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Δ-genera and sectional genera of commutative rings

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Introduction

In algebraic geometry and (complex analytic) singularity theory, various "*genera*" are defined for algebraic varieties and singularities to classify them and to study their structure. So it is natural to consider the same problem in commutative ring theory. In [6], we introduced the notion of *genera* and *arithmetic genera* of commutative rings. On the other hand, the classification of (embedded) projective varieties by their *sectional genera* is a quite classical subject in algebraic geometry studied by Enriques, Castelnuovo, Roth and others. This old subject has been recently resurrected and extended to the classification of polarized varieties by their sectional genera (Fujita, Ionescu, Lanteri, Palleschi and others). T. Fujita, among others, introduced the notions of *Δ-genus* and *sectional genus* of a polarized variety, and studied the structure of polarized varieties with low genera.

The aim of this paper is to introduce the notions of *Δ-genera* and *sectional genera* of commutative rings and to study the structure of commutative rings by these genera.

By the way, the non-negativity of the sectional genus and the Δ-genus of a Cohen-Macaulay local ring traces back to Northcott (1960) and Abhyankar (1967). Moreover, the structure of Cohen-Macaulay local rings with low Δ-genera has been studied by J. Sally in detail. Sally's work generalizes the study of rational surface singularities (due to Artin) and minimally elliptic surface singularities (due to Laufer and Wahl).

§ 1. Δ-genera and sectional genera of polynomial functions

First, we recall some notations and terminologies from [6]. Let $f: \boldsymbol{Z} \to \boldsymbol{Z}$ be a polynomial function, i.e., there is a polynomial $P_f \in \boldsymbol{Q}[t]$ such that $f(n)=P_f(n)$ for all $n \gg 0$. We assume, for simplicity, that $f(n)=0$ for all $n<0$. Then there exist (uniquely determined) integers $d \geqq 0$ and e_i $(0 \leqq i \leqq d)$, $e_0 \neq 0$, such that

$$(\nabla f)(n) := \sum_{i=0}^{n} f(i) = e_0 \binom{n+d}{d} - e_1 \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d,$$

for all $n \gg 0$. Put $d(f)=d$, $e_i(f)=e_i$, $e(f)=e_0$, $g(f)=e_d=(-1)^d P_{\nabla f}(-1)$ and

$p_a(f)=(-1)^d(e_0-e_1+\cdots+(-1)^d e_d-f(0))=(-1)^d(P_{\nabla f}(0)-f(0))$. We call $d(f)$, $e_i(f)$, $e(f)$, $g(f)$, $p_a(f)$ the *dimension*, the i-th *Hilbert coefficient*, the *multiplicity*, the *genus*, the *arithmetic genus* of f respectively. Also we put $n(f)=\min\{n \mid f(m)=P_f(m)$ for all $m>n\}$ (the *postulation number* of f), $m(f)=n(f)+d(f)$ and $F_f(t)=\sum_{n\geqq 0} f(n)t^n$ (the *Hilbert series* of f). Note that $F_f(t)=\varphi_f(t)/(1-t)^d$, $d=d(f)$ for some $\varphi_f\in \boldsymbol{Z}[t]$ and we have $m(f)=\deg(\varphi_f)$. For the other notations used in this paper, see [6].

PROPOSITION 1.1. *Put $d(f)=d$, $e_i(f)=e_i$, $m(f)=m$ and $\varphi_f(t)=\sum_{n=0}^{m} a_n t^n$. Then*:

(1) $F_f(t)=e_0/(1-t)^d-e_1/(1-t)^{d-1}+\cdots+(-1)^{d-1}e_{d-1}/(1-t)+(-1)^d e_d+(1-t)Q(t)$ *for some* $Q\in \boldsymbol{Z}[t]$.

(2) $e_i(f)=\varphi_f^{(i)}(1)/i!=\sum_{i\leqq k\leqq m} a_k\binom{k}{i}$, *where* $\varphi_f^{(i)}=d^i\varphi_f/dt^i$. *In particular,* $e(f)=\varphi_f(1)$, $e_1(f)=\varphi'_f(1)$ *and* $e_i(f)=0$ *for all i such that* $m<i\leqq d$.

PROOF. Put $P_{\nabla f}=P$ and $Q(t)=\sum_{n=0}^{\infty}((\nabla f)(n)-P(n))t^n\in \boldsymbol{Z}[t]$. Then we have $F_f(t)=(1-t)F_{\nabla f}(t)=(1-t)\sum_{n=0}^{\infty}(\nabla f)(n)t^n=(1-t)\{\sum_{n=0}^{\infty}P(n)t^n+Q(t)\}$. Since $P(n)=\sum_{i=0}^{d}(-1)^i e_i\binom{n+d-i}{d-i}$, we have $(1-t)\sum_{n=0}^{\infty}P(n)t^n=(1-t)\sum_{i=0}^{d}(-1)^i e_i\sum_{n=0}^{\infty}\binom{n+d-i}{d-i}t^n=\sum_{i=0}^{d}(-1)^i e_i/(1-t)^{d-i}$. This implies (1). Since $\varphi_f(t)=(1-t)^d F_f(t)=\sum_{i=0}^{d}(-1)^i e_i(1-t)^i+(1-t)^{d+1}Q(t)$, the first equality of (2) follows. The second equality is a result of the equality $i!\varphi_f^{(i)}(t)=\sum_{i\leqq k\leqq m} a_k\binom{k}{i}t^{k-i}$ which can be proved by the induction on i. Q.E.D.

For convenience, we put $e_i(f)=\varphi_f^{(i)}(1)/i!$ for all $i>d$. We say that f is *h-positive* if all the coefficients of φ_f are positive (i.e., $a_i>0$ for all i, $0\leqq i\leqq m(f)$).

COROLLARY 1.2. *If f is h-positive, then $e_i(f)>0$ for all $0\leqq i\leqq m(f)$.*

DEFINITION 1.3. We define the *Δ-genus* $g_\Delta(f)$ and the *sectional genus* $g_s(f)$ of f by

$$g_\Delta(f)=e(f)+(d(f)-1)f(0)-f(1) \quad \text{and}$$

$$g_s(f)=e_1(f)-e(f)+f(0).$$

Note that if $d(f)\geqq 1$, then $g_\Delta(\Delta f)=g_\Delta(f)$ and $g_s(\Delta f)=g_s(f)$, where $(\Delta f)(n)=f(n)-f(n-1)$. If $d(f)=1$, then $g_s(f)=p_a(f)=\sum_{n=1}^{n(f)}(e(f)-f(n))$, and f is h-positive if and only if $f(0)<f(1)<\cdots<f(m-1)<f(m)$ with $m=m(f)$.

The following propositions follow easily from Proposition 1.1 and Definition 1.3. We omit the proof.

