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Positive entire solutions of semilinear biharmonic equations

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1. Introduction

Our main objective is to prove the existence of infinitely many positive, radially symmetric entire solutions $u(x)$ of the fourth order semilinear elliptic differential equation

$$(1.1) \qquad \Delta^2 u = f(|x|, u), \quad x \in \boldsymbol{R}^N, \quad N \geqq 3,$$

where Δ denotes the N-dimensional Laplacian, $|x|$ denotes the Euclidean length of x, and f is a real-valued continuous function in $[0, \infty) \times (0, \infty)$. An *entire solution* of (1.1) is defined to be a function $u \in C^4(\boldsymbol{R}^N)$ satisfying (1.1) pointwise in $\boldsymbol{R}^N$. Detailed hypotheses on (1.1) to be used for existence theorems of three different types are listed in §2.

Emphasis will be placed on the prototype

$$(1.2) \qquad \Delta^2 u = p(|x|)u^{\gamma}, \quad x \in \boldsymbol{R}^N,$$

where $\gamma \neq 1$ is a real constant and $p\colon [0, \infty) \to \boldsymbol{R}$ is a continuous function satisfying one of the following three decay conditions:

$$(1.3) \qquad \int^{\infty} t^{2\gamma+1}|p(t)|dt < \infty, \quad N \geqq 3;$$

$$(1.4) \qquad \int^{\infty} t^3|p(t)|dt < \infty, \qquad N \geqq 5;$$

$$(1.5) \qquad \int^{\infty} t^{\delta}p(t)dt < \infty, \qquad N \geqq 5,$$

where $\delta = N-1-\gamma(N-4)$, $-1<\gamma<1$, and $p(t) \geqq 0$ in (1.5). Our results establish, in particular, the existence of infinitely many positive entire solutions of (1.2) of *each* of the following three types under conditions (1.3), (1.4), or (1.5), respectively:

(I) Unbounded entire solutions $u(x)$ which are bounded above and below by positive constant multiples of $1+|x|^2$;

(II) Entire solutions which are bounded above and below by positive constants; and

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(III) Entire solutions which decay uniformly to zero as $|x|\to\infty$.

Moreover, Corollaries 2.7 and 2.8 and Theorem 2.12 together show, under condition (1.5), that equation (1.2) has infinitely many *triples* (u_1, u_2, u_3) of positive, radially symmetric entire solutions in $\boldsymbol{R}^N$, $N\geqq 5$, where $u_1(x)$ is unbounded, $u_2(x)$ is bounded above and below by positive constants, and $u_3(x)$ decays to zero as $|x|\to\infty$. If $p(t)\geqq 0$ for all $t\geqq 0$, these entire solutions have the asymptotic behaviour

$$\lim_{|x|\to\infty} |x|^{-2}u_1(x) = A_1, \tag{1.6}$$

$$\lim_{|x|\to\infty} u_2(x) = A_2, \tag{1.7}$$

$$\lim_{|x|\to\infty} |x|^{N-4}u_3(x) = A_3 \tag{1.8}$$

for some positive constants A_1, A_2, A_3.

Conditions (1.3)–(1.5) are sharp: In fact, if $p(t)$ has constant sign for all $t\geqq 0$, each condition (1.3), (1.4), or (1.5) is known to be *necessary* for the existence of a radial solution $u(x)$ of (1.2) in some exterior domain satisfying (1.6), (1.7), or (1.8), respectively [1, p. 237].

Entire solutions of the equation $\Delta^m u=f(u)$, $m\geqq 2$, have been investigated by Walter [19, 20] and Walter and Rhee [21]. In particular, conditions on f are given which guarantee that no entire solution in $\boldsymbol{R}^N$ exists. Entire solutions of a class of second order semilinear elliptic systems were studied recently by Kawano [7] and Kawano and Kusano [8]. However, as far as we are aware, no theorems are known which guarantee the existence of positive entire solutions of (1.1) or (1.2) of any of the three types described above. Furthermore, we give explicit criteria such as (1.3)–(1.5) for the existence of such solutions, rather than comparison theorems.

There have been many recent papers dealing with the existence and non-existence of entire solutions of *second order* semilinear elliptic equations of the type $\Delta u=f(x, u)$, $x\in\boldsymbol{R}^N$. For example, the existence of bounded and/or unbounded entire solutions has been proved under various conditions by Gidas and Spruck [4], Joseph and Lundgren [6], Kawano [7], Kusano and Oharu [11], Kusano and Swanson [13], Naito [14], Ni [15], and Toland [18]. We shall not list the many recent studies, outside the scope of our objectives, dealing with the case that $f(\cdot, u)$ changes sign with respect to u, e.g., stationary Klein-Gordon equations.

The problem of existence of *decaying* positive entire solutions of $\Delta u+p(|x|)u^\gamma=0$, $x\in\boldsymbol{R}^N$, $p(t)>0$ for $t>0$, has proved to be elusive. Some sublinear $(0<\gamma<1)$ and singular $(\gamma<0)$ results of this type have been obtained by Fukagai [2, 3], Kusano and Swanson [12]. However, superlinear decaying entire solutions are known to exist only in special cases [4, 6, 10]. Not surprisingly, therefore, it is much more difficult to obtain type III entire solutions of the *fourth* *order* equation (1.2) than type I or II entire solutions. Theorem 2.12 below establishes the existence of positive decaying entire solutions of (1.2) in sublinear and singular cases only; it remains an important open problem to obtain such a result for superlinear equations.

