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Minimal conditions for Lie algebras and finiteness of their dimensions

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Introduction

We shall be concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ unless otherwise specified, and mostly follow [4] for the use of notations and terminology. The classes Min-$\lhd^{\alpha}$ (α is an ordinal), Min-si, Min-asc, Min-ser, Min and $\mathfrak{F}$ are related by the series of inclusions

$$\text{Min-}\lhd \geq \text{Min-}\lhd^2 \geq \text{Min-}\lhd^3 \geq \cdots \geq \text{Min-si} \geq \text{Min-}\lhd^{\omega} \geq \text{Min-}\lhd^{\omega+1}$$
$$\geq \cdots \geq \text{Min-asc} \geq \text{Min-ser} \geq \text{Min} \geq \mathfrak{F},$$

where Min-ser denotes the class of Lie algebras satisfying the minimal condition for serial subalgebras. Concerning these inclusions, Amayo and Stewart have proved that if $\mathfrak{k}$ is of characteristic zero, then Min-$\lhd >$ Min-$\lhd^2 =$ Min-si (cf. [4, Theorem 8.1.4]), and that if $\mathfrak{k}$ is of characteristic $p>0$, then Min-$\lhd^2 >$ Min-$\lhd^3 =$ Min-si (cf. [4, Proposition 8.1.5 and the example in §8.3]). Furthermore, Stewart has proved that Min-si = Min-asc ([13, Theorem]), and that if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L}\mathfrak{F} \cap \mathrm{Min} = \mathfrak{F}$ (cf. [4, Corollary 10.2.2]). The first purpose of this paper is to investigate the relationship among the classes Min-si, Min-ser and Min. The second one is to present sufficient conditions for Lie algebras satisfying minimal conditions to be finite-dimensional.

In Section 1 we shall first prove that if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L}\mathfrak{F} \cap$ Min-$\lhd^2 = \mathrm{L}\mathfrak{F} \cap$ Min-ser (Corollary 1.6), and secondly prove that $\mathrm{L}\mathfrak{F} \cap$ Min-ser $>$ $\mathrm{L}\mathfrak{F} \cap$ Min and so Min-ser $>$ Min (Theorem 1.7). In consequence of these results, we shall conclude that if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L}\mathfrak{F} \cap$ Min-$\lhd > \mathrm{L}\mathfrak{F} \cap$ Min-$\lhd^2 = \mathrm{L}\mathfrak{F} \cap$ Min-ser $> \mathrm{L}\mathfrak{F} \cap$ Min $= \mathfrak{F}$.

In Section 2 we shall prove that $\{\mathrm{J(asc)}, \hat{\mathrm{E}}\mathrm{(asc)}\}\mathfrak{I} \cap$ Min-si $= \{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{I}^* \cap$ Min-ser $= \mathfrak{F}$, where $\mathfrak{I}$ (resp. $\mathfrak{I}^*$) denotes the class of Lie algebras having no infinite-dimensional, simple (resp. absolutely simple) factors of ideals (Theorem 2.5). Especially, if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L(ser)}\mathfrak{F} \leq \mathfrak{I}$ and so $\mathrm{L(ser)}\mathfrak{F} \cap$ Min-$\lhd^2 = \mathfrak{F}$ (Corollary 2.6).

In Section 3 we shall present classes of generalized soluble Lie algebras for which minimal conditions imply finiteness of their dimensions. For example, we shall prove that if $\mathfrak{X}$ is an $\{\mathrm{I}, \mathrm{Q}\}$-closed class of Lie algebras such that every simple (resp. absolutely simple) $\mathfrak{X}$-algebra is abelian, then $\{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathfrak{X} \cap$ Min-si $\leq \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ (resp. $\{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{X} \cap$ Min-ser $\leq \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$) (Theorem 3.2). Moreover, we shall prove that if $\mathfrak{X}$ is a class of Lie algebras L such that for any minimal ideal I of L, $I \in \mathfrak{A}$ and $L/I \in \mathfrak{X}$, then $\{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathfrak{X} \cap$ Min-si $\leq \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ and $\{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{X} \cap$ Min-ser $\leq \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ (Theorem 3.6).

