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Normality, seminormality and quasinormality of $Z[\sqrt[n]{m}]$

Dedicated to Professor Masayoshi Nagata on his 60th birthday

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Introduction

In [11], Ooishi gave a necessary and sufficient condition for $\boldsymbol{Z}[\sqrt{m}]$ to be seminormal. In this paper, we will study $\boldsymbol{Z}[\sqrt[n]{m}]$ and give the criteria for $\boldsymbol{Z}[\sqrt[n]{m}]$ to be normal, p-seminormal, seminormal and quasinormal. First, we treat the normality of $\boldsymbol{Z}[\sqrt[n]{m}]$. Next, we construct some elements which are integral over $\boldsymbol{Z}$. Then using these elements, we study the p-seminormality, the seminormality and the quasinormality of $\boldsymbol{Z}[\sqrt[n]{m}]$.

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§1. Notation, terminology and preliminary results

Let A be a noetherian reduced ring. If the canonical homomorphism $\operatorname{Pic} A \to \operatorname{Pic} A[X]$ (or $\operatorname{Pic} A \to \operatorname{Pic} A[X, X^{-1}]$) is an isomorphism, where X is a variable, A is said to be *seminormal* (or *quasinormal*, resp.), and for an integer p if the kernel of $\operatorname{Pic} A[X] \to \operatorname{Pic} A$ has no p-torsion, A is said to be *p-seminormal*. These are chracterized as follows. The seminormality (or the p-seminormality) of A is equivalent to that if $x \in Q(A)$ satisfies $x^2, x^3 \in A$ (or $x^2, x^3, px \in A$, resp.), then $x \in A$ (cf. [5] or [12]). On the other hand in the case that $\dim A = 1$ and A is a domain, A is quasinormal if and only if the following conditions are satisfied: (1) A is seminormal, and (2) if $x \in Q(A)$ satisfies $x^2 - x, x^3 - x^2 \in A$, then $x \in A$ (cf. [10]). These are our main tools in this paper. Now normality, seminormality and p-seminormality are local properties, that is, A is normal (or seminormal, p-seminormal) if and only if so is $A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A (cf. [12]). If $\dim A = 1$ and A is a domain, quasinormality is a local property (cf. [2]). For an ideal $\mathfrak{a}$ of A, we write $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteqq \mathfrak{p}\}$. And we denote the normalization of A in $Q(A)$ by $\tilde{A}$.

We denote the set of natural numbers by $\boldsymbol{N}$, the set of integers by $\boldsymbol{Z}$, the set of rational numbers by $\boldsymbol{Q}$ and the prime field of characteristic p by $\boldsymbol{F}_p$.

Throughout this paper, m and n are integers with $n \geqq 2$. Moreover when $X^n - m$ is an irreducible polynomial over $\boldsymbol{Z}$, we denote a root of $X^n - m = 0$ by $\sqrt[n]{m}$.

If we write $n=p_1^{e_1}\cdots p_t^{e_t}$, this means the factorization of n into prime factors and $p_i\neq p_j$ for $i\neq j$. For integers p and q if p divides q in $\boldsymbol{Z}$, we write $p|q$. Moreover for a non-negative integer e if $p^e|q$ and $p^{e+1}\nmid q$, we write $p^e\|q$. The greatest common divisor of p and q is denoted by (p, q).

Finally we note the following theorem which plays an important role. We write $m=(\prod_{i=1}^{n-1}a_i^i)b^n$, where a_i, $b\in\boldsymbol{Z}$ $(i=1,\dots, n-1)$ and $\prod_{i=1}^{n-1}a_i$ is square-free. For $j\geqq 0$, put $m_j=\prod_{i=1}^{n-1}a_i^{ij-[ij/n]n}$ and put $\alpha_j=\sqrt[n]{m_j}$ in $\boldsymbol{Q}(\sqrt[n]{m})$ where $[\ ]$ is the Gauss' symbol. Obviously, if $(j, n)=1$, $\boldsymbol{Z}[\alpha_j]^{\sim}=\boldsymbol{Z}[\sqrt[n]{m}]^{\sim}$.

THEOREM 1.1 (cf. [6]). *Assume that n is a prime number. Then if* $m_1^{n-1}\not\equiv 1$ $(\mathrm{mod}\ n^2)$, $\boldsymbol{Z}[\sqrt[n]{m}]^{\sim}=\boldsymbol{Z}[\alpha_1,\dots,\alpha_{n-1}]$, *and if* $m_1^{n-1}\equiv 1$ $(\mathrm{mod}\ n^2)$, $(\boldsymbol{Z}[\sqrt[n]{m}]^{\sim}:\boldsymbol{Z}[\alpha_1,\dots,\alpha_{n-1}])=n$ *as an additive group.*

§2. Normality

Before considering the normality of $\boldsymbol{Z}[\sqrt[n]{m}]$, we will prove the following lemmas.

LEMMA 2.1. *Let p be a prime divisor of n. Then $\boldsymbol{Z}_{(p)}[\sqrt[n]{m}]$ is normal if and only if* $m^p\not\equiv m$ $(\mathrm{mod}\ p^2)$.

