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On the differentiability of Riesz potentials of functions

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In the n-dimensional euclidean space R^n, we define the Riesz potential of order α of a nonnegative measurable function f on R^n by

$$R_\alpha f(x) = \int R_\alpha(x-y)f(y)dy,$$

where $R_\alpha(x)=|x|^{\alpha-n}$ if $\alpha<n$ and $R_n(x)=\log(1/|x|)$. It is known (cf. [2]) that if $f\in L^p(R^n)$, $p\geqq 1$, and $|R_\alpha f|\not\equiv\infty$, then $R_\alpha f$ is (m, p)-semi finely differentiable almost everywhere, where m is a positive integer such that $m\leqq\alpha$. In the case $\alpha p>n$, this fact implies that $R_\alpha f$ is totally m times differentiable almost everywhere. A function u is said to be totally m times differentiable at x_0 if there exists a polynomial P for which $\lim_{x\to x_0}|x-x_0|^{-m}[u(x)-P(x)]=0$.

In this note, we are concerned with the case where $\alpha p=n$ and α is a positive integer m, and aim to give a condition on f which assures the total m times differentiability of $R_\alpha f$.

THEOREM. *Let m be a positive integer, $p=n/m>1$ and f be a nonnegative measurable function on R^n such that $R_m f\not\equiv\infty$ and*

$$\int f(y)^p(\log(2+f(y)))^\delta dy < \infty \qquad \text{for some} \quad \delta > p-1.$$

Then $R_m f$ is totally m times differentiable almost everywhere.

The proof of the theorem will be carried out along the same lines as in that of Theorem 3 in [2].

We first prepare the following lemmas.

LEMMA 1. *If m, p and f are as in the Theorem, then*

$$\int_{E(f)} R_m(x-y)f(y)dy \leqq M\left(\int f(y)^p[\log(2+f(y))]^\delta dy\right)^{1/p}$$

for all $x\in R^n$, where $E(f)=\{y; f(y)\geqq 1\}$ and M is a positive constant independent of f and x.

PROOF. We may assume that $x=0$. We set

$$E_j = \{y; 2^{j-1} \leqq f(y) < 2^j\}$$

for $j=1, 2,\dots$. Then we have

$$\int_{E(f)} R_m(y)f(y)dy \leqq \sum_{j=1}^{\infty} 2^j \int_{E_j} R_m(y)dy \leqq \sum_{j=1}^{\infty} 2^j \int_{B(0,r_j)} R_m(y)dy$$
$$= M_1 \sum_{j=1}^{\infty} 2^j |E_j|^{1/p},$$

where $B(0, r_j)$ denotes the open ball with center at 0 and radius r_j such that $|B(0, r_j)|=|E_j|$ ($|E|$ denotes the n-dimensional Lebesgue measure of a set E) and M_1 is a positive constant independent of j. By Hölder's inequality, we obtain

$$\sum_{j=1}^{\infty} 2^j |E_j|^{1/p} \leqq (\sum_{j=1}^{\infty} 2^{pj} j^{\delta} |E_j|)^{1/p} (\sum_{j=1}^{\infty} j^{-\delta/(p-1)})^{1-1/p}$$
$$\leqq M_2 \left(\int f(y)^p [\log(2+f(y))]^{\delta} dy \right)^{1/p}$$

for some positive constant M_2 independent of f. Thus the lemma is proved.

LEMMA 2. *Under the same assumptions as in the Theorem,*

$$\lim_{x\to x_0} |x-x_0|^{-m} \int_{B(x_0, 2|x-x_0|)} R_m(x-y)\{f(y)-f(x_0)\}dy = 0$$

for almost every $x_0 \in R^n$.

PROOF. Set $E(f, x_0)=\{y; |f(y)-f(x_0)| \geqq 1\}$ and $F(f, x_0)=R^n - E(f, x_0)$. From Lemma 1 it follows that

$$\int_{E(f,x_0)\cap B(x_0,r)} R_m(x-y)|f(y)-f(x_0)|dy$$
$$\leqq M\left(\int_{B(x_0,r)} |f(y)-f(x_0)|^p [\log(2+|f(y)-f(x_0)|)]^{\delta} dy \right)^{1/p}$$
$$\leqq M\left(\int_{B(x_0,r)} |f(y)^p [\log(2+f(y))]^{\delta} - f(x_0)^p [\log(2+f(x_0))]^{\delta}| dy \right)^{1/p}$$

for any x and r, where M is the positive constant given in Lemma 1. Since $\int f(y)^p [\log(2+f(y))]^{\delta} dy < \infty$, we have

$$\lim_{x\to x_0} |x-x_0|^{-m} \int_{E(f,x_0)\cap B(x_0, 2|x-x_0|)} R_m(x-y)|f(y)-f(x_0)|dy = 0$$

for almost every $x_0 \in R^n$.

