim$) respectively.

Then $L \leq \hat{L} \leq \tilde{L}$. If $H \leq L$ then $\hat{H} \leq \hat{L}$ and $\tilde{H} \leq \tilde{L}$.

LEMMA 4.3. *Let L be a subalgebra of* $\mathrm{F}(V)$. *If $L \in \mathfrak{F}$ then $\hat{L}$, $\tilde{L} \in \mathfrak{F}$.*

PROOF. Assume that $L \in \mathfrak{F}$ and let $f_1, \ldots, f_m$ be a basis of L. We set W, U as in Lemma 4.1 and put $K = \{f \in \mathrm{F}(V) \mid Vf \subseteq W, Uf = 0\}$. Since $\mathrm{F}(V)$ is algebraic by Proposition 4.2 a), by Lemma 4.1 K is a finite-dimensional algebraic subalgebra of $F(V)$ containing L. Hence $\tilde{L} \leq K$. Therefore $\tilde{L}$, $\hat{L} \in \mathfrak{F}$.

LEMMA 4.4. *Let a subalgebra L of* $\mathrm{F}(V)$ *be ideally finite and splittable (resp. algebraic). For A, B, $C \leq L$ and $C \leq A \cap B$, if $[A, B] \subseteq C$ then $[\hat{A}, \hat{B}] \subseteq C$* (resp. $[\tilde{A}, \tilde{B}] \subseteq C$).

PROOF. Let $K = \{f \in L \mid [A, f] \subseteq C\}$. Then K is a subalgebra of L. For any element f of K, f_s (resp. $f_{s\mu}$ $(\mu \in M)$) belongs to L. By Corollary 3.6

$$\mathrm{ad}_L f = \mathrm{ad}_L f_s + \mathrm{ad}_L f_n \quad (\text{resp. } \mathrm{ad}_L f_s = \textstyle\sum \xi_\mu \mathrm{ad}_L f_{s\mu})$$

is the Chevalley-Jordan decomposition (resp. the rational decomposition). Hence by Proposition 3.2

$$A(\mathrm{ad}_L f_s) \subseteq \textstyle\sum_{i=1}^{\infty} A(\mathrm{ad}_L f)^i \subseteq C$$

$$(\text{resp. } A(\mathrm{ad}_L f_{s\mu}) \subseteq \textstyle\sum_{i=1}^{\infty} A(\mathrm{ad}_L f)^i \subseteq C).$$

It follows that f_s (resp. $f_{s\mu}$ $(\mu \in M)$) belongs to K. Hence K is splittable (resp. algebraic).

By assumption $B \leq K$. Therefore $\hat{B} \leq K$ (resp. $\tilde{B} \leq K$), whence

$$[A, \hat{B}] \subseteq C \quad (\text{resp. } [A, \tilde{B}] \subseteq C).$$

Next, apply the above reasoning to $\hat{B}$, A, C (resp. $\tilde{B}$, A, C). Then we have $[\hat{B}, \hat{A}] \subseteq C$ (resp. $[\tilde{B}, \tilde{A}] \subseteq C$).

LEMMA 4.5. *Let L be a subalgebra of* F(V). *Assume that* $L=\bigcup_{\lambda\in\Lambda}H_\lambda$ *with* $H_\lambda\leq L$ *and that for any* λ, $\mu\in\Lambda$ *there exists* $\nu\in\Lambda$ *such that* $H_\lambda\cup H_\mu\subseteq H_\nu$. *Then*

$$\hat{L}=\bigcup_{\lambda\in\Lambda}\hat{H}_\lambda \quad and \quad \tilde{L}=\bigcup_{\lambda\in\Lambda}\tilde{H}_\lambda.$$

PROOF. Put $K=\bigcup_{\lambda\in\Lambda}\hat{H}_\lambda$. For any elements f, g of K, take λ, $\mu\in\Lambda$ such that $f\in\hat{H}_\lambda$ and $g\in\hat{H}_\mu$, and take $\nu\in\Lambda$ such that $H_\lambda\cup H_\mu\subseteq H_\nu$. Then $\hat{H}_\lambda\cup\hat{H}_\mu\subseteq\hat{H}_\nu$. It follows that $[f,g]\in\hat{H}_\nu\subseteq K$. Hence $K\leq\hat{L}$. Therefore K is a splittable subalgebra of $\hat{L}$ containing L. Thus $K=\hat{L}$ and $\hat{L}=\bigcup_{\lambda\in\Lambda}\hat{H}_\lambda$.

The other formula is similarly shown.

THEOREM 4.6. *For a subalgebra L of* F(V),

a) $L^{(n)}=\hat{L}^{(n)}=\tilde{L}^{(n)}$ $(n\geq 1)$,

b) $L^n=\hat{L}^n=\tilde{L}^n$ $(n\geq 2)$.

PROOF. By Proposition 4.2 a) we have $L\in\mathrm{L}\mathfrak{F}$. Let $\{F_\lambda|\lambda\in\Lambda\}$ be the set of finite-dimensional subalgebras of L. Then $L=\bigcup_{\lambda\in\Lambda}F_\lambda$ and Λ satisfies the condition of Lemma 4.5. By Lemma 4.5

$$\tilde{L}=\bigcup_{\lambda\in\Lambda}\tilde{F}_\lambda.$$

By Lemma 4.3 $\tilde{F}_\lambda\in\mathfrak{F}$. Applying Lemma 4.4 to $L=\tilde{F}_\lambda$, we have

$$\tilde{F}_\lambda^n=F_\lambda^n \quad (n\geq 2)$$

by induction on n. It follows that

$$\tilde{L}^n=\bigcup_{\lambda\in\Lambda}\tilde{F}_\lambda^n=\bigcup_{\lambda\in\Lambda}F_\lambda^n=L^n \quad (n\geq 2).$$

In particular $\tilde{L}^{(1)}=L^{(1)}$. Now by induction on n we have

$$\tilde{L}^{(n)}=L^{(n)} \quad (n\geq 1).$$

Since $L\leq\hat{L}\leq\tilde{L}$, we have the assertions of the theorem.

