HIROSHIMA MATH. J.
17 (1987), 395–403

Finding boundary for the semistable ends of 3-manifolds

Osamu KAKIMIZU

(Received January 8, 1987)

1. Introduction

The problem of finding boundary for a stable end of a non-compact smooth m-manifold W is studied by Siebenmann [13] for $m \geq 6$, Husch and Price [6] for $m=3$ and so on. On the other hand, the end of the universal covering $\tilde{M}^m$ of a closed aspherical m-manifold M^m is conjectured to be semistable (Mihalik [11]) and to be $\boldsymbol{R}^3$ in case $m=3$. In this paper, we shall find boundary for semistable ends of a P^2-irreducible 3-manifold.

To start with we define the semistability at an end of a manifold. Let W be a non-compact connected m-manifold and ε be an end of W. Let $U_0 \supset U_1 \supset U_2 \supset \cdots$ be a base of neighborhoods of ε consisting of connected m-submanifolds of W, and let r: $[0, \infty) \to W$ be a *base ray*, that is, a proper map with $r[n, \infty) \subset U_n$ for all n. Then we have the inverse sequence of the fundamental groups

$$\mathscr{A}: \pi_1(U_0, a_0) \xleftarrow{\varphi_{0,1}} \pi_1(U_1, a_1) \xleftarrow{\varphi_{1,2}} \pi_1(U_2, a_2) \longleftarrow \cdots$$

where $a_n = r(n)$ and $\varphi_{n,k}$ $(n<k)$ is the homomorphism induced by the path $r \mid [n, k]$: $([n, k], n, k) \to (U_n, a_n, a_k)$. Now ε is defined to be *semistable* if $\mathscr{A}$ is semistable, or satisfies the Mittag-Leffler condition, that is, for each n there is a $k(n)$ $(\geq n)$ such that

$$\operatorname{Im} \varphi_{n,k(n)} = \operatorname{Im} \varphi_{n,k} \qquad \text{for all} \quad k \geq k(n).$$

It is shown in [7] that the semistability of $\mathscr{A}$ is independent of the choice of $\{U_n\}$ and r, and hence the semistability at ε is well defined. Then we have the following main result in this paper:

THEOREM. *Let W be a non-compact connected P^2-irreducible 3-manifold such that ∂W is compact and $\pi_1(W)$ is finitely generated. Let ε be an end of W and suppose that ε is semistable. Then ε has a neighborhood U such that ∂U is a closed surface and U is homeomorphic to $\partial U \times [0, \infty)$.*

Here W is *P^2-irreducible* if it is irreducible and does not contain any 2-sided surface homeomorphic to the projective plane P^2.

We will show that any W that satisfies the hypotheses of the Theorem can have only finitely many ends (cf. Lemma 3). In particular, if W is contractible, then W has only one end and we have the following

COROLLARY. *Let W be a contractible irreducible open 3-manifold. Then W is homeomorphic to $\boldsymbol{R}^3$ if and only if the end of W is semistable.*

This corollary is contained in the recent result of Brin and Thickstun [1, (A)] which says that the same is true even if "the end of W is semistable" is weakened to "*W is end 1-movable*". Furthermore, in the case that W is orientable, our theorem also follows from the combination of [1, Th. I.1] and Lemma 3.

We note that in the Theorem the assumption that $\pi_1(W)$ is finitely generated cannot be removed. For instance, let $H_0 \subset H_1 \subset H_2 \subset \cdots$ be an increasing sequence of handlebodies such that H_n has genus n and H_{n+1} is obtained from H_n by thickening and then by attaching an orientable 1-handle. Put $W_0 = \bigcup_n H_n$. Then W_0 is a connected orientable irreducible open 3-manifold. Furthermore $\pi_1(W_0)$ is infinitely generated, and W_0 has one end which is semistable but not stable.

We also note that the semistability at an end can be expressed in the several ways as follows.