There are many known theorems concerning nonexistence of entire solutions in $\boldsymbol{R}^N$ of second order elliptic equations and inequalities; see, e.g., Haviland [5], Keller [9], Osserman [16], and Redheffer [17]. A few parallel fourth order results of Fink and Kusano [1], Walter [19, 20], and Walter and Rhee [21] have already been mentioned. Use of some of the nonexistence criteria in [1] enables us to obtain necessary and sufficient conditions for the existence of positive entire solutions of (1.1) and/or (1.2), of each type (I), (II), and (III): see Theorems 2.5 and 2.6, Corollaries 2.9 and 2.10, and Remark 2.13.

2. Statement of theorems

The hypotheses to be imposed on the function f in (1.1) will be selected from the list below.

HYPOTHESES

(f_1) f: $[0, \infty)\times(0, \infty)\to \boldsymbol{R}$ is continuous.

(f_2) There exists a continuous function F: $[0, \infty)\times(0, \infty)\to[0, \infty)$ which is nondecreasing in the second variable such that

$$|f(t, u)| \leqq F(t, u) \qquad \text{for all} \quad t \geqq 0, \quad u > 0. \tag{2.1}$$

(f_2^*) There exists a continuous function F: $[0, \infty)\times(0, \infty)\to[0, \infty)$ which is nonincreasing in the second variable and satisfies (2.1).

(f_3) Superlinearity: $u^{-1}F(t, u)$ is nondecreasing in $u\in(0, \infty)$ for each $t\geqq 0$ and satisfies the condition

$$\lim_{u\to 0+} u^{-1}F(u, t)=0, \quad t \geqq 0.$$

(f_3^*) Sublinearity: $u^{-1}F(t, u)$ is nonincreasing in $u\in(0, \infty)$ for each $t\geqq 0$ and satisfies

$$\lim_{u\to \infty} u^{-1}F(t, u) = 0, \quad t \geqq 0.$$

(f_4) There exists a positive constant c such that

$$\int_0^\infty tF(t, c(1+t^2))dt < \infty.$$

(f_5) There exists a positive constant c such that

$$\int_0^\infty t^3F(t, c)dt < \infty.$$

THEOREM 2.1. *Under hypotheses* (f_1), (f_2), *either* (f_3) *or* (f_3^*), *and* (f_4), *equation* (1.1) *has infinitely many positive radial entire solutions* $u(x)$ *which are bounded above and below by positive constant multiples of* $1+|x|^2$ *in* $\boldsymbol{R}^N$, $N\geqq 3$.

THEOREM 2.2. *Under hypotheses* (f_1), (f_2), *either* (f_3) *or* (f_3^*), *and* (f_5), *equation* (1.1) *has infinitely many positive radial entire solutions which are bounded above and below by positive constants in* $\boldsymbol{R}^N$, $N\geqq 5$.

THEOREM 2.3. *Under hypotheses* (f_1), (f_2^*), *and* (f_4), *equation* (1.1) *has infinitely many unbounded positive radial entire solutions in* $\boldsymbol{R}^N$, $N\geqq 3$, *as described in Theorem* 2.1.

THEOREM 2.4. *Under hypotheses* (f_1), (f_2^*), *and* (f_5), *equation* (1.1) *has infinitely many positive radial entire solutions which are bounded above and below by positive constants in* $\boldsymbol{R}^N$, $N\geqq 5$.

The sharpness of conditions (f_4) and (f_5) is indicated by the additional theorems below.

THEOREM 2.5. *If any one of* $\{(f_1), (f_2), (f_3)\}$ *or* $\{(f_1), (f_2), (f_3^*)\}$ *or* $\{(f_1), (f_2^*)\}$ *holds and* $f(t, u)$ *has constant sign for all* $t\geqq 0$, $u>0$, *then condition* (f_4) *is necessary and sufficient for the existence of a positive radial entire solution of* (1.1) *which is asymptotic to a positive constant multiple of* $|x|^2$ *as* $|x|\to\infty$, *uniformly in* $\boldsymbol{R}^N$, $N\geqq 3$.

THEOREM 2.6. *If any one of* $\{(f_1), (f_2), (f_3)\}$ *or* $\{(f_1), (f_2), (f_3^*)\}$ *or* $\{(f_1), (f_2^*)\}$ *holds and* $f(t, u)$ *has constant sign for all* $t\geqq 0$, $u>0$, *then condition* (f_5) *is necessary and sufficient for the existence of a bounded positive radial entire solution of* (1.1) *which is asymptotic to a positive constant as* $|x|\to\infty$, *uniformly in* $\boldsymbol{R}^N$, $N\geqq 5$.

The proofs are given in §4. For example, these theorems can be applied to equation (1.2), where $\gamma\neq 1$ and p: $[0, \infty)\to\boldsymbol{R}$ is continuous. In this case $f(t, u)=p(t)u^\gamma$ and $F(t, u)$ in (f_2) or (f_2^*) can be taken to be

$$F(t, u) = |p(t)|u^\gamma, \quad t \geqq 0, \quad u > 0.$$

Then (f_2) and (f_3) hold if $\gamma>1$, (f_2) and (f_3^*) hold if $0\leqq\gamma<1$, and (f_2^*) holds if $\gamma<0$. Furthermore, (f_4) and (f_5) reduce to (1.3) and (1.4), respectively. Theorems 2.1–2.6 therefore imply the corollaries below.