1.

In this section we shall mainly investigate minimal conditions for locally finite Lie algebras.

Let L be a Lie algebra over any field $\mathfrak{k}$. L is said to be simple if L has no non-trivial ideals. We use $\mathfrak{S}$ to denote the class of simple Lie algebras. In [2, Theorem 3.8] Amayo has proved that if $L \in \mathfrak{S}$ then L has no non-trivial ascendant subalgebras (see also Levič [10]). As in group theory we say that L is absolutely simple if L has no non-trivial serial subalgebras. We use $\mathfrak{S}^*$ to denote the class of absolutely simple Lie algebras. Then clearly $\mathfrak{S}^* \leq \mathfrak{S}$. It is still open whether this inequality is strict. However, it can be easily deduced from the following lemma that $\mathfrak{S}^* = \mathfrak{S}$ provided Min-ser = Min-si.

LEMMA 1.1. *If $L \in$ Min-ser, then every serial subalgebra of L is a subideal of L. In particular, $\mathfrak{S} \cap$ Min-ser $= \mathfrak{S}^*$.*

PROOF. Let $L \in$ Min-ser and H ser L. There are a totally ordered set Σ and a series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ from H to L of type Σ. We may assume that $V_\sigma < \Lambda_\sigma$ for all $\sigma \in \Sigma$. Then $\Lambda_\sigma < \Lambda_\tau$ iff $\sigma < \tau$. Since Λ_σ ser L for all $\sigma \in \Sigma$, Σ must be well-ordered. It follows that H asc L. Owing to [13, Theorem], we have H si L.

By making use of [3, Lemma 4.6 and Theorem 4.7], [4, Theorem 8.2.3] and Lemma 1.1, we have the following

LEMMA 1.2. *Let $L \in$ Min-ser (resp. Min-si). Then every chief factor of L is either absolutely simple (resp. simple) or finite-dimensional. Furthermore, L has an ascending chief series whose factors are either absolutely simple (resp. simple) or finite-dimensional.*

COROLLARY 1.3. *Over any field $\mathfrak{k}$,* Min-ser $\leq \acute{\mathrm{E}}(\lhd)(\mathfrak{S}^* \cup \mathfrak{F})$.

It seems to be a very hard problem whether Min-si implies Min-ser. But by restricting our attention to locally finite Lie algebras satisfying Min-si, we can present an interesting condition for those to satisfy Min-ser in the following theorem:

THEOREM 1.4. *Let $\mathfrak{X}$ be an $\{\mathrm{I}, \mathrm{Q}\}$-closed class of Lie algebras over any* *field* $\mathfrak{k}$ *such that* $\mathfrak{X} \cap \mathrm{L}\mathfrak{F} \cap \mathfrak{S} \leq \mathfrak{S}^*$. *If* $L \in \mathfrak{X} \cap \mathrm{L}\mathfrak{F} \cap$ Min-si, *then every serial subalgebra of* L *is a subideal of* L. *In particular*, $\mathfrak{X} \cap \mathrm{L}\mathfrak{F} \cap$ Min-si $\leq$ Min-ser.

PROOF. Let $L \in \mathfrak{X} \cap \mathrm{L}\mathfrak{F} \cap$ Min-si and $H \operatorname{ser} L$. Then by [4, Proposition 13.2.4] we see that $(H+I)/I \operatorname{ser} L/I$ for any ideal I of L. Let K be a subideal of L minimal with respect to $H \leq K$. We shall prove that $H = K$. Without loss of generality, we may assume that $K = L$ and $H_L = 0$, where H_L denotes the largest ideal of L contained in H. Let F be a subideal of L minimal with respect to F of finite codimension in L. By [4, Lemma 8.2.1] we have $F \lhd L$. Then $(H+F)/F$ $\operatorname{ser} L/F$. It follows that $H + F \operatorname{ser} L$. Since $H+F$ is of finite codimension in L, we can easily see that $H+F \operatorname{si} L$. By the assumption of L we have $L = H+F$. Let $Z = \zeta_1(F)$. In order to prove that $H = L$, it is sufficient to show that $Z = F$. In fact, if $Z = F$ then $F \in \mathfrak{A} \cap$ Min-si $= \mathfrak{A} \cap \mathfrak{F}$. Hence by the minimality of F we have $F = 0$, so that $L = H + F = H$.

