Hiroshima Math. J.
17 (1987), 79–89

An operator theoretic method for solving $u_t = \Delta\psi(u)$

Yoshikazu Kobayashi

(Received May 13, 1986)

1. Introduction

In this paper, we present a new method for solving the Cauchy problem

$$(1.1)\qquad \begin{aligned} &u_t(t, x) = \Delta\psi(u(t, x)), \quad t > 0 \quad \text{and} \quad x \in \boldsymbol{R}^N, \\ &u(0, x) = u_0(x),\ x \in \boldsymbol{R}^N, \end{aligned}$$

where ψ is a locally Lipschitz continuous and nondecreasing function on $\boldsymbol{R}$ such that $\psi(0)=0$; and the method is described from the point of view of the nonlinear semigroup theory.

For $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, a function $u \in L^\infty((0, \infty) \times \boldsymbol{R}^N)$ is called a weak solution of the problem (1.1) if $u \in C([0, \infty); L^1(\boldsymbol{R}^N))$ as an $L^1(\boldsymbol{R}^N)$-valued function on $[0, \infty)$,

$$\int_0^\infty \left(\int_{\boldsymbol{R}^N} u(t, x)f_t(t, x) + \psi(u(t, x))\Delta f(t, x)dx\right)dt = 0$$

for any $f \in C_0^\infty((0, \infty) \times \boldsymbol{R}^N)$ and $u(0, x) = u_0(x)$ a.e.. The existence of weak solutions is established in [1] (in a more general situation) and the uniqueness is proved in [3]. (See also [2] and [11].)

To state the new method for solving the Cauchy problem, let ρ be an arbitrary but fixed rapidly decreasing function on $\boldsymbol{R}^N$ which satisfies

$$(1.2)\qquad \begin{cases} \rho \geq 0, \quad \int_{\boldsymbol{R}^N} \rho(\xi)d\xi = 1, \quad \int_{\boldsymbol{R}^N} \xi_i\rho(\xi)d\xi = 0 \\ \text{and} \\ \int_{\boldsymbol{R}^N} \xi_i\xi_j\rho(\xi)d\xi = \delta_{ij}, \qquad \text{for} \quad i, j = 1, 2, \cdots, N, \end{cases}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. (For example, we can choose the (normalized) Gaussian kernel $(2\pi)^{-N/2}\exp(-|\xi|^2/2)$ as such $\rho(\xi)$.) We set

$$(1.3)\qquad \rho_h(\xi, \eta) = \left(\frac{h}{2\psi_h'(\eta)}\right)^{N/2} \rho\left(\left(\frac{h}{2\psi_h'(\eta)}\right)^{1/2}\xi\right)$$

for $(\xi, \eta) \in \boldsymbol{R}^N \times \boldsymbol{R}$ and $h > 0$, where $\{\psi_h\}_{h>0}$ is a family of smooth strictly increasing functions on $\boldsymbol{R}$ such that $\psi_h(0) = 0$, $\psi_h(\eta) \to \psi(\eta)$ as $h \downarrow 0$, uniformly for bounded

$\eta \in \boldsymbol{R}$, and $\{\psi'_h(\eta)\}$ is uniformly bounded for bounded $h>0$ and bounded $\eta \in \boldsymbol{R}$. Note that, the assumption (1.2) on the function ρ implies

$$\text{(1.4)} \qquad \int_{\boldsymbol{R}^N} \xi_i \rho_h(\xi, h) d\xi = 0 \quad \text{and} \quad \int_{\boldsymbol{R}^N} \xi_i \xi_j \rho_h(\xi, \eta) d\xi = 2h^{-1} \psi'_h(\eta) \delta_{i,j}$$

for $i, j = 1, 2, \cdots, N$.

For each $h>0$, we define an operator C_h on $L^1(\boldsymbol{R}^N)$ by the integral

$$\text{(1.5)} \qquad (C_h w)(x) = \int_{\boldsymbol{R}^N} \left(\int_0^{w(x-h\xi)} \rho_h(\xi, \eta) d\eta \right) d\xi, \quad x \in \boldsymbol{R}^N,$$

where $w \in L^1(\boldsymbol{R}^N)$. Using the properties (1.4) of ρ_h, we see easily that, for $w \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, $h^{-1}(C_h w - w)$ converges to $\Delta\psi(w)$ in the sense of distributions as $h \downarrow 0$. (See the proof of Lemma 3.2 below.) Thus, we can expect that, if $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, then $(C_h^{[t/h]} u_0)(x)$ converges to the weak solution of the Cauchy problem (1.1) as $h \downarrow 0$, where $[t]$ denotes the greatest integer in $t \in \boldsymbol{R}$. In fact, we have the following theorem.