PROPOSITION 4.7. *Let L be a subalgebra of* F(V) *and let* $\mathfrak{X}$ *be any one of the following classes*:

$$\mathfrak{F},\ \mathfrak{A},\ \mathrm{E}\mathfrak{A},\ \mathfrak{N},\ \mathrm{E}\mathfrak{A}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathrm{LE}\mathfrak{A},\ \mathrm{L}\mathfrak{N},$$

$$\mathrm{L}(\lhd)\mathfrak{F},\ \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F}),\ \mathrm{L}(\lhd)(\mathfrak{N}\cap\mathfrak{F}).$$

If $L\in\mathfrak{X}$, *then* $\hat{L}$, $\tilde{L}\in\mathfrak{X}$.

PROOF. The case that $\mathfrak{X}=\mathfrak{F}$ was shown in Lemma 4.3. The cases that $\mathfrak{X}=\mathfrak{A}$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{N}$ follow from Theorem 4.6, the cases that $\mathfrak{X}=\mathrm{E}\mathfrak{A}\cap\mathfrak{F}$, $\mathfrak{N}\cap\mathfrak{F}$ follow from Lemma 4.3 and Theorem 4.6, and the cases that $\mathfrak{X}=\mathrm{LE}\mathfrak{A}$, $\mathrm{L}\mathfrak{N}$ follow from

Lemma 4.5 and Theorem 4.6.

We now show the case that $\mathfrak{X}=\mathrm{L}(\lhd)\mathfrak{F}$. Assume that $L\in\mathrm{L}(\lhd)\mathfrak{F}$ and let $\{F_\lambda|\lambda\in\Lambda\}$ be the set of finite-dimensional ideals of L. Then $L=\cup_{\lambda\in\Lambda}F_\lambda$. By Lemmas 4.3 and 4.5 we have

$$\tilde{L}=\cup_{\lambda\in\Lambda}\tilde{F}_\lambda \quad \text{and} \quad \tilde{F}_\lambda\in\mathfrak{F} \qquad \text{for any} \quad \lambda\in\Lambda.$$

For $F_\lambda\leq F_\mu$, apply Lemma 4.4 to $L=\tilde{F}_\mu$, $A=C=F_\lambda$ and $B=F_\mu$. Then we have $[\tilde{F}_\lambda, \tilde{F}_\mu]\subseteq F_\lambda$. It follows that

$$[\tilde{F}_\lambda, \tilde{F}_\nu]\subseteq F_\lambda \qquad \text{for any} \quad \nu\in\Lambda,$$

which shows that $\tilde{F}_\lambda\lhd\tilde{L}$. Hence $\tilde{L}\in\mathrm{L}(\lhd)\mathfrak{F}$ and therefore $\hat{L}\in\mathrm{L}(\lhd)\mathfrak{F}$.

The remaining cases follow from the facts that

$$\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})=\mathrm{LE}\mathfrak{A}\cap\mathrm{L}(\lhd)\mathfrak{F} \quad \text{and} \quad \mathrm{L}(\lhd)(\mathfrak{N}\cap\mathfrak{F})=\mathrm{L}\mathfrak{N}\cap\mathrm{L}(\lhd)\mathfrak{F}.$$

PROPOSITION 4.8. *Let L be an ideally finite subalgebra of $F(V)$.*

a) *If $H\lhd^\sigma L$, then $\hat{H}\lhd^\sigma\hat{L}$ and $\tilde{H}\lhd^\sigma\tilde{L}$.*

b) *If $H\lhd^\sigma L$ $(\sigma>0)$, then $H\lhd^\sigma\hat{L}$ and $H\lhd^\sigma\tilde{L}$.*

PROOF. By Lemma 4.7 $\hat{L}$, $\tilde{L}\in\mathrm{L}(\lhd)\mathfrak{F}$. Let $H\lhd^\sigma L$ and let $\{H_\alpha|\alpha\leq\sigma\}$ be an ascending series from H to L.

a) Evidently $\hat{H}=H_0\hat{}$ and $\hat{L}=H_\sigma\hat{}$. For any ordinal $\alpha<\sigma$, by Lemma 4.4 we have $H_\alpha\hat{}\lhd H_{\alpha+1}\hat{}$. For any limit ordinal $\lambda\leq\sigma$, by Lemma 4.5 we have $H_\lambda\hat{}=\cup_{\alpha<\lambda}H_\alpha\hat{}$. Hence $\hat{H}\lhd^\sigma\hat{L}$. Similarly $\tilde{H}\lhd^\sigma\tilde{L}$.

b) For $\sigma=1$ by Lemma 4.4 we have $H\lhd\tilde{L}$. For any non-limit ordinal σ, $H_{\sigma-1}\lhd L$. It follows that $H_{\sigma-1}\lhd\tilde{L}$. Hence $H\lhd^\sigma\tilde{L}$. For any limit ordinal σ, by a) we have $H\lhd\tilde{H}\lhd^\sigma\tilde{L}$. Since σ is infinite, $H\lhd^\sigma\tilde{L}$. Hence for all $\sigma\geq 1$ $H\lhd^\sigma\tilde{L}$ and therefore $H\lhd^\sigma\hat{L}$.

§5. Lie algebras of endomorphisms of a finite-dimensional vector space

In this section, we assume that V is a finite-dimensional vector space over $\mathfrak{k}$ and we observe several known properties of subalgebras of [End V].

LEMMA 5.1. *Let L be a subalgebra of* [End V] *and let R denote the soluble radical of L. Then the set R_n of nilpotent elements of R is an ideal of L containing $[R, L]$.*

PROOF. For any element f of L, $R+\langle f\rangle$ is a soluble subalgebra of L and may be triangulated by Lie's theorem. It follows that $[R, f]\subseteq R_n$. Again triangulating R, we see that R_n is a subspace of R and $[R_n, f]\subseteq[R, f]\subseteq R_n$. Hence $R_n\lhd L$.