PROPOSITION 1.4. *Put $m(f)=m$ and $\varphi_f(t)=\varphi(t)=\sum_{n=0}^{m} a_n t^n$. Then we have*

$$g_\Delta(f) = a_2 + a_3 + \cdots + a_m = \varphi(1) - \varphi(0) + \varphi'(0),$$
$$g_s(f) = a_2 + 2a_3 + \cdots + (m-1)a_m = \varphi'(1) - \varphi(1) + \varphi(0), \quad \textit{and}$$
$$g_s(f) - g_\Delta(f) = a_3 + 2a_4 + \cdots + (m-2)a_m$$
$$= \varphi'(1) - 2\varphi(1) + 2\varphi(0) - \varphi'(0).$$

In particular, if $m(f) \leqq 1$ *(resp.* $m(f) \leqq 2$*), then* $g_s(f) = g_\Delta(f) = 0$ *(resp.* $g_s(f) = g_\Delta(f)$*).*

COROLLARY 1.5. *Assume that f is h-positive. Then:*

(1) $g_s(f) \geqq g_\Delta(f) \geqq 0$,
$g_s(f) = 0 \Leftrightarrow g_\Delta(f) = 0 \Leftrightarrow m(f) \leqq 1$, *and*
$g_s(f) = g_\Delta(f) \Leftrightarrow m(f) \leqq 2$.

Assume, moreover, that $f(0) = 1$, *and put* $d(f) = d$, $f(1) = v$. *Then:*

(2) $g_s(f) = 1 \Leftrightarrow g_\Delta(f) = 1 \Leftrightarrow \varphi_f(t) = 1 + (v-d)t + t^2$,
$g_s(f) = 2 \Leftrightarrow g_\Delta(f) = m(f) = 2$
$\Leftrightarrow \varphi_f(t) = 1 + (v-d)t + 2t^2$, *and*
$g_s(f) = 3 \Leftrightarrow g_\Delta(f) = 2$, $m(f) = 3$ *or* $g_\Delta(f) = 3$, $m(f) = 2$
$\Leftrightarrow \varphi_f(t) = 1 + (v-d)t + t^2 + t^3$ *or* $\varphi_f(t) = 1 + (v-d)t + 3t^2$.

(3) $m(f) \leqq g_\Delta(f) + 1$. *The equality holds if and only if* $\varphi_f(t) = 1 + (v-d)t + t^2 + \cdots + t^m$, $m = m(f)$, *and in this case, we have* $e_1(f) = (v-d) + m(m+1)/2 - 1$, $e_i(f) = \binom{m+1}{i+1}$ $(2 \leqq i \leqq d)$, *and* $g_s(f) = \binom{m}{2}$.

EXAMPLE 1.6. Let $M = \bigoplus_{n \geqq 0} M_n$ be a graded module over a ring R and assume that $f(n) = H(M, n) := \ell_R(M_n)$ is a polynomial function. Then we write $e_i(M)$, $F(M, t)$, $\varphi_M(t)$, $n(M)$, $m(M)$, $g(M)$, $p_a(M)$, $g_\Delta(M)$, $g_s(M)$ instead of $e_i(f)$, $F_f(t)$, $\varphi_f(t)$, $n(f)$, $m(f)$, $g(f)$, $p_a(f)$, $g_\Delta(f)$, $g_s(f)$ respectively. If f is h-positive, then we say that M is h-positive.

(1) Let A be a homogeneous algebra over an artinian local ring and $M = \bigoplus_{n \geqq 0} M_n$ a finitely generated graded A-module. If $a \in A_1$ is M-regular, then $g_\Delta(M) = g_\Delta(M/aM)$ and $g_s(M) = g_s(M/aM)$. If M is Cohen-Macaulay, then M is h-positive. This case is treated in §2.

(2) Let X be a projective variety and D an ample Cartier divisor on X. Put $A = \bigoplus_{n \geqq 0} H^0(X, \mathcal{O}_X(nD))$. Then our $g_\Delta(A)$ and $g_s(A)$ coincide with the Δ-genus $\Delta(X, D)$ and the sectional genus $g(X, D)$ of the polarized variety (X, D) introduced by T. Fujita [1].

(3) Let $(R, \mathfrak{m})$ be a noetherian local ring with $\dim(R) = d$ and I an $\mathfrak{m}$-primary ideal of R. Then we put $e_i(I) = e_i(G(I))$, $\varphi_I(t) = \varphi_{G(I)}(t)$, $n(I) = n(G(I))$, $m(I) = m(G(I))$, $g(I) = g(G(I))$, $p_a(I) = p_a(G(I))$, $g_\Delta(I) = g_\Delta(G(I)) = e(I) + (d-1)\ell(R/I) - \ell(I/I^2)$ and $g_s(I) = g_s(G(I)) = e_1(I) - e(I) + \ell(R/I)$, where $G(I) = \bigoplus_{n \geqq 0} I^n/I^{n+1}$. If R is analytically unramified, then we put $\bar{e}_i(I) = e_i(\bar{G}(I))$, $\bar{n}(I) = n(\bar{G}(I))$, $\bar{m}(I) = m(\bar{G}(I))$, $\bar{g}(I) = g(\bar{G}(I))$, $\bar{p}_a(I) = p_a(\bar{G}(I))$, $\bar{g}_\Delta(I) = g_\Delta(\bar{G}(I)) = e(I) + (d-1)\ell(R/\bar{I}) -$

$\ell(\bar{I}/\overline{I^2})$ and $\bar{g}_s(I)=g_s(\bar{G}(I))=\bar{e}_1(I)-e(I)+\ell(R/\bar{I})$, where $\bar{G}(I)=\oplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ and $\bar{J}$ is the integral closure of J. Also we put $G(R)=G(\mathfrak{m})$, $\varphi_R(t)=\varphi_{\mathfrak{m}}(t)$, $g(R)=g(\mathfrak{m})$, $p_a(R)=p_a(\mathfrak{m})$, $g_\Delta(R)=g_\Delta(\mathfrak{m})=e(R)+\dim(R)-\mathrm{emb}(R)-1$, $g_s(R)=g_s(\mathfrak{m})=e_1(R)-e(R)+1$, etc. We call $g_\Delta(I)$, $g_s(I)$, $\bar{g}_\Delta(I)$, $\bar{g}_s(I)$ the *Δ-genus*, the *sectional genus*, the *normal Δ-genus*, the *normal sectional genus* of I respectively. This case is treated in §3.

If $x\in I$ is an R-regular superficial element with respect to I, then we have $e_i(I/xR)=e_i(I)$ $(0\leqq i<d)$ and $g_s(I/xR)=g_s(I)$. If $R=k[\![t^5, t^8, t^{27}, t^{29}]\!]$, then $\varphi_R(t)=1+3t-t^2+t^3+t^4$. Hence $G(R)$ is not h-positive.