THEOREM. *Let $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$. Then, as $h \downarrow 0$, $(C_h^{[t/h]} u_0)(\cdot)$ converges in $L^1(\boldsymbol{R}^N)$ to the unique weak solution $u(t, \cdot)$ of the Cauchy problem* (1.1) *uniformly for bounded $t \geq 0$.*

The method stated above was suggested by the methods of solving the Cauchy problem for the equation

$$u_t + \nabla \cdot \phi(u) = 0, \quad t > 0 \quad \text{and} \quad x \in \boldsymbol{R}^N$$

that were presented in [5], [6] and [7], and the method of solving the Cauchy problem for the equation

$$u_t + \nabla \cdot \phi(u) = \mu \Delta u, \quad t > 0 \quad \text{and} \quad x \in \boldsymbol{R}^N,$$

that was presented in [9], where ϕ is an $\boldsymbol{R}^N$-valued function on $\boldsymbol{R}$, μ is a positive constant and ∇ denotes the spatial nabla. As in these methods, our method is also based on a linearization procedure of the problem (1.1) to the Cauchy problem for a linear equation involving a parameter. In fact, let

$$F(a, \eta) = 2^{-1}(\operatorname{sign}(a-\eta) + \operatorname{sign}(\eta))$$

for $a, \eta \in \boldsymbol{R}$, where $\operatorname{sign}(\eta) = \eta/|\eta|$ if $\eta \neq 0$ and $\operatorname{sign}(0) = 0$. (See [7] for the basic properties of the function F.) Then, we can rewrite (1.4) as follows:

$$(C_h w)(x) = \int_{-\infty}^{\infty} f(t, x, \eta) d\eta,$$

where

$$f(t, x, \eta) = \int_{R^N} \rho(\xi) F(w(x-(2t\psi_h'(\eta))^{1/2}\xi, \eta) d\xi$$

for $t>0$, $\eta \in \boldsymbol{R}$ and $x \in \boldsymbol{R}^N$, and the function $f(t, x, \eta)$ satisfies the linear equation involving the parameter η

$$f_t = \psi_h'(\eta)\Delta f \qquad \text{for} \quad t > 0, \quad x \in \boldsymbol{R}^N,$$

provided ρ is the Gaussian kernel.

The result will be obtaind by applying the approximation theory for nonlinear semigroups. In section 2, an approximation theorem given in [10] will be recalled, the basic properties of the operators C_h will be investigated so as to apply the approximation theorem and a dissipative operator A in $L^1(\boldsymbol{R}^N)$ will be introduced in such a way that $Au = \Delta\psi(u)$ in an appropriate sense. The proof of the Theorem will be given in section 3.

2. Basic properties of C_h

We first recall the approximation theorem for nonlinear semigroups due to Brezis-Pazy [4] and Oharu-Takahashi [10] in a form convenient for our use.

THEOREM 2.1. *Let $\{C_h\}_{h>0}$ be a family of contractions on a real Banach space X. Suppose that the limit*

$$J_\lambda v = \lim_{h\downarrow 0} (I - \lambda h^{-1}(C_h - I))^{-1} v$$

exists for any $v \in X$ and any $\lambda > 0$, where I denotes the identity operator in X. Then there exists an m-dissipative operator A in X such that $J_\lambda = (I - \lambda A)^{-1}$ for $\lambda > 0$ and, for each $v \in \overline{D(A)}$,

$$T(t)v = \lim_{h\downarrow 0} C_h^{[t/h]} v$$

uniformly for bounded $t \geq 0$, where $\{T(t)\}_{t\geq 0}$ is the semigroup generated by A.

For the proof we refer to [10]. We wish to apply this theorem to the operators C_h defined by (1.5) and the Banach space $X = L^1(R^N)$ with the usual norm $\|\cdot\|_1$. For this purpose, we first prepare a few estimates concerning the operators C_h. In what follows, for each $y \in \boldsymbol{R}^N$, we define an operator τ^y on $L^1(\boldsymbol{R}^N)$ by $(\tau^y u)(x) = u(x+y)$ for $x \in \boldsymbol{R}^N$; where $u \in L^1(\boldsymbol{R}^N)$. Let $\|\cdot\|_\infty$ denote the usual norm of the space $L^\infty(\boldsymbol{R}^N)$.

PROPOSITION 2.2. *Let $h>0$. Then:*

(i) *C_h is a contraction operator on $L^1(\boldsymbol{R}^N)$ and $\|C_h u\|_1 \leq \|u\|_1$ for $u \in L^1(\boldsymbol{R}^N)$.*

(ii) *$C_h \tau^y = \tau^y C_h$ for $y \in \boldsymbol{R}^N$.*

(iii) *If* $u \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, *then* $C_h u \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ *and* $\|C_h u\|_\infty \leq \|u\|_\infty$.