PROOF. Put $\vartheta=\sqrt[n]{m}$ and $n=p^eq$, where e, $q\in\boldsymbol{N}$ and $(p, q)=1$. We consider $\boldsymbol{Z}[\vartheta]$ as a homomorphic image of $\boldsymbol{Z}[X]$, where X is a variable. Then $\boldsymbol{Z}[\vartheta]\cong\boldsymbol{Z}[X]/(X^n-m)$. We write $X^n-m\equiv F_1(X)\cdots F_s(X)$ $(\mathrm{mod}\ p\boldsymbol{Z}[X])$, where $F_1(X),\dots, F_s(X)\in\boldsymbol{Z}[X]$ and each image of $F_i(X)$ in $\boldsymbol{F}_p[X]$ is irreducible. Putting $P_i=(p, F_i(X))\boldsymbol{Z}[X]$ $(i=1,\dots, s)$, we have $V((p, X^n-m))=\{P_1,\dots, P_s\}$ in Spec $\boldsymbol{Z}[X]$. Then $\boldsymbol{Z}_{(p)}[\vartheta]$ is normal if and only if X^n-m is a part of a regular system of parameters of $\boldsymbol{Z}[X]_{P_i}$ for all i. Now $X^n-m\equiv(X^q-m)^{p^e}$ $(\mathrm{mod}\ p\boldsymbol{Z}[X])$ and $e\geqq 1$. Hence, we can write $X^n-m=G(X)(X^q-m)^2+a(X^q-m)+m^{p^e}-m$, where $G(X)\in\boldsymbol{Z}[X]$ and a is an integer with $a\equiv 0$ $(\mathrm{mod}\ p)$. Moreover, since $m^{p^{e-1}}=m+kp$ for an integer k, $m^{p^e}=(m+kp)^p\equiv m^p$ $(\mathrm{mod}\ p^2)$. Hence, $X^n-m\equiv m^p-m$ $(\mathrm{mod}\ P_i^2)$ for all i. Thus, X^n-m is a part of a regular system of parameters of $\boldsymbol{Z}[X]_{P_i}$ for all i if and only if $m^p\not\equiv m$ $(\mathrm{mod}\ p^2)$. Q.E.D.

LEMMA 2.2. *Let K be an algebraic extension field of $\boldsymbol{Q}$ and ϑ be an element of K satisfying $\vartheta^n=m\in\boldsymbol{Z}$, where $n\in\boldsymbol{N}$. Assume that m is square-free. If $\boldsymbol{Z}[\vartheta]_{\mathfrak{p}}$ is not normal for some $\mathfrak{p}\in\mathrm{Spec}\,\boldsymbol{Z}[\vartheta]$, then $\mathfrak{p}\not\ni m$ and $\mathfrak{p}\ni n$. In particular $n\boldsymbol{Z}[\vartheta]^{\sim}\subseteqq\boldsymbol{Z}[\vartheta]$.*

PROOF. By [8, (10.18)], $nm\boldsymbol{Z}[\vartheta]^{\sim}\subseteqq\boldsymbol{Z}[\vartheta]$. Hence, we have $\mathfrak{p}\ni nm$. We write $\boldsymbol{Z}[\vartheta]\cong\boldsymbol{Z}[X]/(X^n-m)$. If $\mathfrak{p}\ni m$, then $\mathfrak{p}=(p, X)/(X^n-m)$ for a prime divisor p of m. Since m is square-free, $X^n-m\notin(p, X)^2\boldsymbol{Z}[X]_{(p,X)}$. Hence $\boldsymbol{Z}[\vartheta]_{\mathfrak{p}}$ is normal. This is a contradiction. Therefore, $\mathfrak{p}\not\ni m$ and $\mathfrak{p}\ni n$. Q.E.D.

THEOREM 2.3. *The following statements are equivalent*:
(a) $\boldsymbol{Z}[\sqrt[n]{m}]$ *is normal*;
(b) $\boldsymbol{Z}[\sqrt[t]{m}]$ *is normal for all* $t \in \boldsymbol{N}$ *such that* $t|n^{\alpha}$ *for* $\alpha \in \boldsymbol{N}$;
(c) m *is square-free and* $m^p \not\equiv m \pmod{p^2}$ *for all prime divisors* p *of* n.

PROOF. It is sufficient to prove the equivalence of (a) and (c). Put $\vartheta = \sqrt[n]{m}$. We may assume that m is square-free. Indeed, write $\boldsymbol{Z}[\vartheta] \cong \boldsymbol{Z}[X]/(X^n - m)$ and assume that $p^2|m$ for a prime divisor p of m. Then we have $X^n - m \in (p, X)^2 \boldsymbol{Z}[X]_{(p,X)}$, and $\boldsymbol{Z}[\vartheta]_{(p,\vartheta)}$ is not normal. Hence by (2.2), $\boldsymbol{Z}[\vartheta]$ is normal if and only if $\boldsymbol{Z}_{(p)}[\vartheta]$ is normal for all prime divisors p of n. Therefore equivalence of (a) and (c) is proved by (2.1). Q. E. D.

REMARK 2.4. For an integer m and a prime number p, $m^p \not\equiv m \pmod{p^2}$ if and only if $m \not\equiv r^p \pmod{p^2}$ for all $r = 0, \dots, p-1$.

By (1.1) and the proof of (2.3), we can give a generalization of (2.3) as follows, whose proof is omitted. The notation is as in Section 1.

PROPOSITION 2.5. *For an integer* n, *assume that* $(i, n) = 1$ *for all* i *such that* $a_i \neq \pm 1$. *Then* $\boldsymbol{Z}[\sqrt[n]{m}]^{\sim} = \boldsymbol{Z}[\alpha_1, \dots, \alpha_{n-1}]$ *if and only if* $m_1^p \not\equiv m_1 \pmod{p^2}$ *for all prime divisors* p *of* n *with* $(p, m_1) = 1$.

Next we shall show the following lemma concerning the normalization of $\boldsymbol{Z}[\sqrt[n]{m}]$, which will be used to prove (5.1).

LEMMA 2.6. *Let* $n = p_1^{e_1} \cdots p_t^{e_t}$ *be the factorization of* n *into prime factors and* m *be a square-free integer. Put* $\vartheta = \sqrt[n]{m}$, $q_i = p_i^{e_i}$ *and* $\vartheta_i = \vartheta^{n/q_i} = \sqrt[q_i]{m}$. *Then if* $\boldsymbol{Z}[\vartheta_i]^{\sim} = \boldsymbol{Z}[\vartheta_i, \delta_{i1}, \dots, \delta_{il_i}]$ *for all* i, *we have* $\boldsymbol{Z}[\vartheta]^{\sim} = \boldsymbol{Z}[\vartheta, \{\delta_{ij}\}_{1 \leqq i \leqq t, 1 \leqq j \leqq l_i}]$.