PROPOSITION 5.2. *Every semisimple subalgebra of* [End V] *is algebraic.*

PROOF. Let L be a semisimple subalgebra of [End V]. By Theorem 4.6 we have $L \lhd \tilde{L}$. Regard $\tilde{L}$ as an L-module by $\mathrm{ad}_{\tilde{L}} L$. Then Weyl's theorem says that there exists a subspace A of $\tilde{L}$ such that

$$\tilde{L} = L + A, \quad L \cap A = 0, \quad [A, L] \subseteq A.$$

By Proposition 3.2 b) and Corollary 3.6 we have $[A, \tilde{L}] \subseteq A$. It follows from Theorem 4.6 that $[A, \tilde{L}] \subseteq L \cap A = 0$, that is, $A = \zeta(\tilde{L})$. Again by Weyl's theorem $V = \oplus_{i=1}^{n} V_i$ where each V_i is an irreducible submodule of V. For any element f of A, we have $f_{s\mu} \in \tilde{L}$ and therefore

$$f_{s\mu} = g + h, \quad g \in L, \quad h \in A.$$

By Schur's lemma $h|_{V_i} = \lambda_i 1_{V_i}$ $(1 \leq i \leq n)$. Since $L = L^2$, $\mathrm{tr}(f|_{V_i}) = \mathrm{tr}(g|_{V_i}) = 0$. Hence

$$0 = \mathrm{tr}(f_s|_{V_i}) = \textstyle\sum_{\mu \in M} \xi_\mu \, \mathrm{tr}(f_{s\mu}|_{V_i}).$$

It follows that $\mathrm{tr}(f_{s\mu}|_{V_i}) = 0$ and therefore $\mathrm{tr}(h|_{V_i}) = 0$. Hence $\lambda_i = 0$ $(1 \leq i \leq n)$ and $h = 0$. Thus $f_{s\mu} = g \in L$.

PROPOSITION 5.3. *Let L be a subalgebra of* [End V]. *If L-module V is completely reducible, then L is splittable.*

For the proof, see [8, Chap. 3, Theorem 17].

Let $V_{0,s} = \underbrace{V \otimes \cdots \otimes V}_{s}$ be the space of contravariant tensors of rank s. For $f \in \mathrm{End}\, V$, let $f_{0,1} = f$ and

$$f_{0,s} = \underbrace{f \otimes 1 \otimes \cdots \otimes 1}_{s} + \underbrace{1 \otimes f \otimes \cdots \otimes 1}_{s} + \cdots + \underbrace{1 \otimes \cdots \otimes 1 \otimes f}_{s} \quad (s \geq 2).$$

Then $f_{0,s} \in \mathrm{End}\, V_{0,s}$. Putting $V_{0,r} f_{0,s} = 0$ for $r \neq s$, we may consider that $f_{0,s}$ acts on $\oplus_{r=1}^{t} V_{0,r}$.

LEMMA 5.4. *Let $f \in \mathrm{End}\, V$. Then*

a) $$f_{0,r} = (f_s)_{0,r} + (f_n)_{0,r} \quad \textit{and} \quad (f_s)_{0,r} = \textstyle\sum_{\mu \in M} \xi_\mu (f_{s\mu})_{0,r}$$

are respectively the Chevalley-Jordan decomposition of $f_{0,r}$ and the rational decomposition of $(f_s)_{0,r}$.

b) *Let W be a subspace of $\oplus_{r=1}^{t} V_{0,r}$ which is invariant by $\sum_{r=1}^{t} f_{0,r}$, and let $\bar{f}$ be the restriction of $\sum_{r=1}^{t} f_{0,r}$ to W. Then W is invariant by $\overline{f_s}, \overline{f_n}, \overline{f_{s\mu}}$ and*

$$\bar{f} = \overline{f_s} + \overline{f_n}, \quad \overline{f_s} = \textstyle\sum_{\mu \in M} \xi_\mu \overline{f_{s\mu}}$$

are respectively the Chevalley-Jordan decomposition of $\bar{f}$ *and the rational decomposition of* $\bar{f_s}$.

For a subalgebra L of [End V], we put

$$\mathscr{N}(L_{0,m}) = \cap_{f \in L} \operatorname{Ker} f_{0,m}.$$

THEOREM 5.5. *Let L be an r-dimensional subalgebra of* [End V] *consisting of nilpotent elements. If an element g of* End V *satisfies*

$$\mathscr{N}(L_{0,m}) \subseteq \operatorname{Ker} g_{0,m} \quad (m=1, 2, 3,\dots, 4^r),$$

then g belongs to L.

Outline of the proof is as follows. Assume that f, $f' \in \operatorname{End} V$ and f is nilpotent. Then it is shown that if $\operatorname{Ker} f \subseteq \operatorname{Ker} f'$ and $\operatorname{Ker} f_{0,2} \subseteq \operatorname{Ker} f'_{0,2}$, then f' is expressed as a polynomial of f without constant term. Owing to this it can be shown that if $\operatorname{Ker} f \subseteq \operatorname{Ker} f'$, $\operatorname{Ker} f_{0,2} \subseteq \operatorname{Ker} f'_{0,2}$ and $\operatorname{Ker} f_{0,4} \subseteq \operatorname{Ker} f'_{0,4}$, then $f' = cf$ with $c \in \mathfrak{k}$. Using this fact, the assertion of the theorem may be shown by induction on r.

For detail, see [5, 6].

§6. Splittable and algebraic systems of generators

We begin with

LEMMA 6.1. *Let L be a finite-dimensional subalgebra of* F(V). *Then there exists a finite-dimensional subspace V_0 of V so that we can regard*

$$L \leq [\operatorname{End} V_0] \leq \mathrm{F}(V).$$

PROOF. Let $f_1,\dots, f_n$ be a basis of L. Take W and U as in Lemma 4.1 and put

$$K = \{f \in \mathrm{F}(V) \mid Vf \subseteq W,\ Uf=0\}.$$

Then K is a finite-dimensional subalgebra of F(V) containing L. Let U' be a subspace of V complementary to U and put

$$V_0 = U' + W.$$

Then $\dim V_0 < \infty$ and there exists a subspace V_1 of U such that $V = V_0 \oplus V_1$. We now identify an element f_0 of End V_0 with an element of End V which is obtained from f_0 by putting $V_1 f_0 = 0$. Then we have $\operatorname{End} V_0 \subseteq \operatorname{End} V$ and therefore $L \leq [\operatorname{End} V_0] \leq \mathrm{F}(V)$.