On the other hand, for any $\varepsilon > 0$ we obtain

$$\int_{F(f,x_0)\cap B(x_0,r)} R_m(x-y)|f(y)-f(x_0)|dy$$
$$\leqq \int_{B(x,\varepsilon r)} R_m(x-y)dy + \int_{B(x_0,r)-B(x,\varepsilon r)} R_m(x-y)|f(y)-f(x_0)|dy$$

$$\leqq M'(\varepsilon r)^m + (\varepsilon r)^{m-n}\int_{B(x_0,r)} |f(y)-f(x_0)|dy$$

with a positive constant M' independent of f, x, r and ε. Hence

$$\limsup_{x\to x_0} |x-x_0|^{-m}\int_{F(f,x_0)\cap B(x_0,2|x-x_0|)} R_m(x-y)\,|f(y)-f(x_0)|dy$$

$$\leqq M'\varepsilon^m 2^m + \varepsilon^{m-n} \limsup_{x\to x_0} |x-x_0|^{-n}\int_{B(x_0,2|x-x_0|)} |f(y)-f(x_0)|dy.$$

Since $f\in L^1_{loc}(R^n)$, it follows that

$$\lim_{x\to x_0} |x-x_0|^{-m}\int_{F(f,x_0)\cap B(x_0,2|x-x_0|)} R_m(x-y)\,|f(y)-f(x_0)|dy = 0$$

for almost every $x_0\in R^n$. Thus the lemma is established.

We are now ready to prove the theorem.

PROOF OF THE THEOREM. Let f be as in the theorem. For a multi-index λ with $|\lambda|\leqq m$, define

$$A_\lambda = \lim_{r\to 0}\int_{R^n-B(x_0,r)} [(\partial/\partial x)^\lambda R_m](x_0-y)f(y)dy.$$

If $|\lambda|=m$, then the limit exists and is finite for almost every x_0 as is well-known (cf. [4; Theorem 4 in Chap. II]), and if $|\lambda|<m$, then the limit exists and is finite for x_0 such that $\int |x_0-y|^{m-|\lambda|-n}f(y)dy<\infty$. Thus A_λ exists and is finite for almost every $x_0\in R^n$. In what follows let x_0 be a point such that A_λ exists and is finite for any multi-index λ with $|\lambda|\leqq m$.

On account of Lemma 4 in [2], $\int_{B(0,1)} R_m(x-y)dy$ is infinitely differentiable in $B(0, 1)$. We let $B_\lambda=0$ if $|\lambda|<m$ and $B_\lambda=(\partial/\partial x)^\lambda\int_{B(0,1)} R_m(x-y)dy\Big|_{x=0}$ if $|\lambda|=m$. As in [2; Theorem 3], consider the numbers $C_\lambda=A_\lambda+f(x_0)B_\lambda$ and define

$$P(x) = \sum_{|\lambda|\leqq m} (\lambda!)^{-1}C_\lambda(x-x_0)^\lambda.$$

Letting $K_\ell(x, y)=R_m(x-y)-\sum_{|\lambda|\leqq \ell} (\lambda!)^{-1}(x-x_0)^\lambda[(\partial/\partial x)^\lambda R_m](x_0-y)$, we write

$$|x-x_0|^{-m}\{R_mf(x)-P(x)\}$$

$$= |x-x_0|^{-m}\int_{R^n-B(x_0,1)} K_m(x, y)f(y)dy$$

$$+ |x-x_0|^{-m}\int_{B(x_0,1)-B(x_0,2|x-x_0|)} K_m(x, y)\,[f(y)-f(x_0)]dy$$

$$- |x-x_0|^{-m} \sum_{|\lambda|\leqq m} (\lambda!)^{-1}(x-x_0)^\lambda$$

$$\times \lim_{r\downarrow 0}\int_{B(x_0,2|x-x_0|)-B(x_0,r)} (\partial/\partial x)^{\lambda} R_m(x_0-y)[f(y)-f(x_0)]dy$$

$$+ f(x_0)|x-x_0|^{-m}\left(\int_{B(x_0,1)} K_{m-1}(x, y)dy - \sum_{|\lambda|=m}(\lambda!)^{-1}B_{\lambda}(x-x_0)^{\lambda}\right)$$

$$+ |x-x_0|^{-m}\int_{B(x_0,2|x-x_0|)} R_m(x-y)[f(y)-f(x_0)]dy$$

$$= I_1 + I_2 + I_3 + I_4 + I_5,$$

since $\int_{B(0,r)-B(0,s)} (\partial/\partial x)^{\lambda} R_m(x)dx = 0$ for any r, $s>0$ and any λ with $|\lambda| = m$.

