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Non-triviality of some products of β-elements in the stable homotopy of spheres

Dedicated to Professor Hirosi Toda on his 60th birthday

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§ 1. Introduction

Throughout this paper, p will denote a prime $\geqq 5$. In the p-component of the stable homotopy group π_*S of spheres, H. Toda and L. Smith introduced a family $\{\beta_s;\ s\geqq 1\}$, and then S. Oka introduced another family $\{\beta_{tp/r};\ t\geqq 1,\ 1\leqq r\leqq p$ and $(t, r)\neq(1, p)\}$ (cf. [2]). The products $\beta_s\beta_{tp/r}$ (by composition) is trivial if $r<p$ by [2] and [5]; and some results for $r=p$ are found in [2], [4], [5]. In this paper, we have the following

THEOREM. $\beta_{rp+1}\beta_{tp/p}\neq 0$ *in* π_*S *if* $p\nmid tu(u+1)$ *for* $u=(r+t)/p^n$.

By this theorem and the results in [2], [4], [5], the products a) $\beta_{rp}\beta_{tp/p}$ with $p|r+t$ and b) $\beta_{rp+1}\beta_{tp/p}$ with $r+t=(up-1)p^n$ are not determined to be trivial or not; and we see that the other product $\beta_s\beta_{tp/p}$ is non-trivial if and only if $p\nmid st$. We note that the product a) is trivial in the E_2-term of the Adams-Novikov spectral sequence for π_*S, by [6; Cor. 2.8].

Furthermore, recall the family $\{\beta_{tp^2/p,2};\ t\geqq 2\}$ in π_*S given by S. Oka (cf. [2]). Then the equality $\beta_s\beta_{tp^2/p,2}=\beta_{s+t(p^2-p)}\beta_{tp/p}$ ([2; Prop. 6.1]) in the E_2-term implies

COROLLARY. $\beta_{rp+1}\beta_{tp^2/p,2}\neq 0$ *in* π_*S *if* $p\nmid tu(u+1)$ *for* $u=(r+tp)/p^n$.

We recall [1] the comodule M_0^2 (see (2.3)) over the Hopf algebroid BP_*BP of the Brown-Peterson spectrum BP at p; and we prepare some BP_*BP-comodules in §2. It is proved in [4; §5] that there exists an element b_{s+tp-1} in $\mathrm{Ext}^2_{BP_*BP}(BP_*, M_0^2)$ whose non-triviality implies that of $\beta_s\beta_{tp/p}$ in π_*S (see Lemma 3.1); and we prove the theorem in §3 by showing $b_{(r+t)p}\neq 0$.

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§ 2. The BP_*BP-comodules $M(n,j)$ and $M(n)$

For a given prime $p\geqq 5$, let BP be the Brown-Peterson spectrum at p, and consider the Hopf algebroid

(2.1) $(A, \Gamma) = (BP_*, BP_*BP) = (Z_{(p)}[v_1, v_2,\cdots], BP_*[t_1, t_2,\cdots])$ with

$$|v_i| = |t_i| = e(i) = (p^i-1)/(p-1), \text{ where } |x| = (\deg x)/q \text{ and } q = 2p-2.$$

Then, for a Γ-comodule M with coaction $\psi_M\colon M \to M \otimes_A \Gamma$, $H^*M = \mathrm{Ext}^*_\Gamma(A, M)$ is the homology of the cobar complex (Ω^*M, d_*) defined by

(2.2) $\Omega^s M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma$ (s copies of Γ) and

$$d_s(m \otimes x) = \psi_M m \otimes x + \textstyle\sum_{i=1}^s (-1)^i m \otimes x_1 \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_s - (-1)^s m \otimes x \otimes 1$$

for $m \in M$, $x_i \in \Gamma$ and $x = x_1 \otimes \cdots \otimes x_s$, where $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$ is the diagonal of Γ. In particular, for the Γ-comodule A with $\psi_A = \eta$, the right unit of Γ, we have

(2.2.1) $d_0 v_1 = pt_1$, $d_0 v_2 \equiv v_1 t_1^p - v_1^p t_1 \bmod (p)$ and $d_1 t_1 = 0$ in $\Omega^* A$.

See [3] for details.