§2. The case of graded rings

Let $f\colon \mathbf{Z}\to\mathbf{Z}$ be a polynomial function. Then there is a homogeneous algebra A over a field such that $f(n)=H(A, n)$ for all $n\geqq 0$ if and only if $(f(0), f(1),\dots)$ is an *M-vector* in the sense of [11], i.e., $f(0)=1$ and $0\leqq f(n+1)\leqq f(n)^{\langle n\rangle}$ for all $n\geqq 1$ (for the notation $m^{\langle n\rangle}$, see [11]). In this case, if $1\leqq f(n)\leqq n$ for some n, then $f(m+1)\leqq f(m)$ for all $m\geqq n$.

PROPOSITION 2.1 (*cf.* [3]). *Let A be a one-dimensional homogeneous algebra over a field k which satisfies the following condition*: $H(A, n)<e(A)$ *for all* $n\leqq n(A)$. *Then*:

(1) $m(A)\leqq e(A)-1$, $m(A)\leqq g_s(A)+1$,
$0\leqq g_\Delta(A)\leqq g_s(A)\leqq\binom{e(A)-1}{2}$, *and*
$g_s(A)=g(A)-e(A)+1\leqq g(A)$.

(2) $g(A)=0\Leftrightarrow g(A)=g_s(A)\Leftrightarrow e(A)=1\Leftrightarrow m(A)=0\Leftrightarrow A\cong k[X]$.

(3) $g(A)=1\Leftrightarrow e(A)=2\Leftrightarrow A\cong k[X, Y]/(h)$ *with* $\deg(h)=2$.

(4) $g_s(A)=0\Leftrightarrow g_\Delta(A)=0\Leftrightarrow m(A)\leqq 1$.

(5) $g_s(A)=1\Leftrightarrow g_\Delta(A)=1$ *and A is h-positive*$\Leftrightarrow\varphi_A(t)=1+(v-1)t+t^2$, *where* $v=\mathrm{emb}(A)$.

(6) $g_s(A)=g_\Delta(A)\Leftrightarrow m(A)\leqq 2\Leftrightarrow\varphi_A(t)=1+(v-1)t+g_\Delta(A)t^2$.

(7) $g_s(A)=\binom{e(A)-1}{2}\Leftrightarrow\mathrm{emb}(A)\leqq 2$.

PROOF. Put $f(n)=H(A, n)$, $e(A)=e$, $\mathrm{emb}(A)=v$, $m(A)=m$, $g(A)=g$, $g_\Delta(A)=g_\Delta$ and $g_s(A)=g_s$.

(1) Assume that $m\geqq e$. Then $f(e-1)<e$ by the condition, and we get $e-1\geqq f(e-1)\geqq f(n)=e$ for all $n\gg 0$, which is a contradiction. Therefore $m<e$. Clearly $g_\Delta=e-f(1)\geqq 0$ and $g_s=\sum_{n=1}^{m-1}(e-f(n))\geqq m-1$. For all $n\gg 0$, we have $en-g=(\nabla f)(n-1)=1+v+\sum_{i=2}^{n-1}f(i)\leqq 1+v+(n-2)e$ and $en-g=1+\sum_{i=1}^{e-2}f(i)+\sum_{i=e-1}^{n-1}f(i)\geqq 1+\sum_{i=1}^{e-2}(i+1)+(n-e+1)e=en-e(e-1)/2$ (note that if $i<e$, then $f(i)\geqq i+1$). Hence $g_s-g_\Delta=g-2e+v+1\geqq 0$ and $g\leqq\binom{e}{2}$, i.e., $g_s\leqq\binom{e-1}{2}$.

(2), (3) and (4) are easily shown and we omit the proof.

(6) Assume that $m \geqq 3$. Then $f(2)<e$ and $en-g=1+f(1)+f(2)+\sum_{i=3}^{n-1} f(i)<1+v+e+(n-3)e$ for all $n \gg 0$. Therefore $g_s-g_\Delta=g-2e+v-1>0$. Conversely, if $m \leqq 2$, then $g_s=\sum_{n=1}^{m-1}(e-f(n))=e-v=g_\Delta$.

(5) $g_s=1 \Rightarrow 1=g_s \geqq g_\Delta \geqq 1$ (by (1) and (4)) $\Rightarrow g_s=g_\Delta=1 \Rightarrow \varphi_f(t)=1+(v-1)t+t^2$ (by (6)) $\Rightarrow g_\Delta=1$ and f is h-positive $\Rightarrow m \leqq g_\Delta+1=2$ (by Corollary 1.5, (3)) $g_s=g_\Delta=1$ (by (6)).

(7) Assume that $v \geqq 3$. Then for all $n \gg 0$, we have $en-g \geqq 1+3+\sum_{i=2}^{e-2} f(i)+\sum_{i=e-1}^{n-1} f(i) \geqq en-e(e-1)/2+1$ as in (1). Hence $g_s<\binom{e-1}{2}$. Conversely, assume that $v=2$. Then for all $n<e$, we have $f(n) \geqq n+1$, i.e., $f(n)=n+1$. Therefore $\varphi_f(t)=1+t+\cdots+t^{e-1}$ and we have $g_s=\binom{e-1}{2}$, $A=k[X, Y]/(h)$ with $\deg(h)=e$. Q. E. D.

REMARK. The condition in Proposition 2.1 is satisfied if either A is h-positive or $A \cong G(R)$ for a one-dimensional Cohen-Macaulay local ring R.

Let A be a homogeneous algebra over a field k. We say that A is *numerically Cohen-Macaulay* if $F(A, t)=F(B, t)$ for some Cohen-Macaulay homogeneous k-algebra B, or equivalently $\varphi_A(t)=\sum_{n=0}^{m} a_n t^n$, $m=m(A)$ and $(a_0, a_1,\ldots, a_m)$ is an M-vector, cf. [11]. We say that A is *numerically a complete intersection* of type $(b_1,\ldots, b_r)$, $b_i \geqq 2$ if $\varphi_A(t)=\prod_{i=1}^{r}(1+t+\cdots t^{b_i-1})$. Note that A is a hypersurface of degree $e \geqq 2 \Leftrightarrow \varphi_A(t)=1+t+\cdots+t^{e-1} \Leftrightarrow A$ is numerically Cohen-Macaulay and $\operatorname{emb}(A)=\dim(A)+1$.

If A is numerically Cohen-Macaulay, then A is h-positive and we can apply Corollary 1.5. If A is Cohen-Macaulay, then $m(A)=\operatorname{reg}(A):=\min\{n \mid [H_P^i(A)]_j=0 \text{ if } i+j>n\}$, $P=A_+$, cf. [5].