Now we assume that $Z < F$. Then L/Z has a minimal ideal M/Z contained in F/Z, where $Z < M \leq F$ and $M \lhd L$. By Lemma 1.2 $M/Z \in \mathfrak{S} \cup \mathfrak{F}$. First we show that $M/Z \in \mathfrak{S} \setminus \mathfrak{F}$. Suppose, to the contrary, that $M/Z \in \mathfrak{F}$. Under the adjoint action of F we regard M as an F-module. Then M/Z is a finite-dimensional F-module. It follows that $F/C_F(M/Z) \in \mathfrak{F}$. By the minimality of F we have $F = C_F(M/Z)$. By [4, Lemma 8.1.3] $M \leq \zeta_2(F) \leq \rho(L) \in \mathfrak{F}$. It follows from [4, Corollary 1.4.3] that $F/C_F(M) \in \mathfrak{F}$. By the minimality of F we have $F = C_F(M)$, so that $M \leq Z$. This is a contradiction. Therefore we have $M/Z \in$ $\mathfrak{S} \setminus \mathfrak{F}$. It follows that $M/Z \in \mathfrak{X} \cap \mathrm{L}\mathfrak{F} \cap \mathfrak{S} \leq \mathfrak{S}^*$.

Next we show that $H \cap M = 0$. It is clear that $[H \cap Z, L] = [H \cap Z, H] \subseteq$ $H \cap Z$. Hence $H \cap Z \lhd L$ and so $H \cap Z = 0$ since $H_L = 0$. Since $H \cap M \operatorname{ser} M \in \mathrm{L}\mathfrak{F}$, by [4, Proposition 13.2.4] we have $((H \cap M) + Z)/Z \operatorname{ser} M/Z$. It follows that either $H \cap M \leq Z$ or $M = (H \cap M) + Z$. In the first case $H \cap M = H \cap Z = 0$. Next we consider the second case. Then we have $H \cap M \cong M/Z \in S^* \setminus \mathfrak{F}$. Clearly $H \cap M$ ser L. Let R be the intersection of all the ideals I of $H \cap M$ such that $(H \cap M)/I$ is locally nilpotent. Owing to [14, Theorem 5], we have $R \lhd L$. Since $H \cap M \in \mathfrak{S}^* \setminus \mathfrak{F}$, by using [4, Lemma 8.1.3] we have $R = H \cap M$. Therefore $H \cap M = 0$ since $H_L = 0$.

Now since $Z \lhd H+M$ and $H \operatorname{ser} H+M \in \mathrm{L}\mathfrak{F}$, we have $H+Z$ ser $H+M$. There exists a series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ from $H+Z$ to $H+M$. Then $\{\Lambda_\sigma \cap M,$ $V_\sigma \cap M : \sigma \in \Sigma\}$ is a series from Z to M. Since $M/Z \neq 0$, we can find a $\sigma \in \Sigma$ such that $V_\sigma \cap M < \Lambda_\sigma \cap M$. By [4, Proposition 13.2.4] $(V_\sigma \cap M)/Z$ and $(\Lambda_\sigma \cap M)/Z$ are serial subalgebras of M/Z. It follows that $V_\sigma \cap M = Z$ and $\Lambda_\sigma \cap M = M$. Then by the modular law, we have $H + Z = V_\sigma \lhd \Lambda_\sigma = H + M$. Hence $[H, M] \subseteq (H+Z) \cap$ $M = (H \cap M) + Z = Z$. Regard M as an L-module under the adjoint action of L. Then M/Z is an L-module and $H \leq C_L(M/Z) \lhd L$. By the assumption of L we have $C_L(M/Z) = L$. Hence $M/Z \in \mathfrak{S}^* \cap \mathfrak{A} = \mathfrak{F}_1$. This is the final contradiction.