We now consider the following Γ-comodules $M(n, j)$ and $M(n)$ with coactions η induced from the above η for A:

(2.3) $M(n, j) = v_2^{-1}A/(p^n, v_1^j)$ for $p^{n-1} \mid j$, and

$$M(n) = \mathrm{dirlim}_j M(n, j) = v_2^{-1}A/(p^n, v_1^\infty) = \{x/v_1^j \mid x \in v_2^{-1}A,\ j \geqq 1, \text{ and } x/v_1^j = 0 \text{ if } p^n \mid x \text{ or } v_1^j \mid x\}.$$

Then, the Γ-comodules M_i^j $(i+j=2)$ in [1] are given by

(2.3.1) $M_2^0 = M(1, 1)$, $M_1^1 = M(1)$ and $M_0^2 = \mathrm{dirlim}_n M(n) = \{x/v_0^i v_1^j \mid x \in v_2^{-1}A,\ i, j \geqq 1 \text{ and } x/v_0^i v_1^j = 0 \text{ if } p^i \mid x \text{ or } v_1^j \mid x\}$ $(v_0 = p)$.

Furthermore, we have the short exact sequences

$$0 \longrightarrow M(k, l) \xrightarrow{p} M(k+1, l) \longrightarrow M(1, l) \longrightarrow 0,$$

$$0 \longrightarrow M(1, k) \xrightarrow{1/v_1^k} M(1) \xrightarrow{v_1^k} M(1) \longrightarrow 0,$$

$$0 \longrightarrow M(k, l) \xrightarrow{1/v_1^l} M(k) \xrightarrow{v_1^l} M(k) \longrightarrow 0$$

$$\text{and } 0 \longrightarrow M(k) \xrightarrow{1/v_0^k} M_0^2 \xrightarrow{p^k} M_0^2 \longrightarrow 0$$

for $1 \leqq k \leqq n$ and $l = 2p^n$ of the Γ-comodules. These give rise to the long exact sequences

(2.3.2) $$\cdots \longrightarrow H^{*-1}M(1, l) \xrightarrow{\partial_{k,l}} H^*M(k, l) \xrightarrow{p} H^*M(k+1, l) \longrightarrow H^*M(1, l) \longrightarrow \cdots,$$

(2.3.3) $\cdots \longrightarrow H^{*-1}M(1) \xrightarrow{\partial_k} H^*M(1, k) \xrightarrow{1/v_1^k}$

$$H^*M(1) \xrightarrow{v_1^k} H^*M(1) \longrightarrow \cdots,$$

(2.3.4) $\cdots \longrightarrow H^{*-1}M(1) \xrightarrow{\partial_{k,l}} H^*M(k, l) \xrightarrow{1/v_1^l}$

$$H^*M(k) \xrightarrow{v_1^l} H^*M(k) \longrightarrow \cdots, \quad \text{and}$$

(2.3.5) $\cdots \longrightarrow H^{*-1}M_0^2 \xrightarrow{\partial_k} H^*M(k) \xrightarrow{1/v_0^k} H^*M_0^2 \xrightarrow{p^k} H^*M_0^2 \longrightarrow \cdots.$

We now recall the notations of cycles

(2.4.1) ζ of degree 0 for $i = 1$, g_0 of degree q for $i=2$, $\rho = v_2^{-1}t_1^p \otimes g_0$ of degree 0 for $i = 3$ and $\rho \otimes \zeta$ of degree 0 for $i = 4$ in $\Omega^i v_2^{-1}A$.

which represent some bases of the $F_p[v_2, v_2^{-1}]$-vector space $H^*M(1, 1)=H^*M_2^0$ (cf. [3; Ch. 6]). Then, we have the following:

(2.4.2) [1; Lemma 3.19] $d_1\zeta^{p^n} = 0$ in $\Omega^2M(1, p^n)$ and $\zeta^{p^k} = \zeta^{p^n}$ $(k \geqq n)$ in $H^1M(1, p^n)$.

(2.4.3) [4; Prop. 3.7] There exists an element $G_0 \in \Omega^2 v_2^{-1}A$ such that $G_0 = g_0$ in $\Omega^2M(1, 1)$ and $d_1G_0 = v_1\rho$ in $\Omega^3M(1, 2)$.

v_1 acts on $H^*M(1)$ by (2.2.1), and the $F_p[v_1]$-module $H^*M(1)=H^*M_1^1$ is determined by [4; Th. 4.4]. Besides, the F_p-module $H^*M(1, k)$ is determined by the $F_p[v_1]$-module $H^*M(1)$ and (2.3.3). In particular, [4; Th. 4.4] implies immediately the following:

(2.5.1) Each element in $H^2M(1, p^n)$ $(n \geqq 1)$ at degree 0 is 0 in $H^2M(1, a_{n-1})$, where $a_0 = 1$, $a_i = p^i + p^{i-1} - 1$.