PROOF. Let $u \in L^1(\boldsymbol{R}^N)$. Then

$$\begin{aligned}
&\int_{\boldsymbol{R}^N} |(C_h u)(x)| dx \\
&\quad \leq \int_{\boldsymbol{R}^N} \left(\int_{\boldsymbol{R}^N} \operatorname{sign}(u(x-h\xi)) \int_0^{u(x-h\xi)} \rho_h(\xi, \eta) d\eta \, dx \right) d\xi \\
&\quad = \int_{\boldsymbol{R}^N} \left(\int_{\boldsymbol{R}^N} \operatorname{sign}(u(x)) \int_0^{u(x)} \rho_h(\xi, \eta) d\eta \, dx \right) d\xi .
\end{aligned}$$

Since $\int_{\boldsymbol{R}^N} \rho_h(\xi, \eta) f\xi = 1$, this implies that $C_h u \in L^1(\boldsymbol{R}^N)$ and $\|C_h u\|_1 \leq \|u\|_1$. Let $u, v \in L^1(\boldsymbol{R}^N)$. Then,

$$\begin{aligned}
&(C_h u)(x) - (C_h v)(x) \\
&\quad = \int_{\boldsymbol{R}^N} \left(\int_{v(x-h\xi)}^{u(x-h\xi)} \rho_h(\xi, \eta) \, d\eta \right) d\xi \\
&\quad = \int_{\boldsymbol{R}^N} \left(\int_0^1 \rho_h(\xi, \theta u(x-h\xi) + (1-\theta) v(x-h\xi)) \, d\theta \right) \cdot (u(x-h\xi) - v(x-h\xi)) \, d\xi.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{\boldsymbol{R}^N} |(C_h u)(x) - (C_h v)(x)| dx \\
&\quad \leq \int_{\boldsymbol{R}^N} \left(\int_{\boldsymbol{R}^N} \left(\int_0^1 \rho_h(\xi, \theta u(x-h\xi) + (1-\theta) v(x-h\xi)) d\theta \right.\right. \\
&\qquad\qquad \left.\left. \cdot |(u(x-h\xi) - v(x-h\xi))| \right) dx \right) d\xi \\
&\quad = \int_{\boldsymbol{R}^N} \left(\int_{\boldsymbol{R}^N} \left(\int_0^1 \rho_h(\xi, \theta u(x) + (1-\theta) v(x)) d\theta \cdot |(u(x) - v(x))| \right) dx \right) d\xi \\
&\quad = \int_{\boldsymbol{R}^N} |u(x) - v(x)| dx.
\end{aligned}$$

Assertion (ii) is evident from the definition of C_h. It now remains to prove (iii). Let $u \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ and $k \in \boldsymbol{R}$. Since

$$\int_{\boldsymbol{R}^N} \left(\int_0^k \rho_h(\xi, \eta) d\eta \right) d\xi = k,$$

we have

$$(C_h u)(x) - k = \int_{\boldsymbol{R}^N} \left(\int_k^{u(x-h\xi)} \rho_h(\xi, \eta) d\eta \right) d\xi. \tag{2.1}$$

Taking $k = \pm\|u\|_\infty$ in (2.1), we find that $(C_h u)(x) - \|u\|_\infty \leq 0$ and $(C_h u)(x) + \|u\|_\infty \geq 0$. Hence $C_h u \in L^\infty(\boldsymbol{R}^N)$ and $\|C_h u\|_\infty \leq \|u\|_\infty$. Q. E. D.

The following result will be used to estimate the integrals $\int_{|x|<R} |(C_h u)(x)|dx$ for $R>0$ and $u \in L^1(\boldsymbol{R}^N)$.

PROPOSITION 2.3. *Let $u \in L^1(\boldsymbol{R}^N)$ and $h>0$. Then,*

$$(2.2) \qquad |(C_h u)(x)| - |u(x)| \leq \int_{\boldsymbol{R}^N} \left(\int_{u(x)}^{u(x-h\xi)} \operatorname{sign}(\eta) \rho_h(\xi, \eta) d\eta \right) d\xi,$$

for $x \in \boldsymbol{R}^N$.