COROLLARY 2.7. *For all* $\gamma\neq 1$, *condition* (1.3) *is sufficient for equation* (1.2) *to have infinitely many unbounded positive radial entire solutions in* $\boldsymbol{R}^N$, $N \geqq 3$, *as described in Theorem* 2.1.

COROLLARY 2.8. *For all* $\gamma \neq 1$, *condition* (1.4) *is sufficient for* (1.2) *to have infinitely many positive radial entire solutions bounded above and below by positive constants in* $\boldsymbol{R}^N$, $N \geqq 5$.

COROLLARY 2.9. *If* $\gamma \neq 1$ *and* $p(t)$ *has constant sign in* $[0, \infty)$, *condition* (1.3) *is necessary and sufficient for the existence of a positive radial entire solution of* (1.2) *which is asymptotic to a positive constant multiple of* $|x|^2$ *as* $|x| \to \infty$, *uniformly in* $\boldsymbol{R}^N$, $N \geqq 3$.

COROLLARY 2.10. *If* $\gamma \neq 1$ *and* $p(t)$ *has constant sign in* $[0, \infty)$, *condition* (1.4) *is necessary and sufficient for the existence of a positive radial entire solution of* (1.2) *which is asymptotic to a positive constant as* $|x| \to \infty$, *uniformly in* $\boldsymbol{R}^N$, $N \geqq 5$.

REMARK 2.11. If $\gamma > 1$ and $N \geqq 5$, (1.3) implies (1.4), and consequently (1.3) is sufficient for the existence of infinitely many *pairs* of positive entire solutions $(u_1(x), u_2(x))$, where $u_1(x)$ is bounded above and below by positive constants, and $u_2(x)$ is bounded above and below by constant multiples of $1+|x|^2$ throughout $\boldsymbol{R}^N$. Similarly, if $\gamma < 1$ and $N \geqq 5$, (1.4) is sufficient for the existence of infinitely many such pairs of positive entire solutions.

THEOREM 2.12. *Let* p *be a nonnegative-valued continuous function in* $[0, \infty)$ *satisfying* (1.5), $N \geqq 5$, *and* $-1 < \gamma < 1$. *Then equation* (1.2) *has at least one decaying positive radial entire solution* $u(x)$ *in* $\boldsymbol{R}^N$. *Specifically, there exist positive constants* c_1 *and* c_2 *such that*

$$c_1|x|^{4-N} \leqq u(x) \leqq c_2|x|^{4-N}, \quad |x| \geqq 1.$$

REMARK 2.13. The sharpness of condition (1.5) is demonstrated by the known fact [1, p. 237] that (1.5) is a *necessary* condition for equation (1.2), with $p(t) > 0$ in $[0, \infty)$, to have a positive radial solution $u(x)$ defined in *any exterior domain* in $\boldsymbol{R}^N$, $N \geqq 5$, such that $\lim_{|x| \to \infty} |x|^{N-4} u(x) = \text{constant} > 0$.

REMARK 2.14. If $\gamma < 1$, condition (1.5) implies both (1.4) and (1.3). Therefore, if $N \geqq 5$ and $-1 < \gamma < 1$, Corollaries 2.7 and 2.8 together with Theorem 2.12 show that condition (1.5) is sufficient for the existence of infinitely many *triples* (u_1, u_2, u_3) of positive radial entire solutions of (1.2) in $\boldsymbol{R}^N$, $N \geqq 5$, where $u_1(x)$ is unbounded, $u_2(x)$ is bounded above and below by positive constants, and $u_3(x)$ decays to zero as $|x| \to \infty$.

3. Estimates for iterated intergral operators Φ^2, $\Phi\Psi$, and Ψ^2

The notation $L^1_\lambda(0, \infty)$ will be used for the set of all real-valued measurable functions g in $(0, \infty)$ such that

$$\int_0^\infty t^\lambda |g(t)| dt < \infty.$$

We define integral operators Φ: $C[0, \infty) \to C^2[0, \infty)$ and Ψ: $C[0, \infty) \cap L^1_1(0, \infty) \to C^2[0, \infty)$ by

$$(3.1) \qquad (\Phi h)(t) = \frac{1}{N-2} \int_0^t \left[1 - \left(\frac{s}{t}\right)^{N-2}\right] sh(s) ds, \quad t \geqq 0, \; N \geqq 3,$$

$$(3.2) \qquad (\Psi h)(t) = \frac{1}{N-2} \left[\int_0^t \left(\frac{s}{t}\right)^{N-2} sh(s) ds + \int_t^\infty sh(s) ds\right], \; t \geqq 0, \; N \geqq 3.$$

The operator Φ has been used by Kawano [7], Kusano and Oharu [11], and Ψ has been used by Fukagai [2] in existence theory of entire solutions of second order semilinear elliptic equations.

LEMMA 3.1. *Φ and Ψ have the following properties*:

(A) $(\Delta\Phi h)(|x|) = h(|x|)$ *for all* $h \in C[0, \infty)$;

(B) $(\Delta\Psi h)(|x|) = -h(|x|)$ *for all* $h \in C[0, \infty) \cap L^1_1(0, \infty)$;

(C) $\lim_{t\to\infty} (\Psi h)(t) = 0$ *if* $h \in C[0, \infty) \cap L^1_1(0, \infty)$;

(D) Ψ *maps* $C[0, \infty) \cap L^1_3(0, \infty)$ *into* $C^2[0, \infty) \cap L^1_1(0, \infty)$, $N \geqq 5$.