PROOF. Put $\Delta = \{\delta_{ij}\}$. We assume that $\boldsymbol{Z}[\vartheta, \Delta]_{\mathfrak{p}}$ is not normal for a maximal ideal $\mathfrak{p}$ of $\boldsymbol{Z}[\vartheta, \Delta]$. Since $\boldsymbol{Z}[\vartheta] \subseteqq \boldsymbol{Z}[\vartheta, \Delta] \subseteqq \boldsymbol{Z}[\vartheta]^{\sim}$, we have $\mathfrak{p} \not\ni m$ and $\mathfrak{p} \ni n$ by (2.2). Then $\mathfrak{p} \ni p_i$ for some i. Consider the following ring extensions:

$$\boldsymbol{Z}[\vartheta_i]^{\sim}[\vartheta] \subseteqq \boldsymbol{Z}[\vartheta, \Delta] \subseteqq \boldsymbol{Z}[\vartheta]^{\sim}.$$

Put $l = n/q_i$. Then since $\vartheta^l = \vartheta_i \in \boldsymbol{Z}[\vartheta_i]^{\sim}$, we have $lm\boldsymbol{Z}[\vartheta]^{\sim} \subseteqq \boldsymbol{Z}[\vartheta_i]^{\sim}[\vartheta]$ by [8, (10.18)]. Therefore $\boldsymbol{Z}[\vartheta, \Delta]_{\mathfrak{p}}$ is normal since $\mathfrak{p} \not\ni lm$. This is a contradiction. Q. E. .D

§ 3. Some integral elements

In this section, we construct some integral elements over $\boldsymbol{Z}[\sqrt[n]{m}]$, where m

is not necessarily square-free. One of these will be useful in the following sections.

PROPOSITION 3.1. *Let p be a prime number, e be a positive integer and $n=p^e$. Put $\vartheta=\sqrt[n]{m}$ for an integer m. Moreover let $e=a+b$, where $a, b\in \boldsymbol{Z}$, $a\geqq 0$ and $b\geqq 1$, and put $\alpha=p^a$ and $\beta=p^b$. For each positive integer k, we define an element of $\boldsymbol{Q}(\vartheta)$:*

$$\delta_{b,k} = p^{-k}\sum_{i=0}^{\beta-1}(m^\alpha)^{\beta-1-i}(\vartheta^\alpha)^i\,.$$

Then the minimal polynomial of $\delta_{b,k}$ over $\boldsymbol{Q}$ is

$$f_{b,k}(X) = X^\beta - \sum_{i=0}^{\beta-1}\binom{\beta}{i}p^{k(i-\beta)}m^{\alpha i}(m-m^n)^{\beta-1-i}X^i.$$

Moreover assume that $(m, p)=1$ and let s be a positive integer such that $p^s\| m^{n-1}-1$. Then $\delta_{b,k}$ is integral over $\boldsymbol{Z}$ if and only if $k\leqq\min\{b, s-1\}$.

PROOF. Put $\delta=\delta_{b,k}$, $f=f_{b,k}$, $h=m^\alpha$ and $\tau=\vartheta^\alpha$. Then $h^\beta=m^n$ and $\tau^\beta=m$. Now let ζ be a primitive β-th root of unity. Then the conjugate elements of τ over $\boldsymbol{Q}$ are $\tau, \tau\zeta,\dots, \tau\zeta^{\beta-1}$. Since $\delta(\tau-h)=p^{-k}(m-m^n)$, the conjugate elements of δ over $\boldsymbol{Q}$ are $\delta=l/(\tau-h)$, $l/(\tau\zeta-h),\dots, l/(\tau\zeta^{\beta-1}-h)$, where $l=p^{-k}(m-m^n)$. Since these elements are distinct, we have $[\boldsymbol{Q}(\delta)\colon \boldsymbol{Q}]=\beta=\deg f$. Thus if $f(\delta)=0$, then $f(X)$ is the minimal polynomial of δ over $\boldsymbol{Q}$. Therefore, we will show that $f(\delta)=0$. Now for $0\leqq j\leqq\beta-1$,

$$\tau^j p^k\delta = \sum_{i=0}^{j-1} mh^{j-1-i}\tau^i + \sum_{i=j}^{\beta-1} h^{\beta-1-i+j}\tau^i\,.$$

Hence we have

$$\sum_{i=0}^{j-1} mh^{j-1-i}\tau^i + (h^{\beta-1}-p^k\delta)\tau^j + \sum_{i=j+1}^{\beta-1} h^{\beta-1-i+j}\tau^i = 0.$$

Thus denoting by M the $n\times n$-matrix

$$\begin{pmatrix} h^{\beta-1}-p^k\delta & h^{\beta-2} & h^{\beta-3} & \cdots & 1 \\ m & h^{\beta-1}-p^k\delta & h^{\beta-2} & \cdots & h \\ mh & m & h^{\beta-1}-p^k\delta & \cdots & h^2 \\ \vdots & \vdots & \vdots & & \vdots \\ mh^{\beta-2} & mh^{\beta-3} & mh^{\beta-4} & \cdots & h^{\beta-1}-p^k\delta \end{pmatrix},$$

we have $\det M=0$. Putting $d=m-m^n$, we have

$$\det M=\det\begin{pmatrix} -p^k\delta & & & & 1 \\ d+p^kh\delta & -p^k\delta & & 0 & h \\ & d+p^kh\delta & & & h^2 \\ & & \ddots & & \vdots \\ 0 & & & -p^k\delta & h^{\beta-2} \\ & & & d+p^kh\delta & h^{\beta-1}-p^k\delta \end{pmatrix}$$

$$=(-p^k\delta)^\beta+(-1)^{\beta-1}\sum_{i=0}^{\beta-1}\binom{\beta}{i}h^i d^{\beta-1-i}p^{ki}\delta^i .$$

Therefore, $f(\delta)=0$. This completes the proof of the first assertion.