By this lemma, we can apply the results on Lie algebras of endomorphisms of a finite-dimensional vector space to finite-dimensional subalgebras of $\mathrm{F}(V)$.

PROPOSITION 6.2. *Every semisimple serially finite subalgebra of* $\mathrm{F}(V)$ *is algebraic.*

PROOF. Let L be a semisimple serially finite subalgebra of $\mathrm{F}(V)$. By (1.3) we have

$$L = \oplus_\lambda L_\lambda ,$$

where each L_λ is a finite-dimensional non-abelian simple ideal of L. By Proposition 5.2 and Lemma 6.1 each L_λ is algebraic. Hence any element f of L is expressed as

$$f = f_1 + \cdots + f_k, \quad f_i \in L_{\lambda_i} \quad (1 \le i \le k).$$

Since $f_1, \dots, f_k$ commute with each other, by Lemma 3.3 we have

$$f_{s\mu} = (f_1)_{s\mu} + \cdots + (f_k)_{s\mu} \in \sum_{i=1}^{k} L_{\lambda_i} \le L$$

for any $\mu \in M$. Therefore L is algebraic.

For a subalgebra L of $\mathrm{F}(V)$, we call a system $\{f_\iota | \iota \in I\}$ of generators of L splittable if the semisimple and the nilpotent parts of each f_ι belong to L, and algebraic if the semisimple, the nilpotent and the rationally semisimple parts of each f_ι belong to L. We similarly define splittability and algebraicity of a basis of L.

LEMMA 6.3. *Let V be a finite-dimensional vector space over $\mathfrak{k}$ and let L be a subalgebra of* $[\mathrm{End}\, V]$. *Then L is splittable (resp. algebraic) if L has a splittable (resp. an algebraic) system of generators.*

PROOF. Let L have a splittable (resp. an algebraic) system $G = \{f_1, \dots, f_m\}$ of generators of L. Let R be the soluble radical of L. Then by Lemma 5.1 $R_1 = [L, R]$ consists of nilpotent elements. Denoting $r = \dim R_1$ we put

$$W = \sum_{i=1}^{4r} \mathcal{N}((R_1)_{0,i})$$

and for any element f of $\mathrm{End}\, V$ we define

$$\tilde{f} = \sum_{i=1}^{4r} f_{0,i} .$$

Since R_1 is an ideal of $\tilde{L}$ by Proposition 4.8 b), W is invariant by $\tilde{f}$ for any element f of $\tilde{L}$. Hence put $\bar{f} = \tilde{f}|_W$ and for $A \subseteq \tilde{L}$ put

$$\bar{A} = \{\bar{f} | f \in A\} .$$

Then $\bar{R}$ is the center of $\bar{L}$. Now let $L=R+\Lambda$ be a Levi decomposition of L. Then we have

$$f_i = g_i + h_i, \quad g_i \in R, \quad h_i \in \Lambda \quad (1 \leq i \leq m).$$

Since $[g_i, h_i]=0$, by Lemma 3.3

$$(\bar{f_i})_s = (\bar{g_i})_s + (\bar{h_i})_s \quad (\text{resp.}\ (\bar{f_i})_{s\mu} = (\bar{g_i})_{s\mu} + (\bar{h_i})_{s\mu}).$$

Since Λ is algebraic by Proposition 5.2, it follows from Lemma 5.4 b) that

$$(\bar{h_i})_s = \overline{(h_i)_s} \in \bar{\Lambda} \ (\text{resp.}\ (\bar{h_i})_{s\mu} = \overline{(h_i)_{s\mu}} \in \bar{\Lambda}),$$
$$(\bar{f_i})_s = \overline{(f_i)_s} \in \bar{L} \ (\text{resp.}\ (\bar{f_i})_{s\mu} = \overline{(f_i)_{s\mu}} \in \bar{L}).$$

Hence

$$(\bar{g_i})_s \in \bar{L} \quad (\text{resp.}\ (\bar{g_i})_{s\mu} \in \bar{L}) \quad (1 \leq i \leq m). \tag{1}$$

For any element f of L,

$$\begin{aligned} f &= \sum \alpha_{i_1\cdots i_j}[f_{i_1},\dots,f_{i_j}] \\ &= \sum \alpha_i f_i + \sum_{j\geq 2} \alpha_{i_1\cdots i_j}[f_{i_1},\dots,f_{i_j}] \quad (f_i, f_{i_k} \in G). \end{aligned}$$

Replacing f_i by g_i+h_i, we have

$$f = g + h, \quad h = \sum \alpha_i h_i + \sum_{j\geq 2} \alpha_{i_1\cdots i_j}[h_{i_1},\dots,h_{i_j}],$$
$$g = \text{the sum of remaining terms.}$$

Since $h \in \Lambda$,

$$\bar{h}_s = \overline{h_s} \in \bar{\Lambda} \quad (\text{resp.}\ \bar{h}_{s\mu} = \overline{h_{s\mu}} \in \bar{\Lambda}). \tag{2}$$

On the other hand, $\bar{g} = \sum \beta_k \overline{g_k}$ and therefore by (1)

$$\bar{g}_s = \sum \beta_k(\bar{g_k})_s \in \bar{L} \quad (\text{resp.}\ \bar{g}_{s\mu} = \sum \beta_k(\bar{g_k})_{s\mu} \in \bar{L}). \tag{3}$$

From (2) and (3) it follows that

$$\overline{f_s} = \bar{f}_s = \bar{g}_s + \bar{h}_s \in \bar{L}$$
$$(\text{resp.}\ \overline{f_{s\mu}} = \bar{f}_{s\mu} = \bar{g}_{s\mu} + \bar{h}_{s\mu} \in \bar{L}).$$

Hence there exists an element p (resp. q_μ) of L such that

$$\overline{f_s} = \bar{p} \quad (\text{resp.}\ \overline{f_{s\mu}} = \overline{q_\mu}).$$

We have

$$\overline{f_s - p} = 0 \quad (\text{resp.}\ \overline{f_{s\mu} - q_\mu} = 0),$$

whence by Theorem 5.5

$$f_s - p \in R_1 \quad (\text{resp.}\, f_{s\mu} - q_\mu \in R_1).$$

It follows that $f_s \in L$ (resp. $f_{s\mu} \in L$ for any $\mu \in M$). Therefore L is splittable (resp. algebraic).