PROPOSITION ([4] and [11]). *For W, ε and $\mathscr{A}$ given in the beginning of this section the following three assertions are equivalent*:

(a) *$\mathscr{A}$ is semistable.*
(b) *$\varprojlim^1 \mathscr{A}$ is trivial.*
(c) *Any two base rays $f, g\colon [0, \infty) \to W$ for the end ε are properly homotopic.*

In the results given by Siebenmann [13] and Husch and Price [6], the stability at ε together with some conditions on the fundamental group $\pi_1(\varepsilon) = \varprojlim \mathscr{A}$ at ε is assumed. Note that an end ε of W is defined to be *stable* if the inverse sequence $\mathscr{A}$ is pro-isomorphic to a constant sequence. Moreover, in higher dimensional cases the knowledge of $\pi_1(\varepsilon)$ is indispensable to find boundary, even for a contractible W (see [2]). In contrast with these results the Theorem asserts that we have only to verify that ε is semistable in the 3-dimensional case.

The author is very grateful to Professor Matthew Brin for pointing out an error in the original version of the proof and for various kind comments. The error stated that ∂V_n in Lemma 9 is connected, and is repaired by considering the component $\partial_0 V_n$ of ∂V_n stated in Lemma 4. The author also thanks Professor Takao Matumoto for his encouragement throughout this work.

2. Proof of Theorem

Let W be as in the Theorem and let ε be a semistable end of W. Taking a base $\{U_n\}$ of neighborhoods of ε and a base ray r, we have the inverse sequence $\mathscr{A}$

described in the beginning of §1. We may assume that

$$\mathrm{Int}\, U_n \supset U_{n+1} \text{ and } \partial U_n \text{ is connected for all } n.$$

It is easy to show the following

LEMMA 1. *If for each n there is a $k \geq n$ so that U_k is not irreducible, then* $W \approx \boldsymbol{R}^3$ ("$\approx$" means "is homeomorphic to").

By this lemma, we may assume that U_n is irreducible for all n. Since W is P^2-irreducible, this means that

$$U_n \text{ is } P^2\text{-irreducible for each } n.$$

By the semistability condition on ε, passing to a subsequence if necessary, we may also assume that

$$\mathrm{Im}\,\varphi_{n,k} = \mathrm{Im}\,\varphi_{n,n+1} \qquad \text{for all} \quad k > n. \tag{2}$$

To prove the Theorem we first discuss some consequences of the assumption that $\pi_1(W)$ is finitely generated. By the result of Scott [12], there is an *irreducible core* N of W with $N \subset \mathrm{Int}\, W$, that is, a compact connected irreducible 3-submanifold N of W such that the inclusion $N \subset W$ induces an isomorphism $\pi_1(N) \to \pi_1(W)$ (cf. [8], [9]); and we fix such an N hereafter. In our case both of W and N are aspherical, and *N is a deformation retract of W.* For a given end ε, let

$$Y \text{ denote the component of } W - \mathrm{Int}\, N \text{ with } \varepsilon \text{ as an end.}$$

We may assume $\mathrm{Int}\, Y \supset U_0$ for the base $\{U_n\}$ of neighborhoods of ε.

LEMMA 3. (i) *The inclusions $\partial Y \subset Y$ and $U_n \subset Y$ induce isomorphisms*

(a) $H_*(\partial Y) \xrightarrow{\cong} H_*(Y)$ *and*

(b) $H^*(\partial Y) \xleftarrow{\cong} H^*(Y) \xrightarrow{\cong} \varinjlim H^*(U_n)$.

(ii) *∂Y is connected and ε is the unique end of Y.*

(Throughout this section we consider the (co)homology with coefficient in $\boldsymbol{Z}_2$.)

PROOF. Consider the cohomologies $H_c^*(X)$ of the cochain complex $S_c^*(X)$ with compact support and $H_e^*(X) = H(S^*(X)/S_c^*(X))$. Then, we have $H_c^*(W') \cong H_{3-*}(W', \partial W') \cong H_{3-*}(\mathrm{Int}\, W, N) = 0$ for $W' = \mathrm{Int}\, W - \mathrm{Int}\, N$. Hence $H^*(W') \cong H_e^*(W')$. Consider $Y' = Y \cap W'$. Then Y' is a component of W', and so $H^*(Y') \cong H_e^*(Y')$. In particular $H_e^0(Y') \cong \boldsymbol{Z}_2$, and this means that Y' has exactly one end ε by [3, Th. 1]. Hence $Y = Y'$ since Y is non-compact, and ε is the unique end of Y. Moreover, $H^*(Y) \cong H_e^*(Y) \cong \varinjlim H^*(U_n)$. The rest of the lemma is shown by the facts that $H_*(\partial W') \cong H_*(W')$, $H^*(\partial W') \cong H^*(W')$ and $\partial Y' = \partial Y$. $\square$

Let $F\subset \operatorname{Int} Y$ be a connected closed surface. If $Y-F$ has two components A and B such that $\partial Y\subset A$ and ε is an end of B, then we say that F *separates* ∂Y *from* ε *in* Y.