Now denote the coefficient of X^i of $f(X)$ for $0\leqq i\leqq\beta-1$ by c_i and denote the canonical valuation of $Z_{(p)}$ by v. Then since $(m, p)=1$ by our assumption, we have

$$(\sharp)\quad v(c_i)=v\left(\binom{\beta}{i}\right)-k+(s-k)(\beta-1-i).$$

First, we assume that $k\leqq\min\{b, s-1\}$. To prove that δ is integral over Z, it is sufficient to show that $v(c_i)\geqq 0$ for all i. If $i=0$, then $v(c_0)=-k+(s-k)(\beta-1)$ by ($\sharp$). Since $s-k\geqq 1$ and $\beta=p^b\geqq b+1\geqq k+1$, we have $v(c_0)\geqq 0$. If $i\geqq 1$, then $v\left(\binom{\beta}{i}\right)=v(\beta)-v(i)=b-v(i)$ (cf. [9, Lemma 2]). Hence $v(c_i)=(b-k)-v(i)+(s-k)(\beta-1-i)$ by ($\sharp$). Put $v(i)=t$ and $q=\beta-p^t$. Then since $v(q)=t$ and $q\geqq i$, and since $b\geqq k$ and $s-k\geqq 1$, we have $v(c_i)\geqq -t+(\beta-1-q)=-t+p^t-1\geqq 0$. Thus, δ is integral over Z. Next we assume that $k>\min\{b, s-1\}$. If $k>s-1$, then $v(c_0)\leqq -k<0$ by ($\sharp$). If $k>b$, then $v(c_{\beta-1})=b-k<0$ by ($\sharp$). Therefore δ is not integral over Z. Q.E.D.

COROLLARY 3.2. *Let p be a prime number and suppose that $m^{p-1}\equiv 1 \pmod{p^2}$. Put $\delta=p^{-1}\sum_{i=0}^{p-1}m^i(\sqrt[p]{m})^{p-1-i}$. Then using the notation in Section 1, we have $Z[\sqrt[p]{m}]^\sim=Z[\alpha_1,\dots,\alpha_{p-1},\delta]$.*

PROOF. Since $(m, p)=1$ by the assumption, we have $\delta\in Z[\sqrt[p]{m}]^\sim\setminus Z[\alpha_1,\dots,\alpha_{p-1}]$ by (3.1). Hence we have the conclusion by (1.1). Q.E.D.

REMARK. Let m be a square-free integer, n be an integer with $n\geqq 2$ and $n=p_1^{e_1}\cdots p_t^{e_t}$ be the factorization of n into prime factors. Put $q_i=p_i^{e_i}$. Let $\mathscr{B}=\{i\mid 1\leqq i\leqq t,\ m^{p_i-1}\equiv 1 \pmod{p_i^2}\}$. For each $i\in\mathscr{B}$, letting $p_i^{s_i}\| m^{q_i-1}-1$, put $k_i=\min\{e_i, s_i-1\}$ and $\Delta_i=\{\delta_{j,j}\mid 1\leqq j\leqq k_i\}$. Denote $\bigcup_{i\in\mathscr{B}}\Delta_i$ by Δ. If $\mathscr{B}=\emptyset$, we let $\Delta=\{1\}$. Now if n is square-free, we have $k_i=1$ for all $i\in\mathscr{B}$. Therefore $Z[\sqrt[n]{m}]^\sim=Z[\sqrt[n]{m},\Delta]$ by (2.3), (2.6) and (3.2). From this, the following question arises: Does $Z[\sqrt[n]{m}]^\sim=Z[\sqrt[n]{m},\Delta]$ hold for all $n\geqq 2$? For this question, by (2.6) we may assume that $n=p^e$ for a prime number p and $e\in N$.

When $n=4$, this question can be solved by an elementary calculation. A related topic is also treated in [9].

The following proposition gives some properties of the integral element $\delta_{b,1}$ in (3.1).

PROPOSITION 3.3. *In* (3.1), *let* $a=0$ *and* $b=e$, *and put* $\delta=\delta_{e,1}$. *Assume that* $p^s \| m^{n-1}-1$ *for some integer* $s>1$. *Then* $\delta \in \boldsymbol{Z}[\vartheta]^{\sim} \setminus \boldsymbol{Z}[\vartheta]$, *and we have the following*:

(a) *If* $e \geqq 2$, *then* $\delta^2, \delta^3 \in \boldsymbol{Z}[\vartheta]$ *and hence* $\boldsymbol{Z}[\vartheta]$ *is not seminormal.*

(b) *If* $e=1$ *and* p *is odd, or if* $e=1$, $p=2$ *and* $s \geqq 3$, *then* $\delta^2-\delta$, $\delta^3-\delta^2 \in \boldsymbol{Z}[\vartheta]$ *and hence* $\boldsymbol{Z}[\vartheta]$ *is not quasinormal.*

(c) *If* $e=1$, $p=2$ *and* $s=2$, *then* $\delta^2-\delta$, $\delta^3 \in \boldsymbol{Z}[\vartheta]$.