THEOREM 6.4. *For a subalgebra L of* $\mathrm{F}(V)$ *the following are equivalent*:

a) *L is splittable (resp. algebraic).*

b) *L has a splittable (resp. an algebraic) basis.*

c) *L has a splittable (resp. an algebraic) system of generators.*

PROOF. Assume that L has a splittable system of generators. Replacing each element by its semisimple and nil parts, we may assume that L has a system of generators consisting of semisimple and nilpotent endomorphisms of V. Denote this system of generators by $\{f_\alpha | \alpha \in A\}$.

Let $\{L_\lambda | \lambda \in \Lambda\}$ be the set of finite-dimensional subalgebras of L. Then by Proposition 4.2 a) $L = \cup_{\lambda \in \Lambda} L_\lambda$ and therefore by Lemma 4.5 $\hat{L} = \cup_{\lambda \in \Lambda} \hat{L}_\lambda$. Let $g_1, \dots, g_n$ be a basis of L_λ. Then

$$g_i = \sum \gamma_{\alpha_1 \cdots \alpha_m} [f_{\alpha_1}, \dots, f_{\alpha_m}] \quad (1 \le i \le n).$$

Let G_i be the set of f_{α_j} appearing in this formula and put $G = \cup_{i=1}^n G_i$. Since G is finite, there exists a subalgebra L_μ ($\mu \in \Lambda$) containing G. Hence

$$L_\lambda \le \langle G \rangle \le L_\mu .$$

By Lemma 6.1 there exists a finite-dimensional subspace V_μ of V such that

$$L_\mu \le [\mathrm{End}\, V_\mu] \le \mathrm{F}(V).$$

It follows from Lemma 6.3 that $\langle G \rangle$ is splittable and therefore

$$\hat{L}_\lambda \le \langle G \rangle \le L_\mu .$$

Hence

$$\hat{L} = \cup_\lambda \hat{L}_\lambda = \cup_\lambda L_\lambda = L,$$

that is, $\hat{L} = L$. Therefore L is splittable.

The case of algebraicity is similarly shown.

COROLLARY 6.5. *For a subalgebra L of* $\mathrm{F}(V)$, *$\hat{L}$ (resp. $\tilde{L}$) is a vector space spanned by semisimple (resp. rationally semisimple) and nilpotent parts of elements of L.*

PROOF. Let K be a subspace of $\mathrm{F}(V)$ spanned by semisimple (resp. rationally semisimple) and nil parts of all elements of L. Then

$$L \subseteq K \subseteq \hat{L} \quad (\text{resp. } L \subseteq K \subseteq \tilde{L}).$$

Hence by Theorem 3.6 we have

$$[K, K] \subseteq \hat{L}^2 \subseteq L \subseteq K$$

and therefore K is a subalgebra of $\hat{L}$ (resp. $\tilde{L}$). Since K has a splittable (resp. an algebraic) basis, by Theorem 6.4 K is splittable (resp. algebraic). Thus $K = \hat{L}$ (resp. $K = \tilde{L}$).

COROLLARY 6.6. *The Lie algebra generated by any collection of splittable (resp. algebraic) subalgebras of* $\mathrm{F}(V)$ *is splittable (resp. algebraic).*

PROOF. The Lie algebra L generated by such a collection of subalgebras of $\mathrm{F}(V)$ has a splittable (resp. an algebraic) system of generators. Hence by Theorem 6.4 L is splittable (resp. algebraic).

THEOREM 6.7. *For any subalgebra L of* $\mathrm{F}(V)$ L^2 *is algebraic.*

PROOF. Let $\{L_\lambda | \lambda \in \Lambda\}$ be the set of finite-dimensional subalgebras of L. Then by Proposition 4.2 a) $L = \cup_\lambda L_\lambda$. Hence

$$L^2 = \cup_\lambda L_\lambda^2.$$

For each L_λ, by Lemma 6.1 there exists a finite-dimensional subspace V_λ of V such that

$$L_\lambda \leq [\mathrm{End}\, V_\lambda] \leq \mathrm{F}(V).$$

Hence by Lemma 5.1 the soluble radical of L_λ^2 consists of nilpotent elements and by Proposition 5.2 a Levi subalgebra of L_λ^2 is algebraic. Therefore by Theorem 6.4 L_λ^2 is algebraic. Thus by Corollary 6.6 we conclude that L^2 is algebraic.

§7. Structure theorems

In this section we shall examine the structure of ideally finite subalgebras of $\mathrm{F}(V)$.

THEOREM 7.1. *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$. *Then*

a) $\sigma(L)^\wedge = \sigma(\hat{L})$, $\sigma(L)^\sim = \sigma(\tilde{L})$ *and* $\sigma(L) = \sigma(\hat{L}) \cap L = \sigma(\tilde{L}) \cap L$.

b) *Every Levi subalgebra of L is a Levi subalgebra of $\hat{L}$ and of $\tilde{L}$.*

c) *For any Borel subalgebra B of L, $\hat{B}$ and $\tilde{B}$ are respectively Borel subalgebras of $\hat{L}$ and $\tilde{L}$, and $B = \hat{B} \cap L = \tilde{B} \cap L$.*

PROOF. Let Λ be a Levi subalgebra of L. Then $\sigma(L)^\wedge + \Lambda$ is a subalgebra of $\hat{L}$ by Theorem 4.6 and has a splittable basis by Proposition 6.2. Hence by

Theorem 6.4 $\hat{L}=\sigma(L)^{\wedge}+\Lambda$. Similarly we have $\tilde{L}=\sigma(L)^{\sim}+\Lambda$.