These properties are easily verified from (3.1) and (3.2) and the polar form $\Delta = t^{1-N} \dfrac{d}{dt}\left(t^{N-1} \dfrac{d}{dt}\right)$, $t = |x|$.

We intend to employ the iterates Φ^2, $\Phi\Psi$, and Ψ^2 in the sequel to obtain entire solutions of (1.1) or (1.2) in $\boldsymbol{R}^N$ which are unbounded, bounded, and decaying to zero as $|x| \to \infty$, respectively.

LEMMA 3.2. *If $h(t) \geqq 0$ and $h \in C[0, \infty)$, the function $u(x) = (\Phi^2 h)(|x|)$ is a (radially symmetric) entire solution of $(\Delta^2 u)(x) = h(|x|)$ in $\boldsymbol{R}^N$, $N \geqq 3$, satisfying*

$$(3.3) \qquad 0 \leqq (\Phi^2 h)(t) \leqq \frac{t^2}{2(N-2)^2} \int_0^t sh(s) ds, \quad t \geqq 0.$$

If in addition $h \in L^1_1(0, \infty)$, there exists a positive constant K such that $(\Phi^2 h)(t) \leqq Kt^2$, $t \geqq 0$.

PROOF. Clearly u satisfies $\Delta^2 u = h$ in $\boldsymbol{R}^N$ from property (A) of Lemma 3.1. Furthermore (3.1) implies that

$$0 \leqq (\Phi h)(t) \leqq \frac{1}{N-2}\int_0^t sh(s)ds,$$

$$0 \leqq (\Phi^2 h)(t) \leqq \frac{1}{(N-2)^2}\int_0^t s\int_0^s rh(r)drds$$

$$= \frac{1}{2(N-2)^2}\int_0^t (t^2 - s^2)sh(s)ds,$$

implying (3.3). The last statement of the lemma is obvious.

LEMMA 3.3. *If $h(t)$ is a nonnegative function in $[0, \infty)$ and $h \in C[0, \infty) \cap L_3^1(0, \infty)$, then the function $u(x) = -(\Phi\Psi h)(|x|)$ is a (radially symmetric) entire solution of $(\Delta^2 u)(x) = h(|x|)$ in $\boldsymbol{R}^N$, $N \geqq 5$, such that*

$$0 \leqq (\Phi\Psi h)(t) \leqq \frac{1}{2(N-2)(N-4)}\int_0^\infty s^3 h(s)ds, \quad t \geqq 0. \tag{3.4}$$

PROOF. The first statement follows from Lemma 3.1. The definitions (3.1) and (3.2) show that

$$0 \leqq (\Phi\Psi h)(t) = \frac{1}{(N-2)^2}\int_0^t \left[1 - \left(\frac{s}{t}\right)^{N-2}\right] s(\Psi h)(s)ds$$

$$\leqq \frac{1}{(N-2)^2}\int_0^t s\left(\int_0^s \left(\frac{r}{s}\right)^{N-2} rh(r)dr + \int_s^\infty rh(r)dr\right)ds$$

$$= \frac{1}{(N-2)^2}\left(\int_0^t s^{3-N}\int_0^s r^{N-1}h(r)drds + \int_0^t s\int_s^\infty rh(r)drds\right)$$

$$= \frac{1}{(N-2)^2}\left[\frac{1}{N-4}\int_0^t (r^{4-N} - t^{4-N})r^{N-1}h(r)dr + \frac{1}{2}\int_0^t r^3 h(r)dr + \frac{t^2}{2}\int_t^\infty rh(r)dr\right]$$

$$= \frac{1}{(N-2)^2(N-4)}\left[\int_0^t r^3 h(r)\,dr - \int_0^t \left(\frac{r}{t}\right)^{N-4} r^3 h(r)dr\right]$$

$$+ \frac{1}{2(N-2)^2}\int_0^t r^3 h(r)dr + \frac{t^2}{2(N-2)^2}\int_t^\infty rh(r)dr$$

$$\leqq \frac{1}{2(N-2)(N-4)}\int_0^\infty r^3 h(r)dr, \quad \text{proving (3.4).}$$

For $t > 0$, $N \geqq 5$ define

$$\rho(t) = \min\{1, t^{4-N}\}; \quad \sigma(t) = \min\{t, t^{N+1}\}, \tag{3.5}$$

and $\rho(0)=1$, $\sigma(0)=0$. Let I_1 and I_2 be the functionals defined by

$$I_1(h) = \frac{2}{N(N-2)(N-4)}\int_0^\infty \sigma(t)h(t)dt, \tag{3.6}$$

$$I_2(h) = \max\{I_3(h), I_4(h)\}, \tag{3.7}$$

where

$$I_3(h) = \frac{1}{2(N-2)(N-4)}\int_0^\infty t^3h(t)dt \tag{3.8}$$

and

$$I_4(h) = \frac{1}{(N-2)^2}\left[\frac{1}{2}+\frac{1}{N}+\frac{1}{N-4}\right]\int_0^\infty t^{N-1}h(t)dt, \tag{3.9}$$

with domains defined to be $C[0, \infty)\cap L^1_{N-1}(0, \infty)$, $N\geqq 5$.