PROOF. Since $\min\{e, s-1\} \geqq 1$, we have $\delta \in \boldsymbol{Z}[\vartheta]^{\sim} \setminus \boldsymbol{Z}[\vartheta]$ by (3.1). Now since $\vartheta^n=m$, we have

$$\delta^2 = p^{-2} \textstyle\sum_{k=0}^{n-1} ((k+1)m^{2(n-1)-k} + (n-(k+1))m^{n-1-k})\vartheta^k .$$

Putting $m^{n-1}-1=p^s u$ ($u \in \boldsymbol{Z}$), we have

$$\delta^2 = p^{s-2} u \textstyle\sum_{k=0}^{n-1} (k+1) m^{n-1-k} \vartheta^k + p^{e-1}\delta .$$

Since $s \geqq 2$, we have

$$\delta^2 - p^{e-1}\delta = p^{s-2} u \textstyle\sum_{k=0}^{n-1} (k+1) m^{n-1-k} \vartheta^k \in \boldsymbol{Z}[\vartheta]. \tag{3.3.1}$$

Next by (3.3.1) and the above calculation,

$$\delta^3 - p^{e-1}\delta^2 = p^{2s-3} u^2 \textstyle\sum_{k=0}^{n-1} m^{n-1-k} \sum_{i=0}^{k} (i+1)\vartheta^k + 2^{-1} p^{e+s-2}(p^e+1)u\delta .$$

Since $s \geqq 2$, we have

$$\delta^3 - p^{e-1}\delta^2 - 2^{-1}p^{e+s-2}(p^e+1)u\delta \in \boldsymbol{Z}[\vartheta]. \tag{3.3.2}$$

Thus since $p\delta \in \boldsymbol{Z}[\vartheta]$ and $(u, p)=1$, (a), (b) and (c) easily follow from (3.3.1) and (3.3.2). Q.E.D.

The following proposition can be easily shown. Therefore we omit the proof.

PROPOSITION 3.4. *Let* p *be a prime number and* $n \in \boldsymbol{N}$, *and put* $\vartheta=\sqrt[n]{m}$ *and* $x=\vartheta^{n-1}/p$. *Then if* $n \geqq 3$ *and* $p^2 | m$, *or if* $n=2$ *and* $p^3 | m$, *we have* $x \in \boldsymbol{Z}[\vartheta]^{\sim} \setminus \boldsymbol{Z}[\vartheta]$ *and* $x^2, x^3 \in \boldsymbol{Z}[\vartheta]$. *In particular,* $\boldsymbol{Z}[\vartheta]$ *is not seminormal.*

§4. *p*-seminormality and seminormality

First we will treat the p-seminormality of $\boldsymbol{Z}[\sqrt[n]{m}]$. Since a ring A is p-

seminormal if and only if A is q-seminormal for all prime divisors q of p, we have only to study the case that p is a prime number. Our result on the p-seminormality of $\boldsymbol{Z}[\sqrt[n]{m}]$ is as follows.

THEOREM 4.1. *For a prime number p, $\boldsymbol{Z}[\sqrt[n]{m}]$ is p-seminormal if and only if one of the following conditions holds.*

(a) $p \parallel m$;
(b) $(p, mn) = 1$;
(c) $n \geqq 3, (p, m) = 1$ *and* $p \parallel n$;
(d) $n \geqq 3, (p, m)=1, p \mid n$ *and* $m^p \not\equiv m \pmod{p^2}$;
(e) $n = 2$ *and* $(p, m) = 1$;
(f) $n = 2, p \neq 2$ *and* $p^2 \parallel m$.

Before proving this, we show the following two lemmas.

LEMMA 4.2. *For a prime number p, $\boldsymbol{Z}[\sqrt[n]{m}]$ is p-seminormal if and only if $\boldsymbol{Z}_{(p)}[\sqrt[n]{m}]$ is seminormal.*

In (6.5) this lemma will be generalized and proved.

LEMMA 4.3. *Let $p^e \| n$ for a prime number p and some $e \in \boldsymbol{N}$, and suppose $(p, m)=1$. Then $\boldsymbol{Z}_{(p)}[\sqrt[n]{m}]$ is seminormal if and only if $\boldsymbol{Z}_{(p)}[\sqrt[p^e]{m}]$ is seminormal.*

PROOF. Put $\vartheta=\sqrt[n]{m}$ and $\tau=\vartheta^{n/p^e}=\sqrt[p^e]{m}$. Since $\boldsymbol{Z}_{(p)}[\vartheta]$ is free over $\boldsymbol{Z}_{(p)}[\tau]$, the "only if" part easily follows. Now we can write $\boldsymbol{Z}_{(p)}[\vartheta] \cong \boldsymbol{Z}_{(p)}[\tau][X]/(X^r-\tau)$, where $r=n/p^e$ and the image of X in $\boldsymbol{Z}_{(p)}[\vartheta]$ is ϑ. Then since $\vartheta^n=m$, we see that r and ϑ are invertible in $\boldsymbol{Z}_{(p)}[\vartheta]$ by our assumption. Hence $\boldsymbol{Z}_{(p)}[\vartheta]$ is smooth over $\boldsymbol{Z}_{(p)}[\tau]$. Therefore if $\boldsymbol{Z}_{(p)}[\tau]$ is seminormal, $\boldsymbol{Z}_{(p)}[\vartheta]$ is also seminormal by [13, Prop. 1.5] and [3, 5.8. Th.], because $\boldsymbol{Z}_{(p)}[\tau]$ is a Mori ring. Q. E. D.

PROOF of (4.1). Put $\vartheta=\sqrt[n]{m}$ and $A=\boldsymbol{Z}_{(p)}[\vartheta]$. By (4.2) it is sufficient to consider the condition for A to be seminormal. By the proof of (2.2), if $(p, mn)=1$ or $p\|m$, A is normal. Hence by (3.4) and (4.3) we have only to consider the following two cases.