a) By Propositions 4.7 and 4.8 a) $\sigma(L)^{\wedge}$ is a locally soluble ideal of $\hat{L}$. Let H be any locally soluble ideal of $\hat{L}$. Then $H+\sigma(L)^{\wedge}$ is a locally soluble ideal of $\hat{L}$. In fact, let K be a finitely generated subalgebra of $H+\sigma(L)^{\wedge}$. Since F(V) is locally finite, K is finite-dimensional and therefore $(K+H)/H$ is soluble. It follows that $K^{(n)}\subseteq H$. Hence $K^{(n)}$ is soluble, that is, K is soluble. Therefore $H+\sigma(L)^{\wedge}$ is locally soluble, as asserted. Now

$$\begin{aligned} H+\sigma(L)^{\wedge} &= (H+\sigma(L)^{\wedge}) \cap (\sigma(L)^{\wedge}+\Lambda) \\ &= \sigma(L)^{\wedge} + (H+\sigma(L)^{\wedge}) \cap \Lambda = \sigma(L)^{\wedge}, \end{aligned}$$

whence $H\leq\sigma(L)^{\wedge}$. Thus $\sigma(L)^{\wedge}$ is the largest locally soluble ideal of $\hat{L}$ and $\sigma(L)^{\wedge}=\sigma(\hat{L})$. By maximality of $\sigma(L)$, we have $\sigma(L)=\sigma(\hat{L})\cap L$.

The assertion for $\sigma(\tilde{L})$ is similarly proved.

b) Taking account of the part a), $\hat{L}=\sigma(L)^{\wedge}+\Lambda$ and $\tilde{L}=\sigma(L)^{\sim}+\Lambda$ are Levi decompositions of $\hat{L}$ and $\tilde{L}$ respectively.

c) By (1.3) $\Lambda=\oplus_{\mu}\Lambda_{\mu}$ where each Λ_{μ} is a finite-dimensional non-abelian simple ideal of Λ. Hence by (1.7)

$$B = \sigma(L) + (\oplus_{\mu} B_{\mu})$$

where each B_{μ} is a Borel subalgebra of Λ_{μ}. By Proposition 6.2

$$B_{\mu} \leq \tilde{B}_{\mu} \leq \tilde{\Lambda}_{\mu} = \Lambda_{\mu}$$

and by Proposition 4.7 $\tilde{B}_{\mu}$ is soluble. Hence we have $B_{\mu}=\tilde{B}_{\mu}$ by maximality of B_{μ}, that is, B_{μ} is algebraic. Now by Corollary 6.6 $\sigma(L)^{\wedge}+(\oplus_{\mu}B_{\mu})$ is a splittable subalgebra of $\hat{B}$ containing B and therefore

$$\hat{B} = \sigma(L)^{\wedge} + (\oplus_{\mu} B_{\mu}).$$

From a) and (1.7) it follows that $\hat{B}$ is a Borel subalgebra of $\hat{L}$. By maximality of B we have $B=\hat{B}\cap L$.

The assertion for $\tilde{B}$ is similarly proved.

THEOREM 7.2. *Let L be an ideally finite subalgebra of* F(V). *Then the following are equivalent*:

a) *L is splittable* (*resp. algebraic*).

b) *σ(L) is splittable* (*resp. algebraic*).

c) *A Borel subalgebra of L is splittable* (*resp. algebraic*).

d) *A Cartan subalgebra of L is splittable* (*resp. algebraic*).

PROOF. a)⇔b) If L is splittable (resp. algebraic), then by Theorem 7.1 a)

$$\sigma(L)^{\wedge} = \sigma(\hat{L}) = \sigma(L) \quad (\text{resp. } \sigma(L)^{\sim}=\sigma(\tilde{L})=\sigma(L)),$$

that is, $\sigma(L)$ is splittable (resp. algebraic). Conversely, if $\sigma(L)$ is splittable (resp. algebraic), by (1.5) take a Levi subalgebra Λ of L. Then by Theorem 7.1

$$\hat{L} = \sigma(L)^{\wedge} + \Lambda = \sigma(L) + \Lambda = L$$

$$(\text{resp. } \tilde{L} = \sigma(L)^{\sim} + \Lambda = \sigma(L) + \Lambda = L),$$

that is, L is splittable (resp. algebraic).

a)⇔c) Let B be a Borel subalgebra of L. If L is splittable (resp. algebraic), $\hat{B} \leq L$ (resp. $\tilde{B} \leq L$). By Theorem 7.1 c) $B = \hat{B} \cap L = \hat{B}$ (resp. $B = \tilde{B} \cap L = \tilde{B}$), that is, B is splittable (resp. algebraic). Conversely, let B be splittable (resp. algebraic). By (1.6) $\sigma(L) \leq B$. Taking a Levi subalgebra Λ of L we have $L = B + \Lambda$. Hence by Proposition 6.2 and Corollary 6.6 L is splittable (resp. algebraic).

a)⇔d) Let C be a Cartan subalgebra of L. If L is splittable (resp. algebraic), by Proposition 3.7 $\hat{C}$ (resp. $\tilde{C}$) is a locally nilpotent subalgebra of L. By maximality of C we have $\hat{C} = C$ (resp. $\tilde{C} = C$), that is, C is splittable (resp. algebraic). Conversely, let C be splittable (resp. algebraic). By (1.9) $(C + L^2)/L^2$ is a Cartan subalgebra of L/L^2. Hence $(C + L^2)/L^2 = L/L^2$. It follows that $L = C + L^2$. Since by Theorem 6.7 L^2 is algebraic, by Corollary 6.6 L is splittable (resp. algebraic).

LEMMA 7.3. *Let L be a locally soluble, ideally finite Lie algebra. Then for an element x of L, x belongs to $\rho(L)$ if and only if $\mathrm{ad}_L x$ is nilpotent.*

PROOF. Let $x \in \rho(L)$. Then by (1.2) there exists a finite-dimensional nilpotent ideal K of L containing x. Hence $\mathrm{ad}_K x$ is nilpotent and therefore $\mathrm{ad}_L x$ is nilpotent.

Conversely, let $\mathrm{ad}_L x$ be nilpotent. Take a finite-dimensional ideal F of L containing x. Then $\mathrm{ad}_F x$ is nilpotent. Since F is soluble, it follows that $x \in \rho(F)$. By (1.1) we have $x \in \rho(L)$.