PROPOSITION 2.2. (1) *Let A be a numerically Cohen-Macaulay homogeneous algebra over a field with* $\dim(A)=d \geqq 1$. *Then* $g_s(A) \leqq \binom{e(A)-1}{2}$, *and the equality holds if and only if A is a hypersurface.*

(2) *If A is a Cohen-Macaulay homogeneous domain over a field with $g_s(A)=1$, then A is Gorenstein.*

PROOF. (1) We may assume that A is Cohen-Macaulay. Taking an A-regular sequence $x_1,\ldots, x_{d-1}$ in A_1 and applying Proposition 2.1 to $A/(x_1,\ldots, x_{d-1})A$, we get the assertion.

(2) We have $\varphi_A(t)=1+(\operatorname{emb}(A)-\dim(A))t+t^2$ by Corollary 1.5, (2). Hence A is Gorenstein by [10]. Q. E. D.

EXAMPLE 2.3. (1) Let A be a hypersurface of degree e with $\dim(A)=d$ and $\operatorname{emb}(A)=v$. Then $e_i(A)=\binom{e}{i+1}$, $g_\Delta(A)=e-2$, $g_s(A)=\binom{e-1}{2}$, $g(A)=\binom{e}{v}$

and $p_a(A)=\binom{e-1}{v}$.

(2) Let A be numerically a complete intersection of type $(b_1,\dots, b_r)$. Then $e(A)=\prod_{i=1}^r b_i$, $m(A)=\sum_{i=1}^r b_i-r$ and $g_s(A)=e(A)(m(A)-2)/2+1$.

(3) (Cf. [11], Example 11.4.) Put $A=k[X, Y, Z, W]/(XZ, XW, YW)$ and $B=k[X, Y, Z, W]/(XYZ, XW, YW, ZW)$. Then A is Cohen-Macaulay, B is not Cohen-Macaulay and $F(A, t)=F(B, t)=(1+2t)/(1-t)^2$. Hence $g=p_a=g_s=g_\Delta=0$ for both A and B.

(4) Let A be an artinian homogeneous algebra over a field k with $\mathrm{emb}(A)=v$. Then $g_s(A)=0$ if and only if $A\cong k[X_1,\dots, X_v]/(X_1,\dots, X_v)^2$. $g_s(A)=1$ if and only if $A=A_0\oplus A_1\oplus A_2$ with $A_2\cong k$. Hence to give such an A is equivalent to give a symmetric bilinear form on the k-vector space A_1. Therefore, if k is algebraically closed, then $A\cong k[X_1,\dots, X_v]/((X_1,\dots, X_v)^3, X_iX_j(i\neq j), X_i^2-X_r^2$ $(1\leqq i<r)$, X_j^2 $(r<j\leqq v))$ with $r=\mathrm{emb}(A)-r(A)+1$, and A is Gorenstein if and only if $A\cong k[X_1,\dots, X_v]/((X_1,\dots, X_v)^3, X_iX_j\ (i\neq j), X_i^2-X_v^2\ (1\leqq i<v))$.

The following proposition can be proved easily by using Proposition 1.4 and Corollary 1.5. So we omit the routine proof.

PROPOSITION 2.4. *Let A be a Gorenstein homogeneous algebra over a field which is not a polynomial ring, and let* $\dim(A)=d\geqq 1$, $\mathrm{emb}(A)=v$, $e(A)=e$. *Then*:

(0) $g_s(A)=0\Leftrightarrow g_\Delta(A)=0\Leftrightarrow \mathrm{reg}(A)=1\Leftrightarrow A$ *is a quadric hypersurface.*

(1) $g_s(A)=1\Leftrightarrow g_\Delta(A)=1\Leftrightarrow g_s(A)=g_\Delta(A)\geqq 1\Leftrightarrow \mathrm{reg}(A)=2$.

(2) $g_s(A)=2$ *never occurs.*

(3) $g_s(A)=3\Leftrightarrow g_\Delta(A)=2\Leftrightarrow A$ *is a quartic hypersurface.*

(4) $g_s(A)=4\Leftrightarrow g_\Delta(A)=3$ *and* $e(A)=6\Leftrightarrow A$ *is a complete intersection of type* (2, 3).

(5) $g_s(A)=5\Leftrightarrow g_\Delta(A)=4$ *and* $e(A)=8\Leftrightarrow A$ *is numerically a complete intersection of type* (2, 2, 2).

(6) $g_s(A)=g_\Delta(A)+1\Leftrightarrow \mathrm{reg}(A)=3\Leftrightarrow v=e/2+d-1\Leftrightarrow g_s(A)=e(A)/2+1\Leftrightarrow g_\Delta(A)=e(A)/2$.

(7) $g_s(A)=g_\Delta(A)+2$ *never occurs.*

EXAMPLE 2.5. (1) A is Gorenstein and $g_\Delta(A)=3$ if and only if A is a quintic hypersurface or a complete intersection of type (2, 3). Assume that A is Gorenstein and $\mathrm{reg}(A)\neq 3$. Then, $g_s(A)=6$ if and only if A is a quintic hypersurface; $g_s(A)\neq 7, 8$; $g_s(A)=9$ if and only if A is a complete intersection of type (2, 4); $g_s(A)=10$ if and only if A is a hypersurface of degree 6 or a complete intersection of type (3, 3).

(2) Let C be a non-hyperelliptic smooth projective curve of genus $g\geqq 3$ and $A=\oplus_{n\geqq 0} H^0(C, \mathcal{O}_C(nK))$ be its canonical ring. Then A is a two-dimensional

Gorenstein normal homogeneous domain with $\operatorname{reg}(A)=3$, $\operatorname{emb}(A)=g$, $e(A)=2g-2$, $g_s(A)=g$, $g_\Delta(A)=g-1$, $g(A)=g+1$ and $p_a(A)=1$ (cf. [5], p. 641).

§ 3. The case of local rings

Throughout this section, $(R, \mathfrak{m}, k)$ denotes a Cohen-Macaulay local ring with $\dim(R)=d\geqq 1$, $\operatorname{emb}(R)=v$, $e(R)=e$. Let I be an $\mathfrak{m}$-primary ideal of R. Recall that $g_\Delta(I)=e(I)+(d-1)\ell(R/I)-\ell(I/I^2)$ and $g_s(I)=e_1(I)-e(I)+\ell(R/I)$. If k is an infinite field, then we put $\delta(I)=\min\{n \mid JI^n=I^{n+1}$ for some minimal reduction J of $I\}$ (the *reduction exponent* of I, cf. [6]). We have $\delta(I)\leqq \operatorname{reg}(G(I))$, and the equality holds if $G(I)$ is Cohen-Macaulay. We also put $\delta(\mathfrak{m})=\delta(R)$.