LEMMA 4. *Let* $V\subset \operatorname{Int} Y$ *be a connected* 3-*submanifold of* Y, *and suppose that* V *is a neighborhood of* ε. *Then*:

(i) $H_1(V)$ *is finitely generated.*

(ii) *There is a unique component* F *of* ∂V *such that* F *separates* ∂Y *from* ε *in* Y.

In the rest of this paper, F in (ii) will be denoted by $\partial_0 V$.

PROOF. (i) is seen by the Mayer-Vietoris sequence for $(Y;\, V,\, \mathrm{Cl}(Y-V))$ and by Lemma 3 (i) (a).

(ii) Let X denote the component of $\mathrm{Cl}(Y-V)$ such that $\partial Y\subset\partial X$. Put $F=X\cap V$. It suffices to show that F is connected. Suppose F is not connected. Then there are a circle J in $\operatorname{Int} Y$ and a component S of F such that J traverses S at one point. On the other hand J is homotopic to a circle in N since N is a core of W. By considering the intersection number of J and S, we have a contradiction. Thus F is connected. □

LEMMA 5. *For* V *in* Lemma 4, *the following four assertions are equivalent*:

(a) Y *is orientable.* (b) ∂Y *is orientable.*

(c) V *is orientable.* (d) $\partial_0 V$ *is orientable.*

PROOF. Clearly (a) implies (b), (c) and (d). We now assume that Y is non-orientable and prove that all of ∂Y, V and $\partial_0 V$ are non-orientable. Let $p\colon \tilde{W}\to W$ be the orientable double covering. Then, it is sufficient to show that all of $p^{-1}(\partial Y)$, $p^{-1}(V)$ and $p^{-1}(\partial_0 V)$ are connected, since ∂Y and $\partial_0 V$ are 2-sided surfaces in W.

Note that $\tilde{W}$ is irreducible (cf. [5]) and that $\tilde{N}=p^{-1}(N)$ for the irreducible core N of W is connected and is an irreducible core of $\tilde{W}$.

$p^{-1}(Y)$ is connected by the assumption, and is a component of $\tilde{W}-\operatorname{Int}\tilde{N}$. Thus $p^{-1}(\partial Y)=\partial p^{-1}(Y)$ is also connected by Lemma 3 (ii). Moreover, $p^{-1}(Y)$ has exactly one end, and so does $p^{-1}(V)$ which is connected.

To show that $p^{-1}(\partial_0 V)$ is connected, let Z be the component of $\mathrm{Cl}(Y-V)$ with $\partial Y\subset\partial Z$. Then $p^{-1}(Z)$ is connected since so is $p^{-1}(\partial Y)$. Moreover $\partial Z\supset\partial_0 V$ by the definition of $\partial_0 V$ in Lemma 4. Assume that $p^{-1}(\partial_0 V)$ has two components S and S'. Then there is a circle in $p^{-1}(Y)$ which traverses S at one point since both of $p^{-1}(Z)$ and $p^{-1}(V)$ are connected. On the other hand the circle is homotopic to a circle in $\tilde{N}$ since $\tilde{N}$ is a core of $\tilde{W}$. This is a contradiction, and hence $p^{-1}(\partial_0 V)$ is connected. □

Now let us fix an n for a moment. From $U_{n+1} \subset \operatorname{Int} U_n$ we can construct a 3-submanifold $U' \subset \operatorname{Int} U_n$ so that each component of $\partial U'$ is incompressible in U_n, by applying a finite number of *simple moves* due to McMillan [10, §2] to U_{n+1} in U_n. Here we note that each simple move is done by *adding a* 2- *or* 3-*handle* or *removing a* 1- *or* 0-*handle.* Moreover, we define

$$V_n = \text{the unique non-compact component of } U'.$$

Note that V_n is P^2-irreducible since so is U_n.