PROPOSITION 7.4. *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$.

a) *If L is splittable (resp. algebraic), then $\rho(L)$ is splittable (resp. algebraic).*

b) $\hat{L} = \rho(\hat{L}) + L$. *Therefore L is splittable if and only if $\rho(L) = \rho(\hat{L})$.*

c) $\zeta(L) = \zeta(\hat{L}) \cap L = \zeta(\tilde{L}) \cap L$.

PROOF. a) Let L be splittable (resp. algebraic). Then by Propositions 4.7 and 4.8 a) $\rho(L)^{\wedge}$ (resp. $\rho(L)^{\sim}$) is a locally nilpotent ideal of L. By maximality of $\rho(L)$ we have $\rho(L)^{\wedge} = \rho(L)$ (resp. $\rho(L)^{\sim} = \rho(L)$), that is, $\rho(L)$ is splittable (resp. algebraic).

b) Put $R = \sigma(L)$. By Proposition 4.7 $\hat{R} \in \mathrm{L}\mathfrak{E}\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F}$. Put $R_1 = \rho(\hat{R}) + R$. Then $R_1 \leq \hat{R}$. For any element f of $\rho(\hat{R}) \cup R$, we have $f_n \in \hat{R}$ and therefore

$\mathrm{ad}_{\hat{R}} f_n$ is nilpotent. By Lemma 7.3 it follows that $f_n \in \rho(\hat{R})$. Hence R_1 has a splittable basis. By Theorem 6.4 R_1 is splittable and $R_1 = \hat{R}$. That is,

$$\hat{R} = \rho(\hat{R}) + R.$$

Let $L = R + \Lambda$ be a Levi decomposition of L. Then by (1.1) and Theorem 7.1 we have

$$\hat{L} = \hat{R} + \Lambda = \rho(\hat{R}) + R + \Lambda = \rho(\hat{L}) + L.$$

c) If $f \in \zeta(L)$, by Lemma 4.4 we have $[\langle f \rangle, \tilde{L}] = 0$ and therefore $f \in \zeta(\tilde{L})$. Hence $\zeta(L) \leq \zeta(\tilde{L})$. It follows that $\zeta(L) = \zeta(\tilde{L}) \cap L$. Therefore $\zeta(L) = \zeta(\hat{L}) \cap L$.

PROPOSITION 7.5. *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$. *Then for a Cartan subalgebra C of L there exist Cartan subalgebras C_1 and C_2 of $\hat{L}$ and $\tilde{L}$ respectively such that $C = C_1 \cap L = C_2 \cap L$.*

PROOF. Let B be a Borel subalgebra of L containing C. Then C is a Cartan subalgebra of B. By Theorem 7.1 $\hat{B}$ is a Borel subalgebra of $\hat{L}$ and by Zorn's lemma there exists a maximal locally nilpotent subalgebra C_1 of $\hat{B}$ containing C. Hence by Proposition 4.7 C_1 is splittable. Therefore $\hat{C} \leq C_1$.

Now by (1.11) $B = \rho(B) + C$. Hence by Corollary 6.6 $\hat{B} = \rho(B)^\wedge + \hat{C}$. Since $\rho(B)^\wedge$ is a locally nilpotent ideal of $\hat{B}$ by Propositions 4.7 and 4.8 a), we have $\rho(B)^\wedge \leq \rho(\hat{B})$ and therefore

$$\hat{B} = \rho(\hat{B}) + C_1.$$

From (1.11) it follows that C_1 is a Cartan subalgebra of $\hat{B}$. Therefore by (1.10) C_1 is a Cartan subalgebra of $\hat{L}$. By (1.9) we have $C_1 \cap L = C$.

The existence of a Cartan subalgebra C_2 of $\tilde{L}$ such that $C_2 \cap L = C$ is similarly shown.

For a subalgebra L of $\mathrm{F}(V)$, we denote by L_n and L_s the sets of nilpotent and semisimple elements of L respectively. Then we have

LEMMA 7.6. *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$. *Then $\sigma(L)_n$ is a locally nilpotent ideal of L.*

PROOF. Let $\{F_\lambda | \lambda \in \Lambda\}$ be the set of finite-dimensional ideals of L. Then $L = \bigcup_\lambda F_\lambda$. Putting $N_\lambda = \sigma(F_\lambda)_n$, N_λ is a nilpotent ideal of F_λ by Lemmas 5.1, 6.1 and Engel's theorem. Since $F_\lambda \cap \sigma(L) = \sigma(F_\lambda)$ by (1.1), we have $F_\lambda \cap \sigma(L)_n = N_\lambda$. Hence

$$\sigma(L)_n = \bigcup_\lambda (F_\lambda \cap \sigma(L)_n) = \bigcup_\lambda N_\lambda.$$

For $\lambda, \mu \in \Lambda$, there exists $\nu \in \Lambda$ such that $F_\lambda \cup F_\mu \subseteq F_\nu$ and we have

$$N_\lambda \cup N_\mu = (F_\lambda \cup F_\mu) \cap \sigma(L)_n \subseteq F_\nu \cap \sigma(L)_n = N_\nu,$$
$$[N_\lambda, F_\mu] \subseteq [N_\nu, F_\nu] \subseteq N_\nu.$$

Therefore

$$\sigma(L)_n = \cup_\lambda N_\lambda \lhd \cup_\lambda F_\lambda = L$$

and $\sigma(L)_n \in \mathrm{L}\mathfrak{N}$.

Proposition 7.7. *Let L be a locally nilpotent, ideally finite subalgebra of* $\mathrm{F}(V)$. *Then L is splittable (resp. algebraic) if and only if L is the direct sum of an ideal* L_n *and a central ideal (resp. an algebraic central ideal)* L_s.

Proof. Assume that L is splittable (resp. algebraic). By Lemma 7.6 L_n is an ideal of L. If $f \in L_s$, by Lemmas 3.5 and 7.3 $\mathrm{ad}_L f$ is semisimple and nilpotent. Hence $\mathrm{ad}_L f = 0$ and therefore $f \in \zeta(L)$. By Lemma 3.3 L_s is a central ideal (resp. an algebraic central ideal) of L and $L = L_n \oplus L_s$.