We first note that I_1 tends to zero as $x \to x_0$ since $\int (1+|y|)^{m-n} f(y)dy < \infty$, which follows from the condition that $R_m f \not\equiv \infty$. If $\lim_{r\downarrow 0} r^{-n}\int_{B(x_0,r)} |f(y) - f(x_0)|dy = 0$, then, as in the proof of Theorem 3 in [2], we see that I_2 and I_3 tend to zero as $x \to x_0$. Further, the definition of B_{λ} implies that I_4 tends to zero as $x \to x_0$. Finally, in view of Lemma 2, I_5 tends to zero as $x \to x_0$ for almost every $x_0 \in R^n$. Thus, we infer that $R_m f$ is totally m times differentiable at almost every x_0.

REMARK 1. In the case $p=1$ and $m=n$, if we modify the condition on f in the Theorem, then we obtain the total n times differentiability of $R_n f$. Indeed, if f is a nonnegative measurable function on R^n such that $|R_n f| \not\equiv \infty$ and $\int f(y) \log (2+f(y))dy < \infty$, then $R_n f$ is totally n times differentiable almost everywhere.

Since our definition of differentiability is different from that of [1], we give a sketch of a proof. Instead of Lemmas 1 and 2 we can establish the following results in a way similar to these lemmas, and carry out the proof along the same lines as the proof of Theorem 4 in [3].

LEMMA 1′. *There exists a positive constant M such that*

$$\int_{\{y;f(y)\geqq 1\}} R_n(x-y)f(y)dy \leqq MF \log(1/F)$$

for any nonnegative measurable function f on R^n such that $F \equiv \int f(y) \log(2+f(y))dy < e^{-1}$.

LEMMA 2′. *If f is a nonnegative measurable function on R^n such that $\int f(y) \log(2+f(y))dy < \infty$, then*

$$\lim_{x\to x_0}\int_{B(x_0,2|x-x_0|)} |f(y)-f(x_0)| \log(|x-x_0|/|x-y|)dy = 0$$

for almost every $x_0 \in R^n$.

REMARK 2. We can find a nonnegative measurable function f on R^n such that $\int f(y)^p[\log(2+f(y))]^{p-1}dy<\infty$ and $R_m f$ is not totally m times differentiable at any point of R^n, where m is a positive integer and $p=n/m>1$.

For the construction of such f, take a sequence $\{x_j\}$ which is everywhere dense in R^n. For a sequence $\{r_j\}$ of positive numbers, define

$$f_j(y) = \begin{cases} |y-x_j|^{-m}[\log(1/|y-x_j|)]^{-1}[\log(\log(1/|y-x_j|))]^{-1} & \text{on } B(x_j, r_j) \\ 0 & \text{elsewhere.} \end{cases}$$

If $r_j<e^{-e}$, then it follows that

$$\int f_j(y)^p[\log(2+f_j(y))]^{p-1}dy \leqq M[\log(\log(1/r_j))]^{-p+1}$$

for a positive constant M independent of j and $R_m f_j(x_j)=\infty$. If $\{r_j\}$ is so chosen that $\sum_{j=1}^{\infty} j^{2p-1}[\log(\log(1/r_j))]^{-p+1}<\infty$ and $\max_{k\leqq j} f_k(y)\leqq f_{j+1}(y)$ on $B(x_{j+1}, r_{j+1})$, then $f=\sum_{j=1}^{\infty} f_j$ satisfies the required conditions. In fact, $R_m f(x_j)=\infty$ for each j and

$$\begin{aligned}
&\int(\textstyle\sum_{j=1}^N f_j(y))^p(\log(2+\sum_{j=1}^N f_j(y)))^{p-1}dy \\
&\leqq \textstyle\sum_{k=1}^N \int_{B(x_k,r_k)-\cup_{\ell>k}B(x_\ell,r_\ell)} (\sum_{j=1}^N f_j(y))^p(\log(2+\sum_{j=1}^N f_j(y)))^{p-1}dy \\
&\leqq \textstyle\sum_{k=1}^N \int_{B(x_k,r_k)} (kf_k(y))^p[k\log(2+f_k(y))]^{p-1}dy \\
&\leqq M\textstyle\sum_{k=1}^N k^{2p-1}[\log(\log(1/r_k))]^{-p+1},
\end{aligned}$$

which implies that $\int f(y)^p[\log(2+f(y))]^{p-1}dy<\infty$.

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