LEMMA 3.4. *If $h\in C[0, \infty)\cap L^1_{N-1}(0, \infty)$, $N\geqq 5$, and $h(t)\geqq 0$ in $[0, \infty)$, then $u(x)=(\Psi^2h)(|x|)$ is a (radially symmetric) entire solution of $(\Delta^2u)(x)=h(|x|)$ in $\boldsymbol{R}^N$ such that*

$$I_1(h)\rho(t) \leqq (\Psi^2h)(t) \leqq I_2(h)\rho(t) \tag{3.10}$$

for all $t\geqq 0$. In particular, $\lim_{|x|\to\infty} u(x)=0$ uniformly in $\boldsymbol{R}^N$.

The technically complicated proof will be deferred to the Appendix.

4. Proofs of theorems

PROOF OF THEOREM 2.1. Hypothesis (f_3) shows that

$$\lambda^{-1}tF(t, \lambda(1+t^2)) \leqq c^{-1}tF(t, c(1+t^2))$$

for all $t\geqq 0$, $0<\lambda\leqq c$, where c is as in (f_4), and furthermore that

$$\lim_{\lambda\to 0+} \lambda^{-1}tF(t, \lambda(1+t^2)) = 0, \quad t \geqq 0.$$

In view of (f_4), it follows from the dominated convergence theorem that

$$\lim_{\lambda\to 0+} \frac{1}{\lambda}\int_0^\infty tF(t, \lambda(1+t^2))dt = 0.$$

Then there exists a sufficiently small positive constant k such that

$$\int_0^\infty tF(t, 2k(1+t^2))dt < k. \tag{4.1}$$

Let $C[0, \infty)$ denote the locally convex space of all continuous functions in

$[0, \infty)$ with the topology of uniform convergence on every compact interval in $[0, \infty)$. With k as in (4.1), consider the closed convex subset $\mathscr{Y}$ of $C[0, \infty)$ defined by

$$\mathscr{Y} = \left\{ y \in C[0, \infty) \colon k\left(1 + \frac{t^2}{2}\right) \leqq y(t) \leqq k(1+2t^2),\ t \geqq 0 \right\}. \tag{4.2}$$

With Φ as in (3.1) let $\mathscr{M} \colon \mathscr{Y} \to C[0, \infty)$ be the mapping defined by

$$(\mathscr{M}y)(t) = k(1+t^2) + \Phi^2 f(t, y(t)), \quad t \geqq 0. \tag{4.3}$$

The Schauder-Tychonov fixed point theorem shows that $\mathscr{M}$ has a fixed point $y \in \mathscr{Y}$ after verification that $\mathscr{M}$ is a continuous mapping from $\mathscr{Y}$ into $\mathscr{Y}$ such that $\mathscr{M}\mathscr{Y}$ is relatively compact. To show that $\mathscr{M}\mathscr{Y} \subset \mathscr{Y}$, let $y \in \mathscr{Y}$ and use (f_2) and (4.2) to obtain

$$|f(t, y(t))| \leqq F(t, 2k(1+t^2)), \quad t \geqq 0.$$

Lemma 3.2 shows, for all $t \geqq 0$, that

$$\begin{aligned} |\Phi^2 f(t, y(t))| &\leqq \Phi^2 F(t, 2k(1+t^2)) \\ &\leqq \frac{t^2}{2(N-2)^2} \int_0^\infty tF(t, 2k(1+t^2))dt < \frac{k}{2} t^2 \end{aligned}$$

by (4.1), and hence (4.3) gives

$$|(\mathscr{M}y)(t) - k(1+t^2)| < \frac{k}{2} t^2 \qquad \text{for all} \quad t \geqq 0.$$

This means that $\mathscr{M}y \in \mathscr{Y}$.

If $\{y_n\}$ is a sequence in $\mathscr{Y}$ converging to $y \in \mathscr{Y}$ in the $C[0, \infty)$ topology, use of (f_3) and (f_4) together with the dominated convergence theorem establishes easily that $(\mathscr{M}y_n)(t)$ converges to $(\mathscr{M}y)(t)$ uniformly in $[0, \infty)$ as $n \to \infty$, proving the continuity of $\mathscr{M}$ in $C[0, \infty)$. Ascoli's theorem can be used to show that $\mathscr{M}\mathscr{Y}$ is relatively compact. Then the Schauder-Tychonov fixed point theorem implies the existence of a function $y \in \mathscr{Y}$ such that $(\mathscr{M}y)(t) = y(t)$ for all $t \geqq 0$. Lemma 3.2 applied to (4.3) shows that $u(x) = y(|x|)$ is a positive entire solution of (1.1) with the properties stated in Theorem 2.1.

In the case of the alternative hypothesis (f_3^*), the proof is virtually the same, except that (4.1) holds for a sufficiently *large* positive constant k.