LEMMA 6. *If $n' > n$ is sufficiently large, then $r[n', \infty) \subset \operatorname{Int} V_n$ for a given base ray r, and*

$$\operatorname{Im}[\pi_1(V_n, b) \to \pi_1(U_n, b)] \subset \operatorname{Im}[\pi_1(U_{n+1}, b) \to \pi_1(U_n, b)]$$

for $b = r(n')$ and the homomorphisms induced by the inclusions, where the first one is a monomorphism.

PROOF. The last assertion is valid since ∂V_n is incompressible in U_n. By definition, we have a sequence $U_{n+1} = X_0, X_1, \ldots, X_k = V_n$ of connected non-compact 3-submanifolds of U_n, where X_{i+1} is obtained from X_i by a simple move given above and then by taking the non-compact component if necessary. We may assume that $r[n', \infty) \subset X_i$ for all i. Put $G_i = \operatorname{Im}[\pi_1(X_i, b) \to \pi_1(U_n, b)]$. If $X_{i+1} \subset X_i$, then $G_{i+1} \subset G_i$. If $X_{i+1} = X_i \cup (\text{an } h\text{-handle})$ $(h = 2 \text{ or } 3)$, then $\pi_1(X_i, b) \to \pi_1(X_{i+1}, b)$ is an epimorphism by the van Kampen theorem; hence $G_{i+1} = G_i$. Thus we have the lemma. □

LEMMA 7. *Each loop $f: S^1 \to V_n$ extends to a proper map $\tilde{f}: S^1 \times [0, \infty) \to V_n$ with $\tilde{f} | S^1 \times \{0\} = f$.*

PROOF. Choose a base point $c \in S^1$. We may assume that $f(S^1) \subset \operatorname{Int} V_n$ and $f(c) = b = r(n')$ for n' in Lemma 6. Take $k > \max\{n', n+2\}$ so that $U_{k-1} \subset \operatorname{Int} V_n$. Then $\operatorname{Im}[\pi_1(V_n, b) \to \pi_1(U_n, b)] \subset \operatorname{Im}[\varphi: \pi_1(U_k, a_k) \to \pi_1(U_n, b)]$ by the semistability condition (2) and Lemma 6, where φ is induced by the path $r_k = r | [n', k]: [n', k] \to \operatorname{Int} U_n$. Therefore there is a homotopy $\tilde{f}: S^1 \times [n', k] \to U_n$ such that $\tilde{f} | S^1 \times \{n'\} = f$, $\tilde{f} | \{c\} \times [n', k] = r_k$ and $\tilde{f}_k = \tilde{f} | S^1 \times \{k\}: S^1 \times \{k\} \to U_k$. Moreover, since ∂V_n is incompressible in U_n, we can modify $\tilde{f}$ by the standard cut and paste argument so that $\tilde{f}(S^1 \times [n', k]) \subset V_n$ without changing on $S^1 \times \{n', k\} \cup \{c\} \times [n', k]$. Now, by (2) again, $\tilde{f}_k$ extends to $\tilde{f}: S^1 \times [k, k+1] \to U_{k-1}$ such that $\tilde{f}(S^1 \times \{k+1\}) \subset U_{k+1}$, and so on. Thus $\tilde{f}_k$ extends to a proper map $\tilde{f}: S^1 \times [k, \infty) \to U_{k-1}$ with $\tilde{f} | S^1 \times \{k\} = \tilde{f}_k$; and the lemma is proved. □

We quote the following theorem, which is used in the proof of Lemma 9 below.