PROPOSITION 3.1 (*cf.* [6], Theorem 5.1). *Assume that* $\dim(R)=1$ *and put* $S=\bigcup_{n=0}^{\infty}(I^n : I^n)$. *Then*:

(1) $\ell(R/I^n)=e(I)n-\ell(S/R)+\ell(I^nS/I^n)$ *for all* $n\geqq 0$, *and* $g(I)\geqq g_s(I)\geqq g_\Delta(I)\geqq 0$, $g_s(I)-g_\Delta(I)=\ell(I^2S/I^2)$.

(2) $\delta(I)=0 \Leftrightarrow g(I)=0 \Leftrightarrow I$ *is a principal ideal*,
$\delta(I)\leqq 1 \Leftrightarrow g_s(I)=0 \Leftrightarrow g_\Delta(I)=0$, *and*
$\delta(I)\leqq 2 \Leftrightarrow g_s(I)=g_\Delta(I)$.

LEMMA 3.2 (*cf.* [12]). *Assume that k is an infinite field and let J be a minimal reduction of I. Then* $g_\Delta(I)=\ell(I^2/IJ)$. $g_\Delta(I)=0$ *if and only if* $\delta(I)\leqq 1$, *and in this case, G(I) is Cohen-Macaulay.*

THEOREM 3.3. *We have* $g_s(I)\geqq 0$, $g_\Delta(I)\geqq 0$, *and the following conditions are equivalent*:

(1) $g_s(I)=0$.

(2) $g_\Delta(I)=0$.

(3) $\operatorname{reg}(G(I))\leqq 1$.

(4) $\ell(R/I^{n+1})=e(I)\binom{n+d-1}{d}+\ell(R/I)\binom{n+d-1}{d-1}$ *for all* $n\geqq 0$.

PROOF. We may assume that k is an infinite field. The fact $g_s(I)\geqq 0$ is proved in [4] (see also [6], Lemma 4.2), and we have $g_\Delta(I)\geqq 0$ by Lemma 3.2. The assertions (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Rightarrow$(1) follow from Lemma 3.2 and [6], Theorem 4.3. So we have only to show the assertion (1)$\Rightarrow$(3). Take a superficial system of parameters $x_1,\dots, x_d\in I$ with respect to I and put $J=(x_1,\dots, x_d)$, $I_i=(x_1,\dots, \hat{x}_i,\dots, x_d)$, $1\leqq i\leqq d$. Then we have $g_s(I/I_i)=g_s(I)=0$ and $\dim(R/I_i)=1$. Hence, by Proposition 3.1, we have $x_i(I/I_i)=(I/I_i)^2$, i.e., $x_iI+I_i=I^2+I_i$. Take any element y of I^2. Then for any $j\neq 1$, we have $y=x_1y_1+\cdots+x_dy_d=x_1z_1+\cdots+x_dz_d$ for some y_i, z_i such that y_1, $z_j\in I$. Hence $x_1(y_1-z_1)+\cdots+x_d(y_d-z_d)=0$, and this implies that $y_j-z_j\in J\subset I$. Therefore $y_j\in I$ for all j, $1\leqq j\leqq d$, and we have $y\in JI$. Hence $I^2=JI$, i.e., $\delta(I)\leqq 1$ (or equivalently, $\operatorname{reg}(G(I))\leqq 1$). Q. E. D.

THEOREM 3.4. (1) $0 \leqq g_\Delta(R) \leqq g_s(R) \leqq \binom{e(R)-1}{2}$.

(2) $g_s(R)=0 \Leftrightarrow g_\Delta(R)=0 \Leftrightarrow \operatorname{reg}(G(R)) \leqq 1$. *In this case, we have* $r(R)=e(R)-1$ *if* R *is not a regular local ring* ($r(R)$ *denotes the Cohen-Macaulay type of* R).

(3) $g_s(R)=g_\Delta(R) \Leftrightarrow \operatorname{reg}(G(R)) \leqq 2$

$$\Leftrightarrow \varphi_R(t)=1+(v-d)t+(e+d-v-1)t^2$$

$$\Leftrightarrow \ell(R/\mathfrak{m}^{n+1})=e\binom{n+d}{d}-(2e+d-v-2)\binom{n+d-1}{d-1}+(e+d-v-1)\binom{n+d-2}{d-2}$$

for all $n \geqq 2$. *In this case,* $G(R)$ *is Cohen-Macaulay.*

(4) $g_s(R)=1 \Leftrightarrow g_\Delta(R)=1$ *and* $G(R)$ *is Cohen-Macaulay*

$$\Leftrightarrow g_\Delta(R)=1 \text{ and } \operatorname{reg}(G(R))=2$$

$$\Leftrightarrow \varphi_R(t)=1+(v-d)t+t^2$$

$$\Leftrightarrow \ell(R/\mathfrak{m}^{n+1})=e\binom{n+d-1}{d-1}+\binom{n+d-2}{d-2} \text{ for all } n \geqq 2.$$

(5) *If* $g_s(R)=2$, *then* $g_\Delta(R)=2$ *or* $g_\Delta(R)=1$. *In the first case,* $G(R)$ *is Cohen-Macaulay,* $\varphi_R(t)=1+(v-d)t+2t^2$ *and* $e_2(R)=2$, $e_i(R)=0$ *for all* $i \geqq 3$. *In the second case,* $G(R)$ *is not Cohen-Macaulay and* $r(R)=e(R)-2$.

(6) $g_s(R)=\binom{e(R)-1}{2}$ *if and only if* R *is a hypersurface.* (*We say that* R *is a hypersurface of degree* e *if* $\hat{R} \cong S/(f)$, *where* $(S, \mathfrak{n})$ *is a regular local ring and* $f \in \mathfrak{n}^e - \mathfrak{n}^{e+1}$.)