(2.5.2) $H^3M(1, l)$ $(l=2p^n)$ at degree $m(p+1)q$ $(m=sp^n,\ p \nmid s(s+1))$ is the F_p-vector space spanned by $\partial_l v_{m,i}$ $(i=0, 1)$, where

(2.5.3) $$v_{m,0} = v_2^m g_0/v_1 \quad \text{and} \quad v_{m,1} = v_2^m t_1 \otimes \zeta/v_1 \quad \text{in} \quad H^2M(1).$$

Noticing that $\zeta \otimes \zeta/v_1 = 0$ in $H^2M(1)$, we see by (2.2.1) that

(2.5.4) $$v_{m,1} \otimes \zeta = 0 \quad \text{in} \quad H^3M(1).$$

Furthermore, we have $\partial_l v_{m,0} = (m+1)v_2^m\rho$ in $H^3M(1, 1)$ by (2.4.3), (2.2.1) and the definition of ∂_l and $v_{m,0}$. Then, $v_2^m\rho \otimes \zeta \neq 0$ in $H^4M(1, 1)$ (by (2.4.1)) implies

(2.5.5) $\partial_l v_{m,0} \otimes \zeta^{p^k} \neq 0$ in $H^4M(1, l)$ $(l=2p^n)$ for $k > n$, if $p \nmid m+1$.

LEMMA 2.6. *There exists a cycle* ζ_n *in* $\Omega^1M(n+1, p^n)$ $(n \geqq 0)$ *such that* $\zeta_n = \zeta^{p^{2n}}$ *in* $\Omega^1M(1, p^n)$, *and so* $\zeta_n = \zeta$ *in* $H^1M(1, 1)$ (*by* (2.4.2)).

PROOF. $\zeta_{n,0} = \zeta^{p^{2n}}$ is a cycle in $\Omega^1M(1, p^{2n})$ by (2.4.2). Assume inductively that there is a cycle $\zeta_{n,j}$ in $\Omega^1M(j+1, p^k)$ $(k=2n-j)$ for $0 \leqq j < n$ such that $\zeta_{n,j} = \zeta_{n,0}$ in $\Omega^1M(1, p^k)$. Then $d_1\zeta_{n,j} = p^{j+1}z$ in $\Omega^2M(j+2, p^k)$ for some $z \in \Omega^2M(1, p^k)$, and z is a cycle because p^{j+1}: $\Omega^*M(1, p^k) \to \Omega^*M(j+2, p^k)$ is a monomorphism of complexes (since $k=2n-j>n>j$). Thus $z = v_1^a x$ $(a=a_{k-1})$ in $H^2M(1, p^k)$ for some cycle x by (2.5.1), and so $v_1^a x - z = d_1\phi$ for some ϕ in $\Omega^1M(1, p^k)$. Hence $d_1\zeta' = p^{j+1}v_1^a x$ in $\Omega^2M(j+2, p^k)$ for $\zeta' = \zeta_{n,j} + p^{j+1}\phi$, and so ζ' is a cycle in $\Omega^1M(j+2, p^{k-1})$. Thus we have $\zeta_{n,j+1} = \zeta'$ satisfying the statement for $j+1$. Now, the lemma holds by setting $\zeta_n = \zeta_{n,n}$. q.e.d.

LEMMA 2.7. *For* $i=0, 1$ *and* $m = sp^n$ *with* $p \nmid s$, *there exists a cycle* $N_{m,i}$ *in* $\Omega^2M(n+1)$ *such that* $N_{m,i} = v_{m,i}$ *in* $\Omega^2M(1)$. *Furthermore,* $2d_1N_{m,1} = mpv_{m,0} \otimes \zeta_{n+1}$ *in* $\Omega^3M(n+2)$.

PROOF. Recall [4; Lemma 4.7] an element $z_m \in \Omega^1M(n+2)$ for $m = sp^n$ with $p \nmid s$, such that

(2.7.1) $z_m = 2v_2^m t_1/v_1$ in $\Omega^1M(1)$ and $d_1z_m = mp(v_{m,0} - v_{m,1})$ in $\Omega^2M(n+2)$.