THEOREM 8 (Waldhausen [14, Lemma 5] and Heil [5, Prop. 5]). *Let M be a connected P^2-irreducible 3-manifold, and let F_0 and F_1 be two incompressible components of ∂M (not necessarily $\partial M = F_0 \cup F_1$). If any loop in F_0 is freely homotopic in M to a loop in F_1, then M is homeomorphic to $F_0 \times [0, 1]$.*

LEMMA 9. (i) *The inclusion $\partial_0 V_n \subset V_n$ (for $\partial_0 V_n$ given by* Lemma 4(ii)) *induces a monomorphism $H_1(\partial_0 V_n) \to H_1(V_n)$.*

(ii) *If S is a connected 2-sided incompressible closed surface in* $\operatorname{Int} V_n$ *which separates ∂Y from ε in Y, then S is parallel to $\partial_0 V_n$ in V_n.*

PROOF. (i) Take any $k \geq n+2$ with $\operatorname{Int} V_n \supset U_k$ and put $Z = V_n - \operatorname{Int} U_k \subset V_n$. Then, in the same way as the construction of U' from $U_{n+1} \subset \operatorname{Int} U_n$, we obtain a 3-submanifold $Z' \subset V_n$ so that each component of $\partial Z' - \partial V_n$ is incompressible in V_n by applying simple moves, done by adding a 2- or 3-handle or removing a 1- or 0-handle, to Z in V_n. Now, let V' be the unique non-compact component of $\operatorname{Cl}(V_n - Z')$ and $M_0, M_1, \dots, M_p$ be all components of $M = \operatorname{Cl}(V_n - V')$, where $\partial_0 V_n \subset \partial M_0$. Then, M_0 is a compact connected P^2-irreducible 3-submanifold of W with $\partial M_0 \supset \partial_0 V_n \cup \partial_0 V'$.

Any loop $f: S^1 \to \partial_0 V_n$ extends to a proper map $\tilde{f}: S^1 \times [0, \infty) \to V_n$ by Lemma 7. Here we take $\tilde{f}$ so that it is in general position relative to $\partial V'$. Then each component of $\tilde{f}^{-1}(\partial V')$ is a circle which is inessential or parallel to $S^1 \times \{0\}$ in $S^1 \times [0, \infty)$. Thus we may assume that for some $s > 0$, $\tilde{f}(S^1 \times \{s\}) \subset \partial_0 V'$ and each component of $\tilde{f}^{-1}(\partial_0 V') \cap S^1 \times (0, s)$ is an inessential circle in $S^1 \times [0, s]$. Moreover, since $\partial_0 V'$ is incompressible in V_n, we can modify $\tilde{f} | S^1 \times [0, s]$ by the standard cut and paste argument so that $\tilde{f}^{-1}(\partial_0 V') \cap S^1 \times (0, s) = \emptyset$, i.e., $\tilde{f}(S^1 \times [0, s]) \subset M_0$. Therefore f is homotopic in M_0 to a loop $\tilde{f} | S^1 \times \{s\}$ in $\partial_0 V'$. Thus Theorem 8 implies that $M_0 \approx \partial_0 V_n \times [0, 1]$.

In the construction of Z', for any added 2-handle H and for any h-handle H^h ($h=1$ or 0) removed before, we may assume that $H \cap H^0 = \emptyset$, the attaching boundary of H is disjoint from H^1 and H is in general position relative to H^1. Now consider the union L_k of M and all removed h-handles H^h ($h=1$ or 0). Then by the above assumption, if $H^0 \not\subset M$, then $H^0 \cap M = \emptyset$; and if $H^1 \not\subset M$, then H^1 is cut into smaller 1-handles by 2-handles added after. Therefore L_k is a compact connected 3-submanifold of V_n such that

$$\partial_0 V_n \subset \partial V_n \subset \partial L_k, \quad Z \subset L_k \text{ and } L_k \approx M \cup (\text{1-handles}),$$

where all new 1-handles are attached to $M \cap V' = \partial V'$. These together with $M_0 \approx \partial_0 V_n \times [0, 1]$ imply that

(10) $\qquad \partial_0 V_n \subset L_k$ induces a monomorphism $\alpha_k: H_1(\partial_0 V_n) \longrightarrow H_1(L_k)$.

From the sequence $\{L_k\}$ given above, we choose a subsequence $\{L_{k(j)}\}$ with

$L_{k(j)} \subset L_{k(j+1)}$. Then $V_n = \bigcup_j L_{k(j)}$ since $V_n - \operatorname{Int} U_k = Z \subset L_k$. Consider the commutative diagram

$$\begin{tikzcd}
H_1(\partial_0 V_n) \arrow[r] \arrow[d, "\alpha_{k(j)}"'] \arrow[dr, "\alpha"] & H_1(V_n) \\
H_1(L_{k(j)}) \arrow[r] & \varinjlim H_1(L_{k(j)}) \arrow[u, "\cong"']
\end{tikzcd}$$

induced by the inclusions, where $\alpha = \varinjlim \alpha_{k(j)}$. Then α is a monomorphism by (10), and (i) is proved.