PROOF. We may assume that k is an infinite field. Take a superficial system of parameters $x_1, \ldots, x_d$ such that $x_i \in \mathfrak{m}-\mathfrak{m}^2$. (1) Put $R'=R/(x_1, \ldots, x_{d-1})$. Then $g_s(R)=g_s(R')$, $g_\Delta(R)=g_\Delta(R')$ and $e(R)=e(R')$. Hence the assertion follows from Proposition 3.1, (1) and Proposition 2.1. The proof of (6) is similar. (2) follows from Theorem 3.3. (3) If $\operatorname{reg}(G(R)) \leqq 2$, then $\delta(R) \leqq 2$ and $G(R)$ is Cohen-Macaulay by [7]. (By the way, $\bar{G}(R)$ is Cohen-Macaulay if R is analytically unramified and $\bar{\delta}(R) \leqq 2$.) Therefore we have only to show that if $g_s(R)=g_\Delta(R)$, then $\delta(R) \leqq 2$. Put $\mathfrak{q}=(x_1, \ldots, x_d)$, $\mathfrak{q}_i=(x_1, \ldots, \hat{x}_i, \ldots, x_d)$ and $R_i=R/\mathfrak{q}_i$. Then $g_s(R_i)-g_\Delta(R_i)=g_s(R)-g_\Delta(R)=0$. Hence $\delta(R_i) \leqq 2$ by Proposition 3.1, (2). Therefore $x_i\mathfrak{m}^2+\mathfrak{q}_i=\mathfrak{m}^3+\mathfrak{q}_i$. Take any element y of $\mathfrak{m}^3$. Then for any $j \neq 1$, $y=x_1y_1+\cdots+x_dy_d=x_1z_1+\cdots+x_dz_d$ for some y_i, z_i such that y_1, $z_j \in \mathfrak{m}^2$. As in the proof of Theorem 3.3, we have $y_j-z_j \in \mathfrak{q}$, and y_j is in $(\mathfrak{m}^2, \mathfrak{q})$ for all j. Hence $y=u+w$ with $u \in \mathfrak{q}^2$, $w \in \mathfrak{q}\mathfrak{m}^2$. Since $x_1, \ldots, x_d$ are analytically independent, we have $u \in \mathfrak{q}^2 \cap \mathfrak{m}^3=\mathfrak{q}^2\mathfrak{m}$, and this implies that $y \in \mathfrak{q}\mathfrak{m}^2$, i.e., $\delta(R) \leqq 2$. (4) and (5) follow from (1), (2), (3) and [9]. Q.E.D.

EXAMPLE 3.5. (1) Put $H=\langle e, e+1, \ldots, 2e-1\rangle$, $e \geqq 2$ and $R=k[\![H]\!]$. Then $\bar{g}_s(R)=g_s(R)=g_\Delta(R)=0$, $\bar{g}(R)=g(R)=e-1$, $r(R)=e-1$ and $\bar{\delta}(R)=\delta(R)=1$ (in particular, $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ for all $n \geqq 0$).

(2) Put $H=\langle e, e+1,\dots, e+r, e+r+2,\dots, 2e-1\rangle$, $0\leqq r\leqq e-3$ and $R=k[\![H]\!]$. Then $\bar{g}_s(R)=g_s(R)=g_\Delta(R)=1$, $\bar{g}(R)=g(R)=e$, $c(H)=e+r+2$, R is not Gorenstein and $\bar{\delta}(R)=\delta(R)=2$ (hence $G(R)$ and $\bar{G}(R)$ are Cohen-Macaulay).

(3) Put $H=\langle e, e+1,\dots, 2e-r-1\rangle$, $1\leqq r\leqq (e-1)/2$ and $R=k[\![H]\!]$. Then $\bar{g}_s(R)=g_s(R)=g_\Delta(R)=r$, $\bar{g}(R)=g(R)=e+r-1$, $\varphi_R(t)=1+(e-r-1)t+rt^2$, $c(H)=2e$ and $\bar{\delta}(R)=\delta(R)=2$ (hence $G(R)$ and $\bar{G}(R)$ are Cohen-Macaulay). For example, if $R=k[\![t^5, t^6, t^7]\!]$, then $\bar{g}_s(R)=g_s(R)=g_\Delta(R)=2$.

(4) Put $H=\langle e, e+1,\dots, 2e-3, 3e-1\rangle$, $e\geqq 5$ and $R=k[\![H]\!]$. Then $\bar{g}_s(R)=2$, $g_s(R)=g_\Delta(R)=1$, $\bar{g}(R)=e+1$, $g(R)=e$, $\varphi_R(t)=1+(e-2)t+t^2$, $c(H)=2e$, $\bar{\delta}(R)=2$, $\delta(R)=1$, R is not Gorenstein, and $G(R)$ and $\bar{G}(R)$ are Cohen-Macaulay.

(5) Put $H=\langle e, e+1, e(e-1)-1\rangle$, $e\geqq 4$ and $R=k[\![H]\!]$. Then $\bar{g}_s(R)=g_s(R)=e(e-3)/2$, $\bar{g}(R)=g(R)=(e-2)(e+1)/2$, $g_\Delta(R)=e-3$, $c(H)=e(e-2)$, $\bar{\delta}(R)=e-2$, $\delta(R)=e-1$, $\varphi_R(t)=1+2t+t^3+t^4+\cdots+t^{e-1}$ (hence $G(R)$ is not h-positive), R is not Gorenstein and $G(R)$ is not Cohen-Macaulay. For example, if $R=k[\![t^4, t^7, t^{11}]\!]$, then $\bar{g}_s(R)=g_s(R)=2$, $g_\Delta(R)=1$, $r(R)=2$, $\bar{\delta}(R)=2$, $\delta(R)=3$, $\bar{G}(R)$ is Cohen-Macaulay and $G(R)$ is not Cohen-Macaulay.

(6) Assume that $e(R)=4$. Then $g_s(R)=0$, 1, 2 or 3. $G(R)$ is not Cohen-Macaulay if and only if $g_s(R)=2$, and in this case we have $r(R)=2$. For example, $g_s(k[\![t^4, t^5, t^6, t^{11}]\!])=0$, $g_s(k[\![t^4, t^5, t^7]\!])=1$, $g_s(k[\![t^4, t^5, t^{11}]\!])=2$ and $g_s(k[\![t^4, t^5]\!])=3$.

(7) (cf. [6], Example 6.4). Put $R=k[\![X, Y]\!]$ and $I=(X^4, X^3Y, XY^3, Y^4)$. Then $g(I)=0$, $p_a(I)=-1$, $g_s(I)=1$, $g_\Delta(I)=2$, $n(I)=1$, $\delta(I)=2$ and $F(G(I), t)=(11+3t+3t^2-t^3)/(1-t)^2$. Hence $p_a(I)$ is not necessarily non-negative, $g_s(I)\geqq g_\Delta(I)$ does not hold in general, and $\delta(I)=2$ does not imply that $G(I)$ is Cohen-Macaulay.

THEOREM 3.6. *Assume that R is Gorenstein. Then:*

(1) $g_s(R)=0\Leftrightarrow g_\Delta(R)=0\Leftrightarrow R$ *is a regular local ring or a quadric hypersurface.*

(2) $g_s(R)=1\Leftrightarrow g_\Delta(R)=1\Leftrightarrow g_s(R)=g_\Delta(R)\geqq 1\Leftrightarrow \mathrm{reg}(G(R))=2$. *In this case, $G(R)$ is Gorenstein.*

(3) $g_s(R)=2$ *never occurs.*

(4) $g_s(R)=3\Leftrightarrow g_\Delta(R)=2\Leftrightarrow \varphi_R(t)=1+(v-d)t+t^2+t^3$. *In this case, $G(R)$ is Cohen-Macaulay,* $\mathrm{reg}(G(R))=3$, $e_2(R)=4$, $e_3(R)=1$, $e_i(R)=0$ *for all* $i\geqq 4$, *and $G(R)$ is Gorenstein if and only if R is a quartic hypersurface.*

(5) *If* $g_s(R)=4$, *then* $g_\Delta(R)=3$.