PROOF. Let $k \in \boldsymbol{R}$ and $x \in \boldsymbol{R}^N$. Since the function sign$(\cdot)$ is nondecreasing and $\rho_h \geq 0$, (2.1) implies

$$\begin{aligned} &\operatorname{sign}(k)((C_h u)(x) - u(x)) \\ &= \int_{\boldsymbol{R}^N} \left(\int_k^{u(x-h\xi)} \operatorname{sign}(k) \rho_h(\xi, \eta) d\eta \right) d\xi \\ &\leq \int_{\boldsymbol{R}^N} \left(\int_k^{u(x-h\xi)} \operatorname{sign}(\eta) \rho_h(\xi, \eta) d\eta \right) d\xi. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |k| - |u(x)| &= -\int_k^{u(x)} \operatorname{sign}(\eta) d\eta \\ &= -\int_{\boldsymbol{R}^N} \left(\int_k^{u(x)} \operatorname{sign}(\eta) \rho_h(\xi, \eta) d\eta \right) d\xi. \end{aligned}$$

Hence,

$$\begin{aligned} &|k| - |u(x)| + \operatorname{sign}(k)((C_h u)(x) - k) \\ &\leq \int_{\boldsymbol{R}^N} \left(\int_{u(x)}^{u(x-h\xi)} \operatorname{sign}(\eta) \rho_h(\xi, \eta) d\eta \right) d\xi. \end{aligned}$$

Taking $k = (C_h u)(x)$ in this inequality, we obtain (2.2). Q. E. D.

We set

$$A_h = h^{-1}(C_h - I) \quad \text{and} \quad J_{\lambda,h} = (I - \lambda A_h)^{-1}$$

for λ, $h>0$. Assertion (i) of Proposition 2.1 implies that each A_h is m-dissipative and the resolvent $J_{\lambda,h}$ is nonexpansive in $L^1(\boldsymbol{R}^N)$. The following result can be proved in the same way as in the proof of Proposition 2.2 in [7].

PROPOSITION 2.4. *Let λ, $h>0$. Then:*

(i) *$J_{\lambda,h}$ is a contraction operator on $L^1(\boldsymbol{R}^N)$ and $\|J_{\lambda,h}v\|_1 \leq \|v_1\|$ for $v \in L^1(\boldsymbol{R}_N)$.*

(ii) *$J_{\lambda,h}\tau^y = \tau^z J_{\lambda,H}$ for $y \in \boldsymbol{R}^N$.*

(iii) *If $v \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, then $J_{\lambda,h}v \in L^\infty(\boldsymbol{R}^N)$ and $\|J_{\lambda,h}v\|_\infty \leq \|v\|_\infty$.*

We define an operator A_0 in $L^1(\boldsymbol{R}^N)$ by

$$A_0 u = \Delta\psi(u) \qquad \text{for} \quad u \in D(A_0),$$

where the domain $D(A_0)$ of A_0 is the set of $u \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ such that $\Delta\psi(u)$ in the sense of distribution is an integrable function on $\boldsymbol{R}^N$. Let A be the closure of A_0 in $L^1(\boldsymbol{R}^N)$. The following fact is proved in [1] but we here gives its proof for completeness.

PROPOSITION 2.5. *The operators A_0 and A are dissipative in $L^1(\boldsymbol{R}^N)$.*

PROOF. It is sufficient to show that A_0 is dissipative in $L^1(\boldsymbol{R}^N)$. To this end, define a linear operator L by $Lu = \Delta u$ for $u \in D(L)$, where $D(L)$ is the set of $u \in L^1(\boldsymbol{R}^N)$ such that the distributional derivative Δu is in $L^1(\boldsymbol{R}^N)$. Since the assumptions on ψ imply that $\psi(u) \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ if $u \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, we have $A_0 u = L\psi(u)$ for $u \in D(A_0)$. It is known that the operator L is densely defined (linear) m-dissipative operator in $L^1(\boldsymbol{R}^N)$. (See, for example, Lemma 1.1 in [1].)

Let u, $v \in D(A_0)$ and λ, $\varepsilon > 0$. Since ψ is nondecreasing and $(I - \varepsilon L)^{-1}$ is nonexpansive in $L^1(\boldsymbol{R}^N)$, we have

$$\begin{aligned}
&\int_{\boldsymbol{R}^N} [|(u-v)(x) - \lambda\varepsilon^{-1}((I-\varepsilon L)^{-1}I)(\psi(u)-\psi(v))(x)| - |(u-v)(x)|]dx \\
&\geq -\lambda\varepsilon^{-1}\int_{\boldsymbol{R}^N} \operatorname{sign}((u-v)(x))\,[((I-\varepsilon L)^{-1}-I)(\psi(u)-\psi(v))(x)]dx \\
&\geq -\lambda\varepsilon^{-1}\int_{\boldsymbol{R}^N} |(I-\varepsilon L)^{-1}(\psi(u)-\psi(v))(x)| - |(\psi(u)-\psi(v))(x)|dx \\
&\geq 0.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ in this inequality gives

$$\int_{\boldsymbol{R}^N} [|(u-v)(x) - \lambda L(\psi(u)-\psi(v))(x)| - |(u-v)(x)|]dx \geq 0.$$

Q. E. D.

3. Proof of the Theorem

We start with the following lemma.