(ii) For a given S in (ii) we carry out the construction of Z' in the proof of (i) by taking k with $U_k \cap S = \emptyset$ in addition. Then we can assume that $S \subset \operatorname{Int} Z'$ in addition, since $S \subset \operatorname{Int} Z$ and S is incompressible in V_n. Moreover we see that $S \subset \operatorname{Int} M_0$ by the definition of $\partial_0 V_n$ and M_0 and the assumption that S separates ∂Y from ε in Y. Since $M_0 \approx \partial_0 V_n \times [0, 1]$, S is parallel to $\partial_0 V_n$ in M_0 and so in V_n. □

Now we proceed to the next step of the proof. For $\{V_n\}$ constructed above, choose a sequence $n(0) < n(1) < \cdots$ such that

(11) $$\operatorname{Int} V_{n(i)} \cap \operatorname{Int} U_{n(i)+1} \supset U_{n(i+1)} \supset V_{n(i+1)} \qquad \text{for all } i.$$

LEMMA 12. *The inclusion $V_{n(i)} \subset Y$ induces an isomorphism $H_1(V_{n(i)}) \to H_1(Y)$ for all i.*

PROOF. For a base ray r, we may assume that $r[2i-1, \infty) \subset U_{n(i)}$ and $r[2i, \infty) \subset V_{n(i)}$. Then, by taking $a_i = r(2i-1)$ and $b_i = r(2i)$ as base points of $U_{n(i)}$ and $V_{n(i)}$, respectively, we have the commutative diagram

(13) $$\begin{tikzcd}
\pi_1(U_{n(0)}, a_0) & & \pi_1(U_{n(1)}, a_1) \arrow[ll, "\varphi_0"'] \arrow[dl, "\beta_0"] & & \pi_1(U_{n(2)}, a_2) \arrow[ll, "\varphi_1"'] \arrow[dl, "\beta_1"] & \cdots \arrow[l] \\
& \pi_1(V_{n(0)}, b_0) \arrow[ul, "\alpha_0"] & & \pi_1(V_{n(1)}, b_1) \arrow[ll, "\lambda_0"'] \arrow[ul, "\alpha_1"] & & \cdots \arrow[ll, "\lambda_1"']
\end{tikzcd}$$

where α_i, β_i, φ_i and λ_i are induced by the base ray r. Here we have

(14) $$\lambda_i \text{ is an epimorphism for all } i,$$

by a simple diagram chasing method using (2), Lemma 6 and (11). Further consider the commutative diagram

(15) $$\begin{tikzcd}
\pi_1(V_{n(0)}) \arrow[d] & \pi_1(V_{n(1)}) \arrow[l, "\lambda_0"'] \arrow[d] & \pi_1(V_{n(2)}) \arrow[l, "\lambda_1"'] \arrow[d] & \cdots \arrow[l] \\
H_1(V_{n(0)}) & H_1(V_{n(1)}) \arrow[l, "\eta_0"] & H_1(V_{n(2)}) \arrow[l, "\eta_1"] & \cdots \arrow[l]
\end{tikzcd}$$

where each η_i is induced by $V_{n(i+1)} \subset V_{n(i)}$ and each vertical Hurewicz homomorphism is an epimorphism. Then (14) shows that η_i is also an epimorphism for all i. Thus $H^1(V_{n(i)}) \to H^1(V_{n(i+1)})$ is a monomorphism for all i. Therefore, the upper homomorphism in the commutative diagram

$$\begin{tikzcd} H^1(V_{n(i)}) \arrow[rr] & & \varinjlim H^1(V_{n(i)}) \\ & H^1(Y) \arrow[ul] \arrow[ur, "\cong"'] & \end{tikzcd}$$

is also a monomorphism. Since the right oblique homomorphism is an isomorphism by Lemma 3 (i) (b), so is the left one. Hence we have the lemma by Lemma 4 (i). □