PROOF. (1) follows from Theorem 3.4, (2). (2) follows from Theorem 3.4, (3), (4) and from the fact that $G(R)$ is Gorenstein if R is Gorenstein and $g_\Delta(R)=1$ (cf. [7]). (3) If $g_s(R)=2$, then we have $2=g_s(R)\geqq g_\Delta(R)\geqq 2$, i.e., $g_s(R)=g_\Delta(R)=2$. This contradicts (2). (4) If $g_s(R)=3$, then $2\leqq g_\Delta(R)<g_s(R)=3$. Hence $g_\Delta(R)=2$.

Conversely, if $g_\Delta(R)=2$, then $G(R)$ is Cohen-Macaulay by [8], and we have $\operatorname{reg}(G(R))=g_\Delta(R)+1=3$. Hence $\varphi_R(t)=1+(v-d)t+t^2+t^3$ and we get $g_s(R)=3$.
(5) Since $3\leqq g_\Delta(R)<g_s(R)=4$, we have $g_\Delta(R)=3$. Q. E. D.

EXAMPLE 3.7. (1) Put $H=\langle e, e+1,\dots, 2e-2\rangle$, $e\geqq 2$ and $R=k[\![H]\!]$. Then R is Gorenstein and $\bar{g}_s(R)=g_s(R)=g_\Delta(R)=1$.

(2) Put $H=\langle 2a, 2a+1,\dots, 3a-1\rangle$, $a\geqq 2$ and $R=k[\![H]\!]$. Then $G(R)$ is Gorenstein, $\operatorname{reg}(G(R))=3$, $\varphi_R(t)=1+(a-1)t+(a-1)t^2+t^3$, $\bar{g}_s(R)=g_s(R)=a+1$, $g_\Delta(R)=a$, $\bar{g}(R)=g(R)=3a$ and $\bar{\delta}(R)=\delta(R)=3$. For example, if $R=k[\![t^6, t^7, t^8]\!]$, then $\bar{g}_s(R)=g_s(R)=4$ and $g_\Delta(R)=3$.

(3) Put $H=\langle 2a+1, 2a+2,\dots, 3a, 4a+1\rangle$, $a\geqq 2$ and $R=k[\![H]\!]$. Then R is Gorenstein, $\varphi_R(t)=1+at+(a-1)t^2+t^3$, $\bar{g}_s(R)=g_s(R)=a+1$, $g_\Delta(R)=a$, $G(R)$ is Cohen-Macaulay and is not Gorenstein and $\bar{\delta}(R)=\delta(R)=3$. For example, if $R=k[\![t^5, t^6, t^9]\!]$, then R is Gorenstein and $\bar{g}_s(R)=g_s(R)=3$, $g_\Delta(R)=2$.

(4) Put $R=k[\![t^6, t^7, t^{15}]\!]$. Then R is Gorenstein and $g_s(R)=7$, $g_\Delta(R)=3$. Hence the converse of Theorem 3.6, (5) does not hold.

(5) Let R be a Gorenstein local ring with $e(R)=5$. Then $g_s(R)=1$, 3 or 6. $G(R)$ is always Cohen-Macaulay, and $G(R)$ is Gorenstein if and only if $g_s(R)=1$ or 6. For example, $g_s(k[\![t^5, t^6, t^7, t^8]\!])=1$, $g_s(k[\![t^5, t^6, t^9]\!])=3$ and $g_s(k[\![t^5, t^6]\!])=6$.

PROPOSITION 3.8. *Assume that* $\dim(R)=2$. *Then* $g(I)=p_a(I)+g_s(I)\geqq p_a(I)\geqq -g_s(I)$, *and the following conditions are equivalent*:

(1) $g_s(I)=0$.

(2) $n(I)<0$.

(3) $g(I)=p_a(I)$ (*resp.* $g(I)=p_a(I)=0$).

PROOF. The first assertion and (3)⇒(1) are clear. (1)⇒(2): By Theorem 3.3, we have $n(I)+2=\operatorname{reg}(G(I))\leqq 1$. (2)⇒(3) follows from [6], Theorem 1.3, (7).
Q. E. D.

COROLLARY 3.9. *Assume that* $\dim(R)=2$. *Then*:

(1) $g_s(R)=0\Leftrightarrow g(R)=p_a(R)=0\Leftrightarrow g(R)=0$ *and* $G(R)$ *is Cohen-Macaulay.*

(2) $g_s(R)=1\Leftrightarrow g(R)=1$ *and* $p_a(R)=0$.

(3) *If* $g(R)=0$, *then* $p_a(R)\leqq -2$.

(4) *If* R *is Gorenstein, then* $g_s(R)=3$ *if and only if* $g(R)=4$ *and* $p_a(R)=1$.

PROOF. (1) If $g(R)=0$ and $G(R)$ is Cohen-Macaulay, then $0=g(R)\geqq p_a(R)\geqq 0$, which implies that $p_a(R)=0$. The other assertions are clear.

(2) If $g_s(R)=1$, then $\varphi_R(t)=1+(v-d)t+t^2$ by Theorem 3.4, (4). Hence $g(R)=e_2(R)=1$ and $p_a(R)=g(R)-g_s(R)=0$. The converse is clear.

(3) We have $p_a(R)\leqq g(R)=0$ by Proposition 3.8. If $p_a(R)=-1$, then $g_s(R)=g(R)-p_a(R)=1$. Hence $G(R)$ is Cohen-Macaulay by Theorem 3.4, (4).

Therefore $p_a(R) \geqq 0$, which is a contradiction.

(4) If $g_s(R)=3$, then $g(R)=e_2(R)=4$ by Theorem 3.6, (4). Hence $p_a(R)=g(R)-g_s(R)=1$. The converse is clear. Q. E. D.

Next, we consider the normal genera. Henceforth we assume that R is analytically unramified. Recall that $\bar{g}_\Delta(I)=e(I)+(d-1)\ell(R/\bar{I})-\ell(\bar{I}/\overline{I^2})$ and $\bar{g}_s(I)=\bar{e}_1(I)-e(I)+\ell(R/\bar{I})$. We have $\bar{g}_\Delta(I) \geqq g_\Delta(\bar{I}) \geqq 0$ and $\bar{g}_s(I) \geqq g_s(\bar{I}) \geqq 0$. We put $\bar{\delta}(I)=\min\{n \mid$ there exists a minimal reduction J of $\bar{I}$ such that $J\overline{I^m}=\overline{I^{m+1}}$ for all $m \geqq n\}$ (cf. [6]). The following lemma is analogous to Lemma 3.2. We omit the proof.