LEMMA 3.1. *Let $v \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$. Then*

$$\int_{|x|>R} |(J_{\lambda,h}v)(x)|dx \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty,$$

uniformly for bounded $h>0$ and bounded $\lambda>0$.

PROOF. Set $u_{\lambda,h}=J_{\lambda,h}v$ for λ, $h>0$. Then it follows from Proposition 2.4 that $u_{\lambda,h} \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ and

$$\|u_{\lambda,h}\|_p \leq \|v\|_p, \quad p = 1, \infty. \tag{3.1}$$

Since

$$h^{-1}(C_h u_{\lambda,h} - u_{\lambda,h}) = \lambda^{-1}(u_{\lambda,h} - v), \tag{3.2}$$

we have

$$\begin{aligned} &\lambda^{-1}(|u_{\lambda,h}(x)| - |v(x)|) \\ &\quad \leq \operatorname{sign}(u_{\lambda,h}(x))\lambda^{-1}(u_{\lambda,h}(x) - v(x)) \\ &\quad = \operatorname{sign}(u_{\lambda,h}(x))h^{-1}((C_h u_{\lambda,h})(x) - u_{\lambda,h}(x) \\ &\quad \leq h^{-1}(|(C_h u_{\lambda,h})(x)| - |u_{\lambda,h}(x)|). \end{aligned}$$

Hence, (2.2) with $u=u_{\lambda,h}$ implies

$$\begin{aligned} &\lambda^{-1}(|u_{\lambda,h}(x)| - |v(x)|) \\ &\quad \leq h^{-1}\int_{\boldsymbol{R}^N}\left(\int_{u_{\lambda,h}(x)}^{u_{\lambda,h}(x-h\xi)} \operatorname{sign}(\eta)\rho_h(\xi, \eta)d\eta\right)d\xi. \end{aligned} \tag{3.3}$$

Choose a function $g \in C^\infty(\boldsymbol{R})$ such that

$$g(s) = 1 \quad \text{if} \quad s \geq 1; \quad g(s) = 0 \quad \text{if} \quad s \leq 0; \quad \text{and} \quad 0 \leq g(s) \leq 1 \quad \text{for} \quad s \in \boldsymbol{R},$$

and define, for $R>r>0$, the function $f^{R,r} \in C^\infty(\boldsymbol{R}^N)$ by

$$f^{R,r}(x) = g((R-r)^{-1}(|x|-r)) \qquad \text{for} \quad x \in \boldsymbol{R}^N.$$

Since $f^{R,r}(x)=1$ if $|x| \geq R$; $f^{R,r}(x)=0$ if $|x| \leq r$; and $0 \leq f^{R,r}(x) \leq 1$ for $x \in \boldsymbol{R}^N$, it follows from (3.3) that

$$\begin{aligned} &\int_{|x|>R} |u_{\lambda,h}(x)|dx - \int_{|x|>r} |v(x)|dx \\ &\leq \int_{\boldsymbol{R}^N} (|u_{\lambda,h}(x)| - |v(x)|)f^{R,r}(x)dx \\ &\leq \lambda h^{-1}\int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_{u_{\lambda,h}(x)}^{u_{\lambda,h}(x-h\xi)} \operatorname{sign}(\eta)\rho_h(\xi, \eta)d\eta\right)d\xi\right]f^{R,r}(x)dx \\ &= \lambda h^{-1}\int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_0^{u_{\lambda,h}(x)} \operatorname{sign}(\eta)\rho_h(\xi, \eta)d\eta\right)(f^{R,r}(x+h\xi) - f^{R,r}(x))dx\right]d\xi. \end{aligned} \tag{3.4}$$

By Taylor's formula, we have

$$h^{-1}(f^{R,r}(x+h\xi)-f^{R,r}(x)) - \xi\cdot\nabla f^{R,r}(x)$$
$$\leq 2^{-1}Nh\sum_{i=1}^{N}|\xi_i|^2\sup_{i,j}\left\|\frac{\partial^2}{\partial x_i\partial x_j}f^{R,r}\right\|_\infty .$$

Therefore, in view of (1.4), (3.4) implies

$$\int_{|x|>R}|u_{\lambda,h}(x)|dx - \int_{|x|>r}|v(x)|dx$$
$$\leq \lambda N^2\int_{\boldsymbol{R}^N}\left[\int_0^{u_{\lambda,h}(x)}\operatorname{sign}(\eta)\psi'_h(\eta)d\eta\right]dx\cdot\sup_{i,j}\left\|\frac{\partial^2}{\partial x_i\partial x_j}f^{R,r}\right\|_\infty .$$