Case 1. $n \geqq 3$, $(m, p)=1$ and $n=p^e$, where $e \in \boldsymbol{N}$. If $m^p \not\equiv m \pmod{p^2}$, A is normal by (2.1). On the other hand if $m^p \equiv m \pmod{p^2}$ and $e \geqq 2$, A is not seminormal by (3.3). Thus we reduce ourselves to the case that $n=p$ and $m^{p-1} \equiv 1 \pmod{p^2}$. We will prove that A is seminormal. First by (3.2) and (3.3), putting $\delta=p^{-1}\sum_{i=0}^{p-1} m^{p-1-i}\vartheta^i$, we have $\tilde{A}=A+A\delta$. Let $x=\alpha+\beta\delta \in \tilde{A}$ $(\alpha, \beta \in A)$ satisfy $x^2, x^3 \in A$. Since $\operatorname{Max} A=\{(p, \vartheta-m)\}$ and since $p\delta \in A$ and $(\vartheta-m)\delta=p^{-1}(m-m^p) \in A$, we may suppose that $\beta=1$. Now p is odd, because $n=p \geqq 3$. Hence by $x^2, x^3 \in A$ and (3.3), we have $(2\alpha+1)\delta$, $(3\alpha^2+3\alpha+1)\delta \in A$. From this it follows

easily that $\delta \in A$, and hence $x \in A$. Thus A is seminormal. This completes the proof of Case 1.
Case 2. $n=2$. If $p=2$ and $m=4r$ for an odd integer r, then $x=1+\sqrt{r}$ satisfies $x \in \tilde{A} \setminus A$ and x^2, $x^3 \in A$. Hence A is not seminormal. Therefore by (3.4) and the above, we may assume that $(m, p)=1$ or $p^2 \| m$, $p \neq 2$. We will prove that A is seminormal in either case. Let $x \in \tilde{A}$ satisfy x^2, $x^3 \in A$. First suppose that $(m, p)=1$. Then if $m=ab^2$ for a square-free integer a and some $b \in \boldsymbol{Z}$, we have $A=\boldsymbol{Z}_{(p)}[\sqrt{a}]$. Hence we may assume that m is square-free. Moreover by a well-known result, we may also assume that $m \equiv 1 \pmod 4$ and $p=2$. Then $\tilde{A}=A+A\delta$ for $\delta=(m+\vartheta)/2$. Hence we can write $x=\alpha+\beta\delta$ for some α, $\beta \in A$. By the same manner as in Case 1, we may suppose that $\beta=1$. Then since $x^2 \in A$, we have $\delta \in A$ by (3.3). Hence $x \in A$. Therefore A is seminormal. Next we suppose that $p^2 \| m$ and $p \neq 2$. Put $m=p^2 r$ $(r \in \boldsymbol{Z})$. Then $\sqrt{m}=p\sqrt{r}$ and $(r, p)=1$. Since $p \neq 2$, it follows that $\boldsymbol{Z}_{(p)}[\sqrt{r}]$ is normal and $x \in \boldsymbol{Z}_{(p)}[\sqrt{r}]$. Hence we can write $x=\alpha+\beta\sqrt{r}$ for some α, $\beta \in \boldsymbol{Z}_{(p)}$. Then since x^2, $x^3 \in A$, it follows easily that $x \in A$. Thus A is seminormal. Q.E.D.

$\boldsymbol{Z}[\sqrt[n]{m}]$ is seminormal if and only if $\boldsymbol{Z}[\sqrt[n]{m}]$ is p-seminormal for all prime numbers p. Therefore we obtain the following criterion for the seminormality of $\boldsymbol{Z}[\sqrt[n]{m}]$ by (4.1).

THEOREM 4.4. *$\boldsymbol{Z}[\sqrt[n]{m}]$ is seminormal if and only if one of the following conditions holds.*

(a) *m is square-free, and for each prime divisor p of n,*

(i) $m^p \not\equiv m \pmod{p^2}$, *or*

(ii) $p \| n$;

(b) *m is not square-free, $n=2$ and $m=ab^2$, where a and b are square-free integers, b is odd and $(a, b)=1$.*

REMARK. The result of the case $n=2$ in (4.4) has been already obtained by Ooishi (cf. [11]).

§5. Quasinormality

Let s and t be integers with $(s, t)=1$ and $t \geqq 2$. We denote the order of s in the unit group of $\boldsymbol{Z}/t\boldsymbol{Z}$ by $\mathrm{ord}_t(s)$. For the sake of convenience we define $\mathrm{ord}_1(s)=1$. Then we have the following theorem.

THEOREM 5.1. *$\boldsymbol{Z}[\sqrt[n]{m}]$ is quasinormal if and only if one of the following conditions holds.*

(a) *m is square-free, and for each prime divisor p of n,*

(i) $m^p \not\equiv m \pmod{p^2}$, *or*

(ii) $p=2$, $2 \| n$, $m \equiv 5 \pmod 8$ *and* $\mathrm{ord}_{n/2}(2)$ *is odd*;

(b) *m is not square-free, $n=2$ and $m=ab^2$ for relatively prime square-free integers a and b satisfying that $a \not\equiv 1 \pmod 8$, b is odd and $\left(\frac{a}{p}\right)=-1$ for all prime divisors p of b, where $\left(\frac{a}{p}\right)$ stands for the Legendre's symbol.*

First, we note the following lemma.

LEMMA 5.2. *Let K be a finite algebraic extension field of $\boldsymbol{F}_2$. Then $X^2+X+1=0$ has a root in K if and only if $[K: \boldsymbol{F}_2]$ is even.*