The converse follows from Corollary 6.6 and the fact that L_n is algebraic.

When L is a subalgebra of $\mathrm{F}(V)$, an abelian subalgebra T of L is called a torus of L if every element of T is semisimple. When L is a not necessarily linear Lie algebra, an abelian subalgebra T of L is called an ad-torus if every element of T is ad-semisimple.

Lemma 7.8. *Let L be a torus of* $\mathrm{F}(V)$ *and let V be a locally finite L-module. Then V is completely reducible.*

Proof. Let U be a finite-dimensional submodule of V. Then U is an $L/C_L(U)$-module. Here $C_L(U) = \{f \in L \mid Uf = 0\} \lhd L$ and by Lemma 2.3 $L/C_L(U)$ is a finite-dimensional torus of $[\mathrm{End}\, U]$. Hence there exists a basis of U consisting of common eigenvectors of elements of $L/C_L(U)$. Namely, U is a direct sum of 1-dimensional submodules. Each 1-dimensional submodule of U is a submodule of L-module V. Since V is locally finite, V is a sum of 1-dimensional submodules. By (1.12) V is completely reducible.

Theorem 7.9. *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$. *If L is splittable (resp. algebraic), then there exist a torus (resp. an algebraic torus) T and a Levi subalgebra Λ of L such that*

$$\sigma(L) = \sigma(L)_n + T, \quad \sigma(L)_n \cap T = 0, \quad [\Lambda, T] = 0.$$

Proof. Let L be splittable. Put $R = \sigma(L)$. Then by (1.1) $\rho(R) = \rho(L)$ and by Lemma 7.6 $\rho(L)_n = R_n$. Since $\rho(L)$ is splittable by Proposition 7.4 a), as in the proof of Proposition 7.7 we see that $\rho(L)_s$ is a central ideal of R and

$$\rho(L) = R_n + \rho(L)_s.$$

Now let T be a maximal torus of R containing $\rho(L)_s$. Then by Lemma 3.5 T is an ad-torus of R. Taking a maximal ad-torus T_0 of R containing T, we have

$$C_R(T) \supseteq C_R(T_0).$$

By Theorem 7.2 R is splittable and therefore by Proposition 4.2 b) R is ad-splittable. Hence by [11, Theorem 13.2] $C_R(T_0)$ is a Cartan subalgebra of R and by (1.11)

$$R = \rho(L) + C_R(T_0) = \rho(L) + C_R(T).$$

For any element f of $C_R(T)$ we have $f_n \in R_n$. Since by Corollary 3.6 $[T, f_s]=0$, by Lemma 2.5 $T+\langle f_s \rangle$ is a torus of R and by maximality of T we have $f_s \in T$. Hence

$$R = \rho(L) + T = R_n + T, \quad R_n \cap T = 0.$$

Next, since $\mathrm{ad}_L T$ is completely reducible by Lemma 7.8, there exists a subspace $\varLambda_1$ of L such that

$$L = R + \varLambda_1, \quad R \cap \varLambda_1 = 0, \quad [\varLambda_1, T] \subseteq \varLambda_1.$$

It follows that $[T, \varLambda_1] \subseteq R \cap \varLambda_1 = 0$. Putting $L_1 = C_L(T)$, L_1 is a subalgebra of L containing $\varLambda_1$. Hence $L_1 = (R \cap L_1) + \varLambda_1$. Put $R_1 = R \cap L_1$. Then

$$L_1/R_1 \cong (L_1 + R)/R = (R + \varLambda_1)/R = L/R,$$

whence L_1/R_1 is semisimple and therefore $R_1 = \sigma(L_1)$. Now let $\varLambda$ be a Levi subalgebra of L_1. Then

$$L = R + \varLambda_1 \subseteq R + L_1 = R + \varLambda,$$

that is, $L = R + \varLambda$. Here $R \cap \varLambda = 0$, since $R \cap \varLambda$ is semisimple as an ideal of $\varLambda$ by (1.4). Therefore $\varLambda$ is a Levi subalgebra of L. We also have $[T, \varLambda] \subseteq [T, L_1] = 0$, that is, $[T, \varLambda] = 0$.

Especially if L is algebraic, by Theorem 7.2 R is algebraic. Hence $\tilde{T} \leq R$. By Theorem 4.6 and Corollary 6.5 $\tilde{T}$ is a torus. By maximality of T we have $T = \tilde{T}$.

THEOREM 7.10. *Let L be an ideally finite subalgebra of* $\mathrm{F}(V)$. *Then there exists a torus A of $\tilde{L}$ such that*

$$\tilde{L} = L + A, \quad \hat{L} = L + (\hat{L} \cap A), \quad L \cap A = 0.$$

PROOF. Put $R = \sigma(L)$. Then by Theorem 7.1 $\tilde{R} = \sigma(\tilde{L})$. By Theorem 7.9 there exists an algebraic torus T of $\tilde{L}$ such that

$$\tilde{R} = \tilde{R}_n + T, \quad \tilde{R}_n \cap T = 0.$$

Putting $R_1 = R + T$, by Theorem 3.6 we have $R_1 \leq \tilde{R}$. Hence

$$R_1 = (R_1 \cap \tilde{R}_n) + T.$$

Since T is algebraic, R_1 has an algebraic basis and therefore by Theorem 6.4 R_1 is algebraic. Hence $R_1 = \tilde{R}$, that is,

$$\tilde{R} = R + T.$$

Take a subspace A of T complementary to $R \cap T$. Then A is a torus of $\tilde{L}$ such that $R \cap A = 0$. By (1.5) L has a Levi subalgebra Λ and by Theorem 7.1 b)

$$\tilde{L} = \tilde{R} + \Lambda = (R + A) + \Lambda = L + A.$$

Since $\tilde{R} \cap L = R$ by Theorem 7.1 a), it follows that

$$L \cap A = R \cap A = 0.$$

Finally, since $L \leq \hat{L} \leq \tilde{L}$, we have $\hat{L} = L + (\hat{L} \cap A)$.

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