Thus, using (3.1), we obtain the estimate

$$\int_{|x|>R}|u_{\lambda,h}(x)|dx \leq \int_{|x|>r}|v(x)|dx$$
$$+ \lambda N^2[\sup_{|\eta|\leq\|v\|_\infty}\psi'_h(\eta)]\cdot\|v\|_1\cdot\sup_{i,j}\left\|\frac{\partial^2}{\partial x_i\partial x_j}f^{R,r}\right\|_\infty .$$

By the hypothesis on ψ'_h and the definition of $f^{R,r}$, we see that the second term on the hand side of the above inequality converges to zero as $R\to\infty$, uniformly for bounded λ, $h>0$. Therefore,

$$\limsup_{R\to\infty}\left(\sup_{0<\lambda<\lambda_0,0<h<h_0}\int_{|x|>R}|u_{\lambda,h}(x)|dx\right)\leq\int_{|x|>r}|v(x)|dx$$

for $\lambda_0>0$ and $h_0>0$. Since $v\in L^1(\boldsymbol{R}^N)$, this shows the desired assertion.

Q. E. D.

LEMMA 3.2. *Let $v\in L^1(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$ and $\lambda>0$. Then we have:*

(i) *The set $\{J_{\lambda,h}v;\ 0<h<h_0\}$ is precompact in $L^1(\boldsymbol{R}^N)$ for $h_0>0$.*

(ii) *If $\{h(n)\}$ is a null sequence such that $J_{\lambda,h(n)}v$ converges to a limit $u\in L^1(\boldsymbol{R}^N)$ as $n\to\infty$, then $u\in D(A_0)$ and $\lambda^{-1}(u-v)=A_0u$.*

PROOF. Let $h_0>0$ and set $u_{\lambda,h}=J_{\lambda h}v$ for λ, $h>0$. Assertions (i), (ii) of Proposition 2.4 together imply

$$\|\tau^y u_{\lambda,h}-u_{\lambda,h}\|_1 = \|J_{\lambda,h}\tau^y v-J_{\lambda,h}v\|_1 \leq \|\tau^y v-v\|_1 \tag{3.5}$$

for $h>0$ and $y\in\boldsymbol{R}^N$. Hence

$$\sup_{0<h<h_0}\|\tau^y u_{\lambda,h}-u_{\lambda,h}\|_1 \longrightarrow 0 \quad as \quad y\longrightarrow 0. \tag{3.6}$$

Furthermore, Lemma 3.1 implies that

(3.7) $$\sup_{0<h<h_0}\int_{|x|>R}|u_{\lambda,h}(x)|dx \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

In view of (3.1), (3.6) and (3.7), the well-known compactness criterion in $L^1(\boldsymbol{R}^N)$ can be applied to get the first assertion (i).

To prove (ii), let $\{h(n)\}$ be a null sequence such that $u_{\lambda,h(n)}$ converges a.e. to a limit $u_\lambda \in L^1(\boldsymbol{R}^N)$ as $n \to \infty$. It follows from (3.1) that $u_\lambda \in L^\infty(\boldsymbol{R}^N)$ and $\|u_\lambda\|_\infty \leq \|v\|_\infty$. Let $f \in C_0^\infty(\boldsymbol{R}^N)$. By (3.2) and the definition of C_h, we have

(3.8) $$\begin{aligned}&\int_{\boldsymbol{R}^N}\lambda^{-1}(u_{\lambda,h}(x)-v(x))f(x)dx\\
&= h^{-1}\int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_{u_{\lambda,h}(x)}^{u_{\lambda,h}(x-h\xi)}\rho_h(\xi,\eta)d\eta\right)d\xi\right]f(x)dx\\
&= h^{-1}\int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_0^{u_{\lambda,h}(x)}\rho_h(\xi,\eta)d\eta\right)\cdot(f(x+h\xi)-f(x))dx\right]d\xi.\end{aligned}$$

Using (1.4), we can rewrite the right side of the above equality as

$$\begin{aligned}&\int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_0^{u_{\lambda,h}(x)}\rho_h(\xi,\eta)d\eta\right)\cdot\{h^{-1}(f(x+h\xi)-f(x))-\xi\cdot\nabla f(x)\}dx\right]d\xi\\
&= \int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_0^{u_{\lambda,h}(x)}\rho_h(\xi,\eta)d\eta\right)\left\{\int_0^1\xi\cdot(\nabla f(x+\theta h\xi)-\nabla f(x))d\theta\right\}dx\right]d\xi\end{aligned}$$

and by a change of variables the above integral is transformed into

$$\begin{aligned}&\int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_0^{u_{\lambda,h}(x)}(2\psi_h'(\eta))(2h\psi_h'(\eta))^{-1/2}\right.\right.\\
&\qquad\left.\left.\cdot\left\{\int_0^1\xi\cdot(\nabla f(x+\theta(2h\psi_h'(\eta))^{1/2}\xi)-\nabla f(x))d\theta\right\}d\eta\right)\rho(\xi)d\xi\right]dx.\end{aligned}$$