PROOF of (5.1). We put $\vartheta=\sqrt[n]{m}$ and $A=\boldsymbol{Z}[\vartheta]$.
Case 1. m is square-free. For $q \in \boldsymbol{N}$ with $q|n$, A is free over $\boldsymbol{Z}[\sqrt[q]{m}]$. Hence if A is quasinormal, $\boldsymbol{Z}[\sqrt[q]{m}]$ is also quasinormal. Moreover for a prime number p, if p is odd and $\boldsymbol{Z}[\sqrt[p]{m}]$ is not normal, or if $p=2$ and $m \equiv 1 \pmod 8$, then $\boldsymbol{Z}[\sqrt[p]{m}]$ is not quasinormal by (3.3). Therefore by the implications normal$\Rightarrow$quasinormal$\Rightarrow$seminormal and by (2.3) and (4.4), we may assume that, for an odd natural number r, $n=2r$, $\boldsymbol{Z}[\sqrt[r]{m}]$ is normal and $m \equiv 5 \pmod 8$. Then putting $\delta=(m+\sqrt{m})/2$, we have $\tilde{A}=A+A\delta$ by (2.6), (3.2) and (3.3). Hence for each $x \in \tilde{A}$ we can write $x=\alpha+\beta\delta$ for some α, $\beta \in A$. Then by $2\delta \in A$ and (3.3), we have x^2-x, $x^3-x^2 \in A$ if and only if $(\beta^2-\beta)\delta$, $\beta(\alpha^2+\alpha+1)\delta \in A$. Now for $z \in A$, it can be easily seen that $z\delta \in A$ if and only if $z \in (2, \vartheta^r-1)$. Moreover since $A \cong \boldsymbol{Z}[X]/(X^n-m)$ and $(2, X^r-1) \ni X^n-m$, we have $\boldsymbol{Z}[\vartheta]/(2, \vartheta^r-1) \cong \boldsymbol{F}_2[X]/(X^r-1)$. Therefore A is not quasinormal if and only if there exist f and g in $\boldsymbol{F}_2[X]$ such that g^2-g, $g(f^2+f+1) \in (X^r-1)\boldsymbol{F}_2[X]$ and $g \notin (X^r-1)\boldsymbol{F}_2[X]$. We write $X^r-1=F_1(X)\cdots F_u(X)$ in $\boldsymbol{F}_2[X]$, where each $F_i(X)$ is irreducible. Then since $F_1(X),\dots, F_u(X)$ are relatively prime, putting $K_i=\boldsymbol{F}_2[X]/(F_i(X))$, we have $\boldsymbol{F}_2[X]/(X^r-1) \cong K_1 \times \cdots \times K_u$. Hence by (5.2), A is not quasinormal if and only if $[K_i: \boldsymbol{F}_2]$ is even for some i. Now for the minimal splitting field K of $X^r-1=0$ over $\boldsymbol{F}_2$, we have $[K: \boldsymbol{F}_2]=\mathrm{ord}_r(2)$ by [1, chap. V, §11 Exercices 1)], and $K=K_i$ for some i. Therefore $[K_i: \boldsymbol{F}_2]$ is even for some i if and only if $\mathrm{ord}_r(2)$ is even. Thus the proof for Case 1 is completed.
Case 2. m is not square-free. Since quasinormality implies seminormality, by (4.4) we may assume that $n=2$ and $m=ab^2$, where a and b are square-free integers, b is odd and $(a, b)=1$. Then $\vartheta=b\sqrt{a}$. First note that if $a \equiv 1 \pmod 8$, A is not quasinormal by (3.3). Therefore we may assume that $a \not\equiv 1 \pmod 8$. Now let x be an element of $\tilde{A}$ satisfying x^2-x, $x^3-x^2 \in A$. Then $x \in \boldsymbol{Z}[\sqrt{a}]$, because $A \subseteq \boldsymbol{Z}[\sqrt{a}]$ and $\boldsymbol{Z}[\sqrt{a}]$ is quasinormal by Case 1. Hence we can write $x=\alpha+\beta\sqrt{a}$ $(\alpha, \beta \in \boldsymbol{Z})$. Therefore A is not quasinormal if and only if there exist α and β in $\boldsymbol{Z}$ such that $b \nmid \beta$ and $\beta(2\alpha-1)$, $\beta(3\alpha^2+\beta^2 a-2\alpha) \in b\boldsymbol{Z}$. Thus A is quasinormal if and only if $X^2 a \equiv 1 \pmod{b_0}$ does not have any roots in $\boldsymbol{Z}$ for all b_0 such that $b_0>1$ and $b_0|b$. This is equivalent to that $\left(\frac{a}{p}\right)=-1$ for all prime divisors p of b.

Q.E.D.

§ 6. Some remarks

As stated in Section 1, we know the implications normal$\Rightarrow$quasinormal$\Rightarrow$ seminormal$\Rightarrow$$p$-seminormal. But the converse of each implication does not necessarily hold as shown by well-known examples. But by (2.3), (4.1), (4.4) and (5.1), we can easily construct such examples in the case of dimension 1, though they do not contain a field.

EXAMPLE 6.1. Let $A_1=\boldsymbol{Z}[\sqrt{5}]$ and $A_2=\boldsymbol{Z}[\sqrt{17}]$, and let $A_3=\boldsymbol{Z}[\sqrt{pq^3}]$ for distinct prime numbers p and q. Then A_1 is quasinormal but not normal, A_2 is seminormal but not quasinormal and A_3 is p-seminormal but not seminormal. Moreover A_3 is p-seminormal but not q-seminormal.

Next let $n=p_1^{e_1}\cdots p_t^{e_t}$ be the factorization of n into prime factors and put $q_i=p_i^{e_i}$ for each i. Then by (2.3) (or (4.4)) we see that $\boldsymbol{Z}[\sqrt[n]{m}]$ is normal (or seminormal, resp.) if and only if $\boldsymbol{Z}[\sqrt[q_i]{m}]$ is normal (or seminormal, resp.) for all i. If m is square-free, this result can be generalized as follows. Let $\boldsymbol{P}$ be a property concerning noetherian rings satisfying the following four axioms.

(1) A ring R has $\boldsymbol{P}$ if and only if $R_{\mathfrak{p}}$ has $\boldsymbol{P}$ for all $\mathfrak{p}\in \operatorname{Max}(R)$;

(2) Regularity implies $\boldsymbol{P}$;

(3) Let $(A, \mathfrak{m})\rightarrow(B, \mathfrak{n})$ be a flat local homomorphism. Then if B has $\boldsymbol{P}$, A also has $\boldsymbol{P}$;

(4) In (3), if A has $\boldsymbol{P}$ and the canonical homomorphism $A\rightarrow B$ is regular, B also has $\boldsymbol{P}$.