Proof of Theorem 2.2. In view of (f_5), the same argument leading to (4.1) shows that there exists a sufficiently small (if (f_3) holds) or large (if (f_3^*) holds) positive constant k such that

$$\int_0^\infty t^3 F(t, 2k)dt \leqq k. \tag{4.4}$$

In analogy with (4.2) and (4.3), let $\tilde{\mathscr{Y}}$ be the subset of $C[0, \infty)$ and let $\tilde{\mathscr{M}}$ be the mapping defined, respectively, by

$$(4.5)\qquad \tilde{\mathscr{Y}} = \left\{ y \in C[0, \infty) : \frac{k}{2} \leqq y(t) \leqq 2k \text{ for all } t \geqq 0 \right\},$$

$$(4.6)\qquad (\tilde{\mathscr{M}} y)(t) = k - \Phi\Psi f(t, y(t)), \quad t \geqq 0.$$

If $y \in \tilde{\mathscr{Y}}$, it follows from ($f_2$) that

$$|f(t, y(t))| \leqq F(t, 2k), \quad t \geqq 0,$$

and hence $f(\cdot, y) \in C[0, \infty) \cap L_3^1(0, \infty)$ by (4.4). Then Lemma 3.3 can be applied to give

$$\begin{aligned} |\Phi\Psi f(t, y(t))| &\leqq \Phi\Psi F(t, 2k) \\ &\leqq \frac{1}{2(N-2)(N-4)} \int_0^\infty t^3 F(t, 2k) dt \\ &< \frac{k}{2} \qquad \text{for all} \quad t \geqq 0 \end{aligned}$$

by (4.4) and (4.5). Then (4.6) implies that

$$|(\tilde{\mathscr{M}} y)(t) - k| < \frac{k}{2} \qquad \text{for all} \quad t \geqq 0,$$

from which $\tilde{\mathscr{M}} y \in \tilde{\mathscr{Y}}$. Therefore $\tilde{\mathscr{M}}$ maps $\tilde{\mathscr{Y}}$ into itself. The continuity of $\tilde{\mathscr{M}}$ in the $C[0, \infty)$ topology and the relative compactness of $\tilde{\mathscr{M}}\tilde{\mathscr{Y}}$ are verified as in the proof of Theorem 2.1, and hence the Schauder-Tychonov theorem guarantees the existence of a function $y \in \tilde{\mathscr{Y}}$ such that $(\tilde{\mathscr{M}} y)(t) = y(t)$ for all $t \geqq 0$. Application of Lemma 3.3 to (4.6) completes the proof of Theorem 2.2.

The proofs of Theorems 2.3 and 2.4 are virtually the same as those of Theorems 2.1 and 2.2, respectively.

PROOF OF THEOREM 2.5. For any one-signed function $h \in C[0, \infty)$, calculation yields

$$(I^2 h)''(t) = (\Phi h)(t) + \frac{1-N}{t^N} \int_0^t s^{N-1} (\Phi h)(s) ds, \quad t > 0.$$

Then use of L'Hospital's rule three times shows that

$$(4.7)\qquad \lim_{t\to\infty} \frac{(\Phi^2 h)(t)}{t^2} = \frac{1}{2N} \lim_{t\to\infty} (\Phi h)(t),$$

the limit being finite if and only if $h \in L_1^1(0, \infty)$ in view of (3.1). Let $u(x) = y(|x|)$

be a solution of (1.1) obtained in Theorem 2.1 or 2.3. Then $y(t)=(\mathscr{M}y)(t)$ for $t\geqq 0$, i.e.,

$$y(t) = k(1+t^2) + (\Phi^2 h)(t),$$

where $h=f(\cdot, y(\cdot))\in L_1^1(0, \infty)$ by (4.1) and (4.2). It follows from (4.7) that

$$\lim_{t\to\infty}\frac{y(t)}{t^2} = \lim_{|x|\to\infty}\frac{u(x)}{|x|^2} = A > 0 \tag{4.8}$$

exists and is positive. Therefore (f_4) is sufficient for (1.1) to have a positive radial entire solution $u(x)$ which is asymptotic to a positive constant multiple of $|x|^2$ as $|x|\to\infty$.

Conversely, if $f(t, u)$ has constant sign for all $t\geqq 0$, $u>0$, a theorem of Fink and Kusano [1, p. 237] states that (f_4) is necessary for the existence of a radial solution of (1.1) defined in an exterior domain and satisfying (4.8). This proves Theorem 2.5.

Theorem 2.6 is proved similarly: A solution $u(x)=y(|x|)$ of (1.1) obtained in Theorem 2.2 or 2.4 has the property, replacing (4.8),

$$\lim_{|x|\to\infty} u(x) = \text{constant} > 0. \tag{4.9}$$

On the other hand, if $f(t, u)$ has constant sign, (f_5) is known to be necessary for (1.1) to have a radial solution $u(x)$ in an exterior domain in $\boldsymbol{R}^N$, $N\geqq 5$, satisfying (4.9) [1, p. 237].