Put $h=h(n)$ in (3.8) and let n tend to the infinity in the resultant equality. Since $\|u_{\lambda,h}\|_\infty$ is uniformly bounded for $h>0$ and $\psi_h'(\eta)$ is uniformly bounded for bounded $h>0$ and bounded $\eta \in \boldsymbol{R}$ by the assumption, the Lebesgue convergence theorem yields

$$\begin{aligned}&\int_{\boldsymbol{R}^N}\lambda^{-1}(u_\lambda(x)-v(x))f(x)dx\\
&= \int_{\boldsymbol{R}^N}\left[\int_{\boldsymbol{R}^N}\left(\int_0^{u_\lambda(x)}(2\psi_h'(\eta))\cdot\left\{\int_0^1\theta\, d\theta\right\}d\eta\right)\right.\\
&\qquad\qquad\left.\cdot\left(\textstyle\sum_{i,j=1}^N\xi_i\xi_j\dfrac{\partial^2}{\partial x_i\partial x_j}f(x)\right)\rho(\xi)d\xi\right]dx.\\
&= \int_{\boldsymbol{R}^N}\psi_h(u_\lambda(x))\Delta f(x)dx,\end{aligned}$$

where we have used (1.2). Hence $u_\lambda \in D(A_0)$ and $\lambda^{-1}(u_\lambda - v) = A_0 u_\lambda$. Q. E. D.

In the same way as in the proof of the consistency condition (d_2) in [7] (p. 504–505), we can show that the lemma above implies the following result.

Proposition 3.3. *The operator A is m-dissipative in $L^1(\boldsymbol{R}^N)$ and*

$$(I-\lambda A)^{-1}v = \lim_{h\downarrow 0} J_{\lambda,h}v \quad in \quad L^1(\boldsymbol{R}^N)$$

for $\lambda>0$ and $v\in L^1(\boldsymbol{R}^N)$.

We have also the following result.

Proposition 3.4. *The domain $D(A)$ of A is dense in $L^1(\boldsymbol{R}^N)$.*

Proof. Let $v\in L^1(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$ and set $u_\lambda=(I-\lambda A)^{-1}v$ for $\lambda>0$. Since $u_\lambda=\lim_{h\downarrow 0} J_{\lambda,h}v$ in $L^1(\boldsymbol{R}^N)$, (3.1) and (3.5) imply

$$\|u_\lambda\|_1 \le \|v_1\|_1 \quad \text{and} \quad \|\tau^y u_\lambda - u_\lambda\|_1 \le \|\tau^y v - v\|_1,$$

respectively. Also, Lemma 3.1 implies

$$\sup_{0<\lambda<\lambda_0}\int_{|x|>R}|u_\lambda(x)|dx \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty$$

for any $\lambda_0>0$. So, the Fréchet-Kolmogorov theorem implies that the set $\{u_\lambda;\ 0<\lambda<\lambda_0\}$ is precompact in $L^1(\boldsymbol{R}^N)$. Let $\{\lambda(n)\}$ be a null sequence such that $u_{\lambda(n)}$ converges in $L^1(\boldsymbol{R}^N)$ to a limit $u\in L^1(\boldsymbol{R}^N)$ as $n\to\infty$. Since $\|u_\lambda\|_\infty$ is bounded for $\lambda>0$ by (3.1), so is $\|\psi(u_\lambda)\|_\infty$ and $\lambda(n)\psi(u_{\lambda(n)})$ converges in the sense of distribution to zero as $n\to\infty$. Therefore, $u_{\lambda(n)}-v=\lambda(n)\Delta\psi(u_{\lambda(n)})$ converges in the sense of distribution to zero as $n\to\infty$ and $v=u=\lim_{n\to\infty} u_{\lambda(n)}$ in $L^1(\boldsymbol{R}^N)$. This implies $v\in\overline{D(A)}$. Since $L^1(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$ is dense in $L^1(\boldsymbol{R}^N)$, we see that $\overline{D(A)}=L^1(\boldsymbol{R}^N)$. Q.E.D.