The last equality gives us an element w in $\Omega^2M(n+1)$ such that $d_1z_m = mpw$ in $\Omega^2M(2n+2)$ and $w = v_{m,0} - v_{m,1}$ in $\Omega^2M(1)$. Since p^{n+1}: $\Omega^*M(n+1) \to \Omega^*M(2n+2)$ is monomorphic, we see that w is a cycle. Thus the lemma holds for $N_{m,0} = w + N_{m,1}$ and $N_{m,1} = 2^{-1}z_m \otimes \zeta_{n+1}$ by the first equality in (2.7.1) and Lemma 2.6. In fact, the last equality in the lemma follows from (2.5.4), Lemma 2.6 and the last equality of (2.7.1). q.e.d.

§3. Proof of Theorem

We recall the following

LEMMA 3.1 [4; §5]. $\beta_s\beta_{tp/p} \neq 0$ *in* π_*S *if* $tb_{s+tp-1} \neq 0$ *in* $H^2M_0^2$, *where* $b_m = v_2^m t_1 \otimes \zeta/v_0v_1 = N_{m,1}/v_0 \in H^2M_0^2$.

In fact, the last equality follows from Lemma 2.7.

In view of this lemma, the theorem in §1 follows immediately from the following

PROPOSITION 3.2. $b_m \neq 0$ *in* $H^2M_0^2$ *if* $p \nmid s(s+1)$ *for* $s = m/p^n$.

REMARK. $b_m = 0$ in $H^2M_0^2$ for $m = (rp^i - 1)p^n$ with $1 \leqq i \leqq n+2$ by [1; Prop. 6.9].

Now, we prove this proposition.

LEMMA 3.3. *For* $1\leqq k\leqq n$ *and* $l=2p^n$, *the map* p: $H^4M(k, l)\to H^4M(k+1, l)$ *in* (2.3.2) *is monomorphic at degree* $m(p+1)q$ *for* $m=sp^n$ *with* $p\nmid s(s+1)$.

PROOF. Consider the Γ-comodule $B=v_2^{-1}A/(p^{n+1})$, the short exact sequence $0\longrightarrow B\overset{\lambda}{\underset{\subset}{\longrightarrow}}v_1^{-1}B\overset{\mu}{\longrightarrow}M(n+1)\longrightarrow 0$ and the projection p_r: $B\to M(r, l)$ $(1\leqq r\leqq n+1)$. Then we have a cycle $c_i=\lambda^{-1}d_2\mu^{-1}N_{m,i}$ $(i=0, 1)$ in Ω^3B for the cycle $N_{m,i}$ in Lemma 2.7, and $p_1c_i=\partial_l v_{m,i}$ for ∂_l: $H^2M(1)\to H^3M(1, l)$ by definition since $N_{m,i}=v_{m,i}$ in $\Omega^2M(1)$ by Lemma 2.7. Thus, for $\partial_{k,l}$: $H^3M(1, l)\to H^4M(k, l)$ in (2.3.2),

$$\partial_{k,l}(\partial_l v_{m,i}) = \partial_{k,l}(p_1c_i) = p^{-1}d_3(p_{k+1}c_i) = p^{-1}p_{k+1}(d_3c_i) = 0.$$

Hence, $\partial_{k,l}=0$ at degree $m(p+1)q$ by (2.5.2), which shows the lemma by (2.3.2). q. e. d.

PROPOSITION 3.4. $p^nv_{m,0}\otimes\zeta_{n+1}\neq 0$ *in* $H^3M(n+1)$ *for* m *in Lemma* 3.3.

PROOF. Note that $p^nv_{m,0}\otimes\zeta_{n+1}=p^nv_{m,0}\otimes\zeta^k$ $(k=p^{2n+2})$ in $H^3M(n+1)$ and ζ^k is a cycle in $\Omega^1M(1, k)$ by (2.4.2). Then, for $\delta_{n+1,l}$ in (2.3.4) with $l=2p^n$, $\delta_{n+1,l}(p^nv_{m,0}\otimes\zeta^k)=p^n\partial_l v_{m,0}\otimes\zeta^k$ holds by the definition of δ and ∂. On the other hand, $p^n\partial_l v_{m,0}\otimes\zeta^k\neq 0$ in $H^4M(n+1, l)$ by (2.5.5) and Lemma 3.3. These show the proposition. q. e. d.

PROOF OF PROPOSITION 3.2. For δ_{n+1} in (2.3.5) and b_m in Lemma 3.1, $2\delta_{n+1}b_m=2d_1N_{m,1}/p=mv_{m,0}\otimes\zeta_{n+1}$ in $H^3M(n+1)$ by Lemma 2.7. Thus, $b_m\neq 0$ by the above proposition. q. e. d.

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