Proof of Theorem 2.12. Suppose first that $0\leqq\gamma<1$. Let $\rho(t)=\min\{1, t^{4-N}\}$, $N\geqq 5$, as in §3, and define

$$\hat{\mathscr{Y}} = \{y\in C[0, \infty)\colon k_1\rho(t)\leqq y(t)\leqq k_2\rho(t),\ t\geqq 0\}, \tag{4.10}$$

where k_1 and k_2 are positive constants satisfying the inequalities

$$k_1 \leqq [I_1(p\rho^\gamma)]^{1/(1-\gamma)} \leqq [I_2(p\rho^\gamma)]^{1/(1-\gamma)} \leqq k_2. \tag{4.11}$$

It follows from (1.5) and (3.6)–(3.9) that $I_1(p\rho^\gamma)$ and $I_2(p\rho^\gamma)$ are finite. Let $\hat{\mathscr{M}}\colon\hat{\mathscr{Y}}\to C[0, \infty)$ be the mapping defined by

$$(\hat{\mathscr{M}}y)(t) = \Psi^2(py^\gamma)(t), \quad t\geqq 0. \tag{4.12}$$

If $y\in\hat{\mathscr{Y}}$, then by (4.10),

$$t^{N-1}p(t)y^\gamma(t) \leqq k_2^\gamma t^\delta p(t), \quad \delta = N-1-\gamma(N-4),$$

for $t\geqq 1$, $N\geqq 5$. Therefore $py^\gamma\in L_{N-1}^1(0, \infty)$ and Lemma 3.4 is applicable to $h=py^\gamma$. It then follows from (3.10) and (4.12) that

$$(4.13)\qquad I_1(py^\gamma)\rho(t) \leqq (\hat{\mathscr{M}}y)(t) \leqq I_2(py^\gamma)\rho(t), \quad t \geqq 0.$$

For $y \in \hat{\mathscr{Y}}$, (4.10) and (4.13) imply that

$$(4.14)\qquad k_1^\gamma I_1(p\rho^\gamma)\rho(t) \leqq (\hat{\mathscr{M}}y)(t) \leqq k_2^\gamma I_2(p\rho^2)\rho(t)$$

for all $t \geqq 0$. Consequently (4.11) yields the inequalities

$$(\hat{\mathscr{M}}y)(t) \leqq k_2^\gamma k_2^{1-\gamma}\rho(t) = k_2\rho(t)$$

and

$$(\hat{\mathscr{M}}y)(t) \geqq k_1^\gamma k_1^{1-\gamma}\rho(t) = k_1\rho(t),$$

from which $\hat{\mathscr{M}}y \in \hat{\mathscr{Y}}$. Therefore $\hat{\mathscr{M}}\hat{\mathscr{Y}} \subset \hat{\mathscr{Y}}$, and it can be verified that $\hat{\mathscr{M}}$ is continuous in the $C[0, \infty)$ topology and that $\hat{\mathscr{M}}\hat{\mathscr{Y}}$ is relatively compact. The Schauder-Tychonov fixed point theorem then implies the existence of $y \in \hat{\mathscr{Y}}$ such that $(\hat{\mathscr{M}}y)(t) = y(t)$ for all $t \geqq 0$. From Lemma 3.4, $u(x) = y(|x|)$ is a positive decaying entire solution of (1.2), as described in Theorem 2.12, for a choice of k_1 and k_2 satisfying (4.11).

The proof in the singular case $-1 < \gamma < 0$ is virtually the same, except that the constants k_1 and k_2 in the definition (4.10) of $\hat{\mathscr{Y}}$ are replaced by

$$k_1 = [I_1(p\rho^\gamma)I_2^\gamma(p\rho^\gamma)]^{1/(1-\gamma^2)}$$

and

$$k_2 = [I_1^\gamma(p\rho^\gamma)I_2(p\rho^\gamma)]^{1/(1-\gamma^2)},$$

respectively. Again the desired decaying entire solution is obtained as a fixed point of $\hat{\mathscr{M}}$ in $\hat{\mathscr{Y}}$.

5. Appendix: Proof of Lemma 3.4.

The notation in (3.5)–(3.9) will be used in the proof.

The first part of Lemma 3.4 is a consequence of Lemma 3.1. The definition (3.2) shows, if $h \in C[0, \infty) \cap L_{N-1}^1(0, \infty)$, $N \geqq 5$, that

$$(\Psi^2 h)(t) = J_1(t) + J_2(t), \quad t \geqq 0,$$

where

$$J_1(t) = \frac{1}{N-2}\int_0^t \left(\frac{s}{t}\right)^{N-2} s(\Psi h)(s)\,ds,$$

$$J_2(t) = \frac{1}{N-2}\int_t^\infty s(\Psi h)(s)ds.$$

One easily finds upon application of Fubini's theorem that

$$J_1(t) = \frac{t^{2-N}}{(N-2)^2}\left[\frac{1}{2}\int_0^t (t^2-r^2)r^{N-1}h(r)\,dr + \frac{1}{N}\int_0^t r^{N+1}h(r)\,dr + \frac{t^N}{N}\int_t^\infty rh(r)\,dr\right];$$

$$J_2(t) = \frac{1}{(N-2)^2}\left[\frac{t^{4-N}}{N-4}\int_0^t r^{N-1}h(r)\,dr + \frac{1}{N-4}\int_t^\infty r^3h(r)\,dr + \frac{1}{2}\int_t^\infty (r^2-t^2)rh(r)\,dr\right].$$

Define $G(t)=(\Psi^2h)(t)=J_1(t)+J_2(t)$, i.e.,

$$\begin{aligned}(N-2)^2G(t) &= \frac{1}{2}t^{2-N}\int_0^t (t^2-r^2)r^{N-1}h(r)\,dr + \frac{t^{2-N}}{N}\int_0^t r^{N+1}h(r)\,dr \\ &+ \frac{t^2}{N}\int_t^\infty rh(r)\,dr + \frac{t^{4-N}}{N-4}\int_0^t r^{N-1}h(r)\,dr + \frac{1}{N-4}\int_t^\infty r^3h(r)\,dr \\ &+ \frac{1}{2}\int_t^\infty (r^2-t^2)rh(r)\,dr. \end{aligned} \tag{5.1}$$