Lemma 3.10. *Assume that k is an infinite field and $\bar{I}=I$. Let J be a minimal reduction of I. Then $\bar{g}_\Delta(I)=\ell(\overline{I^2}/IJ)$, and $\bar{g}_\Delta(I)=0$ if and only if $g_\Delta(I)=0$ and $\overline{I^2}=I^2$. (In particular, $\bar{g}_\Delta(I)=0$ if $\bar{\delta}(I) \leqq 1$.)*

Proposition 3.11 (*cf.* [6], Theorem 5.4). *Assume that* $\dim(R)=1$. *Then*:

(1) $\ell(R/\overline{I^n})=e(I)n-\ell(\bar{R}/R)+\ell(I^n\bar{R}/\overline{I^n})$ *for all* $n \geqq 0$, *and*
$\bar{g}(I)=\ell(\bar{R}/R) \geqq \bar{g}_s(I) \geqq \bar{g}_\Delta(I) \geqq 0$.

(2) $\bar{\delta}(I)=0 \Leftrightarrow R$ *is a DVR*,
$\bar{\delta}(I) \leqq 1 \Leftrightarrow \bar{g}_s(I)=0$, *and*
$\bar{\delta}(I) \leqq 2 \Leftrightarrow \bar{g}_s(I)=\bar{g}_\Delta(I)$.

Theorem 3.12. *Assume that* $\dim(R)=2$. *Then* $\bar{g}(I)=\bar{p}_a(I)+\bar{g}_s(I) \geqq \bar{p}_a(I) \geqq -\bar{g}_s(I)$, *and the following conditions are equivalent*:

(1) $\bar{g}_s(I)=0$.

(2) $\bar{g}(I)=0$.

(3) $\mathrm{reg}(\bar{G}(I)) \leqq 1$.

(4) $\bar{n}(I)<0$.

(5) $\bar{g}(I)=\bar{p}_a(I)$ (*resp.* $\bar{g}(I)=\bar{p}_a(I)=0$).

Proof. The assertions (2)$\Leftrightarrow$(3)$\Rightarrow$(5)$\Rightarrow$(1) and (2)$\Rightarrow$(4) follow from [6], Theorem 4.4 and Theorem 6.1. (4)$\Rightarrow$(5) follows from [6], Theorem 1.3, (7). (1)$\Rightarrow$(2): We may assume that $\bar{I}=I$. If $\bar{g}_s(I)=0$, then $\bar{g}_s(I)=g_s(I)=0$ and we get $\bar{e}_1(I)=e_1(I)$, $g(I)=0$ (cf. Theorem 3.3). Hence for all $n \gg 0$, we have $0 \leqq \ell(R/I^n)-\ell(R/\overline{I^n})=g(I)-\bar{g}(I)=-\bar{g}(I) \leqq 0$ (cf. [6], Theorem 3.1). Therefore $\bar{g}(I)=0$.

Q. E. D.

Corollary 3.13. *Assume that R is normal and* $\dim(R)=2$. *Then the following conditions are equivalent*:

(1) *R is pseudo-rational (see* [6] *for the definition).*

(2) $\bar{g}(I)=0$ *for all $\mathfrak{m}$-primary ideal I or R.*

(3) $\bar{g}_s(I)=0$ *for all $\mathfrak{m}$-primary ideal I or R.*

(4) $\bar{g}_\Delta(I)=0$ *for all $\mathfrak{m}$-primary ideal I of R.*

(5) $I\bar{I}=\overline{I^2}$ *for all $\mathfrak{m}$-primary ideal I of R.*

(6) $g_s(I)=0$ *and* Proj $(R(I))$ *is normal for all integrally closed* $\mathfrak{m}$*-primary ideal* I *of* R.

(7) $g_s(I)=0$ *and* $R(I)$ *is normal for all integrally closed* $\mathfrak{m}$*-primary ideal* I *of* R.

PROOF. The equivalence of (1), (2), (3), (7) and the assertions (2)$\Rightarrow$(4)$\Leftrightarrow$(5) follow from Theorem 3.12, Lemma 3.10 and [6]. (5)$\Rightarrow$(6): We have $g_s(I)=0$ by Theorem 3.3 and Lemma 3.10. By the induction, I^{2^n} is integrally closed for all $n\geqq 1$. Hence I^n is integrally closed for all $n\gg 0$. (6)$\Rightarrow$(2): Since $\overline{I^n}=I^n$ for all $n\gg 0$, we get $\bar{g}(I)=g(I)=0$. Q. E. D.

EXAMPLE 3.14. Assume that R is Gorenstein, $\dim(R)=2$, $\bar{g}_s(R)=1$ and $e(R)\geqq 3$. Then $g_s(R)=\bar{g}(R)=g(R)=g_\Delta(R)=1$. In fact, since $e(R)\geqq 3$ and R is Gorenstein, we have $1=\bar{g}_s(R)\geqq g_s(R)\geqq 1$. Hence $g_s(R)=1$ and we get $g_\Delta(R)=g(R)=1$ by Theorem 3.6, (2) and Corollary 3.9, (2).

REMARK. After completing this work, the author learned that C. Huneke [2] had obtained results similar to our Theorem 3.3 and Theorem 3.12 independently.

References

[1] T. Fujita, On the structure of polarized varieties with Δ-genus zero, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 103–115.

[2] C. Huneke, Hilbert functions and symbolic powers, preprint.

[3] D. Kirby, The reduction number of a one-dimensional local ring, J. London Math. Soc. **10** (1975), 471–481.

[4] D. G. Northcott, A note on the coefficients of the abstract Hilbert function, J. London Math. Soc. **35** (1960), 209–214.

[5] A. Ooishi, Castelnuovo's regularity of graded rings and modules, Hiroshima Math. J. **12** (1982), 627–644.

[6] A. Ooishi, Genera and arithmetic genera of commutative rings, Hiroshima Math. J. **17** (1987), 47–66.

[7] J. Sally, Tangent cones at Gorenstein singularities, Compositio Math. **40** (1980), 167–175.

[8] J. Sally, Good embedding dimensions for Gorenstein singularities, Math. Ann. **249** (1980), 95–106.

[9] J. Sally, Cohen-Macaulay local rings of embedding dimension $e+d-2$, J. Algebra **83** (1983), 393–408.

[10] R. P. Stanley, Hilbert functions of graded algebras, Adv. in Math. **28** (1978), 57–83.

[11] R. P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[12] G. Valla, On form rings which are Cohen-Macaulay, J. Algebra **58** (1979), 247–250.

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