Now we are ready to prove the Theorem. By Lemmas 3 (i) (a), 9 (i) and 12, we have

$$\dim H_1(\partial Y) \geq \dim H_1(\partial_0 V_{n(i)}) \qquad \text{for all} \quad i.$$

This means that there are only finitely many possibilities of the value of $\dim H_1(\partial_0 V_{n(i)})$ for all i. Thus we have a subsequence $m(0) < m(1) < \cdots$ of $n(0) < n(1) < \cdots$ such that

$$H_1(\partial_0 V_{m(0)}) \cong H_1(\partial_0 V_{m(i)}) \qquad \text{for all} \quad i.$$

By Lemma 5, this implies that

(16) $$\partial_0 V_{m(0)} \approx \partial_0 V_{m(i)} \qquad \text{for all} \quad i.$$

We claim now that $\partial_0 V_{m(i+1)}$ is incompressible in $V_{m(i)}$. If not, by applying the cut and paste argument to $\partial_0 V_{m(i+1)}$ in $V_{m(i)}$, we get a connected 2-sided incompressible closed surface S in Int $V_{m(i)}$ so that $V_{m(i)} - S$ has two components $A \supset \partial V_{m(i)}$ and B with ε as an end, and that $\chi(S) > \chi(\partial_0 V_{m(i+1)})$. Hence S separates ∂Y from ε in Y since so is $\partial_0 V_{m(i)}$. Thus, by Lemma 9 (ii), S is parallel to $\partial_0 V_{m(i)}$ in $V_{m(i)}$, and $\chi(\partial_0 V_{m(i)}) = \chi(S)$. These contradicts with (16). Thus $\partial_0 V_{m(i+1)}$ is incompressible in $V_{m(i)}$.

Therefore, by the definition of $\partial_0 V_{m(i+1)}$ and Lemma 9 (ii), $\partial_0 V_{m(i+1)}$ is parallel to $\partial_0 V_{m(i)}$ in $V_{m(i)}$. Thus $\mathrm{Cl}\,(V_{m(i)} - V_{m(i+1)}) \approx \partial_0 V_{m(i)} \times [0, 1]$. This and (11) together with the fact that $\partial U_{m(i+1)}$ is connected show that $\partial V_{m(i)} = \partial_0 V_{m(i)}$ for all i. Thus $V_{m(0)} \approx \partial V_{m(0)} \times [0, \infty)$ as desired.

The proof of the Theorem is now completed.

References

[1] M. G. Brin and T. L. Thickstun, Open, irreducible 3-manifolds which are end 1-movable, to appear in Topology.

[2] M. W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. of Math. **117** (1983), 293–324.
[3] D. Epstein, Ends, Topology of 3-Manifolds and Related Topics, ed. by M. K. Fort, Prentice-Hall, 1962, 110–117.
[4] R. Geoghegan, A note on the vanishing of $\lim^1$, J. Pure Appl. Algebra **17** (1980), 113–116.
[5] W. Heil, On P^2-irreducible 3-manifolds, Bull. Amer. Math. Soc. **75** (1969), 772–775.
[6] L. S. Husch and T. M. Price, Finding boundary for a 3-manifold, Ann. of Math. **91** (1970), 223–235.
[7] B. Jackson, End invariants of group extensions, Topology **21** (1982), 71–81.
[8] O. Kakimizu, Cores of non-compact irreducible 3-manifolds, to appear in Kobe J. Math.
[9] D. McCullough, A. Miller and G. A. Swarup, Uniqueness of cores of non-compact 3-manifolds, J. London Math. Soc. **32** (1985), 548–556.
[10] D. R. McMillan, Jr., Compact, acyclic subsets of three-manifolds, Michigan Math. J. **16** (1969), 129–136.
[11] M. L. Mihalik, Semistability at the end of a group extensions, Trans. Amer. Math. Soc. **277** (1983), 307–321.
[12] G. P. Scott, Compact submanifolds of 3-manifolds, J. London Math. Soc. **7** (1973), 246–250.
[13] L. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, Ph. D. thesis, Princeton University, 1965.
[14] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. **87** (1968), 56–88.

Department of Mathematics,
Faculty of Science,
Hiroshima University