Differentiation of (5.1) gives

$$\begin{aligned}(N-2)^2G'(t) &= \frac{2-N}{2}t^{1-N}\int_0^t (t^2-r^2)r^{N-1}h(r)\,dr + t^{3-N}\int_0^t r^{N-1}h(r)\,dr \\ &+ \frac{2-N}{N}t^{1-N}\int_0^t r^{N+1}h(r)\,dr + \frac{t^{2-N}}{N}t^{N+1}h(t) + \frac{2t}{N}\int_t^\infty rh(r)\,dr - \frac{t^2}{N}th(t) \\ &- t^{3-N}\int_0^t r^{N-1}h(r)\,dr + \frac{t^{4-N}}{N-4}t^{N-1}h(t) - \frac{1}{N-4}t^3h(t) - t\int_t^\infty rh(r)\,dr.\end{aligned}$$

It follows that

$$(N-2)^2G'(t) \leqq -\frac{(N-2)}{2}t^{1-N}\int_0^t (t^2-r^2)r^{N-1}h(r)\,dr - \frac{N-2}{N}t^{1-N}\int_0^t r^{N+1}h(r)\,dr,$$

from which $G'(t)<0$ for all $t>0$. In particular $G(t)$ is decreasing in the interval $0\leqq t\leqq 1$, and accordingly $G(1)\leqq G(t)\leqq G(0)$ in this interval. From (5.1),

$$G(0) = \frac{1}{2(N-2)(N-4)}\int_0^\infty r^3h(r)dr = I_3(h)$$

and

$$\begin{aligned}(N-2)^2G(1) &= \frac{1}{2}\int_0^1 (1-r^2)r^{N-1}h(r)\,dr + \frac{1}{N}\int_0^1 r^{N+1}h(r)\,dr + \frac{1}{N}\int_1^\infty rh(r)\,dr \\ &+ \frac{1}{N-4}\int_0^1 r^{N-1}h(r)\,dr + \frac{1}{N-4}\int_0^\infty r^3h(r)\,dr + \frac{1}{2}\int_1^\infty (r^2-1)rh(r)\,dr \\ &\geqq \left(\frac{1}{N}+\frac{1}{N-4}\right)\left(\int_0^1 r^{N+1}h(r)\,dr + \int_1^\infty rh(r)\,dr\right).\end{aligned}$$

Hence

$$G(1) \geqq \frac{2}{N(N-2)(N-4)} \int_0^\infty \sigma(r)h(r)dr = I_1(h).$$

This proves the estimates (3.10) in the case $0 \leqq t \leqq 1$.

In order to estimate $G(t)$ for $t \geqq 1$, we use the notation $H(t) = t^{N-4}G(t)$. From (5.1), calculation gives

$$\begin{aligned}(N-2)^2H'(t) = &- t^{-3}\int_0^t (t^2-r^2)r^{N-1}h(r)dr + t^{-1}\int_0^t r^{N-1}h(r)dr \\ &- \frac{2}{N}t^{-3}\int_0^t r^{N+1}h(r)dr + \frac{t^{-2}}{N}t^{N+1}h(t) + \frac{N-2}{N}t^{N-3}\int_t^\infty rh(r)dr \\ &- \frac{t^{N-2}}{N}th(t) + \frac{1}{N-4}t^{N-1}h(t) + t^{N-5}\int_t^\infty r^3h(r)dr - \frac{t^{N-4}}{N-4}t^3h(t) \\ &+ \frac{N-4}{2}t^{N-5}\int_t^\infty (r^2-t^2)rh(r)dr - t^{N-3}\int_t^\infty rh(r)dr.\end{aligned}$$

Then

$$(N-2)^2H'(t) \geqq \frac{N-2}{N}t^{N-3}\int_t^\infty rh(r)dr + \frac{N-4}{2}t^{N-5}\int_t^\infty (r^2-t^2)rh(r)dr$$

from which $H'(t) > 0$ for all $t > 0$, and in particular $H(1) \leqq H(t) \leqq H(\infty)$ for all $t \geqq 1$. We note that $H(1) = G(1) \geqq I_1(h)$, and since $h \in L^1_{N-1}(0, \infty)$, $H(\infty)$ is estimated as follows:

$$(N-2)^2H(\infty) \leqq \frac{1}{2}\int_0^\infty r^{N-1}h(r)dr + \frac{1}{N}\int_0^\infty r^{N-1}h(r)dr + \frac{1}{N-4}\int_0^\infty r^{N-1}h(r)dr,$$

and so

$$H(\infty) \leqq \frac{1}{(N-2)^2}\left[\frac{1}{2} + \frac{1}{N} + \frac{1}{N-4}\right]\int_0^\infty r^{N-1}h(r)dr = I_4(h).$$

We combine the above estimates to obtain

$$I_1(h) \leqq H(1) \leqq H(t) = t^{N-4}G(t) \leqq H(\infty) \leqq I_4(h),$$

i.e., for $t \geqq 1$,

$$I_1(h)t^{4-N} \leqq G(t) \leqq I_2(h)^{4-N}.$$

This proves (3.10) in the case $t \geqq 1$.

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