REMARK. (a) If B is smooth over A, then $A\rightarrow B$ is regular (cf. [13, Prop. 1.5]).

(b) If $\boldsymbol{P}$=normality or seminormal Mori, $\boldsymbol{P}$ satisfies the above four axioms (cf. [4, IV (6.5.4)] or [3, 5.8. Th.]).

Then we have the following proposition, which is proved easily by (2.2) and (4.3).

PROPOSITION 6.2. *Let $n=p_1^{e_1}\cdots p_t^{e_t}$ be the factorization of n into prime factors and $\boldsymbol{P}$ be a property satisfying the above four axioms, and assume that m is square-free. Put $q_i=p_i^{e_i}$. Then $\boldsymbol{Z}[\sqrt[n]{m}]$ has $\boldsymbol{P}$ if and only if $\boldsymbol{Z}[\sqrt[q_i]{m}]$ has $\boldsymbol{P}$ for all i.*

For quasinormality, (6.2) does not hold as follows.

EXAMPLE 6.3. By (5.1), $\boldsymbol{Z}[\sqrt[6]{13}]$ is not quasinormal, but $\boldsymbol{Z}[\sqrt{13}]$ is quasinormal and $\boldsymbol{Z}[\sqrt[3]{13}]$ is normal. Indeed, quasinormality does not satisfy the axiom (4). For example, $\boldsymbol{Z}_{(2)}[\sqrt{13}]$ is quasinormal by [2, 4.7. Cor.] and

$Z_{(2)}[\sqrt[6]{13}]$ is etale over $Z_{(2)}[\sqrt{13}]$, but $Z_{(2)}[\sqrt[6]{13}]$ is not quasinormal. Now in [2, 4.13. Example], Greco also shows such an example due to Ferrand which contains the real number field. But as above, we can construct many such examples by (5.1), though they do not contain a field.

Finally, for p-seminormality, we can prove the following proposition. Since p-seminormality satisfies the axioms (1), (2) and (3) (cf. [12]), (6.2) holds for $\boldsymbol{P}=p$-seminormality.

PROPOSITION 6.4. *Let A and B be noetherian reduced rings. Assume that A is a Mori ring satisfying (S_2) and the canonical homomorphism $A \to B$ is normal. Then for a prime number p, if A is p-seminormal, B is also p-seminormal.*

For the proof, we first show the following lemma which is a generalization of (4.2).

LEMMA 6.5. *Let R be a noetherian reduced ring satisfying (S_2) and let p be a prime number. Then R is p-seminormal if and only if $R_{\mathfrak{p}}$ is seminormal for all height 1 prime ideals $\mathfrak{p}$ containing p.*

PROOF. By the proof of [12, Th. 6.1], R is p-seminormal if and only if $R_{\mathfrak{p}}$ is p-seminormal for all $\mathfrak{p} \in \operatorname{Spec} R$ with depth $R_{\mathfrak{p}}=1$. Now seminormality implies p-seminormality, and if the image of p in R is not zero, $R[1/p]$ is always p-seminormal. Moreover since R satisfies (S_2), depth $R_{\mathfrak{p}}=1$ implies ht $\mathfrak{p}=1$. Therefore it is sufficient to show that if R is a noetherian reduced p-seminormal local ring of dimension 1 with p not invertible, R is seminormal. Assume that R is not seminormal. Then there exists $x \in Q(R) \setminus R$ such that x^2, $x^3 \in R$. We will show that R is not p-seminormal.
Case 1. p is a non-zero divisor in R. Then $Q(R)=R[1/p]$. Hence we can assume that $px \in R$. Thus R is not p-seminormal.
Case 2. p is a zero divisor in R. Put $R_1=R/\sqrt{pR}$ and $R_2=R/0: pR$. Since R is reduced, we have the following ring extensions:

$$R \longrightarrow R_1 \times R_2 \longrightarrow \tilde{R}_1 \times \tilde{R}_2 .$$

Then $\tilde{R}=\tilde{R}_1 \times \tilde{R}_2$. Now since $x \in \tilde{R}$, we can write $x=(x_1, x_2)$, where $x_i \in \tilde{R}_i$, $i=1, 2$. By Case 1, $p^e x_2 \in R_2$ for some non-negative integer e. Let e be the minimal non-negative integer satisfying this. If $p^e x \in R$, then $e \geqq 1$ by our assumption. Then since $p^{e-1} x_2 \notin R_2$ by the minimality of e, we have $p^{e-1} x \notin R$. Since $(p^{e-1} x)^2$, $(p^{e-1} x)^3 \in R$, we see that R is not p-seminormal. On the other hand, if $p^e x \notin R$, then $p^{e+1} x \in R$. Indeed, since $p^{e+1} x=(0, p^{e+1} x_2)$ and since $p^e x_2 \in R_2=R/0: pR$ and $pR_1=0$, we have $p^{e+1} x \in pR$. Since $(p^e x)^2$, $(p^e x)^3 \in R$, it follows that R is not p-seminormal. Q.E.D.

PROOF of (6.4). Since $A \to B$ is normal, B is also a Mori ring satisfying (S_2). Therefore by (6.5), it is sufficient to show that $B_{\mathfrak{p}}$ is seminormal for all height 1 prime ideals $\mathfrak{P}$ containing p. Put $\mathfrak{p} = \mathfrak{P} \cap A$. Since B is flat over A, we have $\mathrm{ht}\,\mathfrak{p} \leqq 1$. Then $A_{\mathfrak{p}}$ is seminormal by (6.5). Since $A_{\mathfrak{p}} \to B_{\mathfrak{P}}$ is normal, it follows that $B_{\mathfrak{P}}$ is seminormal by [3, 5.8. Th.]. Q. E. D.

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