By Propositions 3.3 and 3.4, the dissipative operator A generates a contraction semigroup $\{T(t)\}_{t\ge 0}$ on $L^1(\boldsymbol{R}^N)$. On the other hand, Theorem 2.1 and Proposition 3.2 together imply that, for $u_0\in L^1(\boldsymbol{R}^N)$

$$T(t)u_0 = \lim_{h\downarrow 0} C_h^{[t/h]}u_0 \tag{3.9}$$

holds in $L^1(\boldsymbol{R}^N)$ uniformly for bounded $t\ge 0$. Hence, it suffices to show that $u(t, x)=(T(t)u_0)(x)$ is a weak solution of the problem (1.1) provided $u_0\in L^1(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$. To this end, let $u_0\in L^1(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$ and set $u(t, x)=(T(t)u_0)(x)$. Obviously, $u\in C([0, \infty); L^1(\boldsymbol{R}^N))$ and, by Proposition 2.2 (iii) and (3.9), $u\in L^\infty((0, \infty)\times(\boldsymbol{R}^N)$ with $|u(t, x)|\le\|u_0\|_\infty$. Set $u_h(t, x)=(C_h^{[t/h]}u_0)(x)$. Since $(C_h u_h)(t,\cdot)=u_h(t+h,\cdot)$, we have $h^{-1}(u_h(t+h,\cdot)-u_h(t,\cdot))=(A_h u_h)(t,\cdot)$, so that

$$h^{-1}\int_0^\infty\left[\int_{R^N} u_h(t, x)(f(t-h, x)-f(t, x))dx\right]dt$$
$$= h^{-1}\int_0^\infty\left[\int_{R^N}\left(\int_{R^N}\left(\int_0^{u_{\lambda,h}(x)} \rho_h(\xi, \eta)d\eta\right)\cdot(f(t, x+h\xi)-f(t, x))dx\right)d\xi\right]dt$$

for $f\in C_0^\infty((0, \infty)\times R^N)$ and $h>0$ sufficiently small. Since $u_h(t,\cdot)$ converges to $u(t,\cdot)$in $L^1(R^N)$ as $h\downarrow 0$, uniformly for bounded $t\geq 0$, and $|u_h(t, x)|\leq\|v\|_\infty$, the same argument as in the proof of assertion (ii) of Lemma 3.2 yields

$$-\int_0^\infty\left[\int_{R^N} u(t, x)f_t(t, x)dx\right]dt = \int_0^\infty\left[\int_{R^N}\psi(u(t, x))\Delta f(t, x)dx\right]dt$$

for $f\in C_0^\infty((0, \infty)\times R^N)$. Thus, $u(t, x)$ is a weak solution of (1.1). This completes the proof of the Theorem.

References

[1] Ph. Bénilan, H. Brezis and M. Crandall, A semilinear equation in L^1, Ann. Scu. Norm. Sup. Pisa **2** (1975), 523–555.

[2] Ph. Bénilan and M. Crandall, The continuous dependence on ϕ of solutions of $u_t-\Delta\phi(u)=0$, Indiana Univ. Math. J. **30** (1981), 162–177.

[3] H. Brezis and M. Crandall, Uniqueness of solutions of the initial-value problem for $u_t-\Delta\phi(u)=0$, J. Math. Pure Appl. **58** (1979), 153–163.

[4] H. Brezis and A. Pazy, Convergence and approximation of nonlinear semigroups in Banach spaces, J, Funct. Anal. **9** (1972), 63–74.

[5] Y. Giga and T. Miyakawa, A kinetic constructions of global solutions of first order quasilinear equations, Duke Math. J. **50** (1983), 505–515.

[6] Y. Kobayashi, A new operator theoretic algorithm for solving fiirst order scalar quasilinear equations, Proc. of Symposia in Pure Mathematics, Vol. 43, Part 2 (1986), 65–75, (F. E. Browder, ed.), AMS, Providence.

[7] Y. Kobayashi, A product formula approach to first order quasilinear equations, Hiroshima Math. J. **14** (1985), 589–509.

[8] I. Miyadera, Nonlinear Semi-Groups, Kinokuniya, Tokyo, 1977, (in Japanese).

[9] T. Miyakawa, Construction of solutions of a semilinear parabolic equation with the aid of the linear Boltzman equation, Hiroshima Math. J. **14** (1984), 299–310.

[10] S. Oharu and T. Takahashi, A convergence theorem of nonlinear semigroups and its application to first order quasilinear equations, J. Math. Soc. Japan **26** (1974), 124–160.

[11] M. Watanabe, An approach by difference to a quasi-linear parabolic equations, Proc. Japan Acad. **59** (1983), 375–378.

[12] A. Vol'pert and S. Hudajev, Cauchy problems for degenerate second order quasilinear parabolic equation, Math. USSR Sbornik **7** (1969), 365–387.

Department of Mathematics,
Faculty of Enginnering,
Niigata University