Therefore we have $Z=F$.

COROLLARY 1.5. *Over any field* $\mathfrak{k}$, $\mathrm{L}\mathfrak{F} \cap \mathfrak{S} = \mathrm{L}\mathfrak{F} \cap \mathfrak{S}^*$ *if and only if* $\mathrm{L}\mathfrak{F} \cap$ Min-si $= \mathrm{L}\mathfrak{F} \cap$ Min-ser.

PROOF. If $\mathrm{L}\mathfrak{F} \cap \mathfrak{S} = \mathrm{L}\mathfrak{F} \cap \mathfrak{S}^*$, then since the class $\mathfrak{D}$ of all Lie algebras is an $\{\mathrm{I}, \mathrm{Q}\}$-closed class satisfying $\mathfrak{D} \cap \mathrm{L}\mathfrak{F} \cap \mathfrak{S} \leq \mathfrak{S}^*$, by Theorem 1.4 we have $\mathrm{L}\mathfrak{F} \cap$ Min-si $= \mathrm{L}\mathfrak{F} \cap$ Min-ser. Conversely, if $\mathrm{L}\mathfrak{F} \cap$ Min-si $= \mathrm{L}\mathfrak{F} \cap$ Min-ser, then by Lemma 1.1 we have

$$\mathrm{L}\mathfrak{F} \cap \mathfrak{S} = \mathrm{L}\mathfrak{F} \cap \text{Min-si} \cap \mathfrak{S} = \mathrm{L}\mathfrak{F} \cap \text{Min-ser} \cap \mathfrak{S} = \mathrm{L}\mathfrak{F} \cap \mathfrak{S}^*.$$

It is not known whether over a field of positive characteristic $\mathrm{L}\mathfrak{F} \cap \mathfrak{S}$ coincides with $\mathrm{L}\mathfrak{F} \cap \mathfrak{S}^*$. However, Stewart has proved in [14, Theorem 8] that $\mathrm{L}\mathfrak{F} \cap \mathfrak{S} = \mathrm{L}\mathfrak{F} \cap \mathfrak{S}^*$ over any field of characteristic zero. By making use of this result, [4, Theorem 8.1.4] and Corollary 1.5, we obtain

COROLLARY 1.6. *Over any field* $\mathfrak{k}$ *of characteristic zero,*

$$\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^2 = \mathrm{L}\mathfrak{F} \cap \text{Min-ser}.$$

Concerning the other minimal conditions for locally finite Lie algebras, we can prove the following

THEOREM 1.7. *Over any field* $\mathfrak{k}$,

(1) $\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd > \mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^2$;

(2) $\mathrm{L}\mathfrak{F} \cap \text{Min-ser} > \mathrm{L}\mathfrak{F} \cap \text{Min}$. *In particular*, Min-ser $>$ Min.

PROOF. (1) We here consider a well-known example. Let A be a vector space over $\mathfrak{k}$ with basis $\{a_0, a_1, \ldots\}$ and think of A as an abelian Lie algebra. Let x be the downward shift on A, that is, x be the derivation of A defined by $a_0 x = 0$ and $a_{i+1} x = a_i$ $(i \geq 0)$. Form the split extension $L = A \dotplus \langle x \rangle$ of A by $\langle x \rangle$. Then it is well known (cf. [4, p. 119]) that $\zeta_{n+1}(L) = \langle a_0, \ldots, a_n \rangle$ $(n < \omega)$, $\zeta_\omega(L) = A$ and $\zeta_{\omega+1}(L) = L$. Hence $L \in \mathfrak{Z} \leq \mathrm{L}\mathfrak{F}$. Moreover, $\{I : I \lhd L\} = \{\zeta_\alpha(L) : \alpha \leq \omega+1\}$ and so $L \in \text{Min-}\lhd$. But $L \notin \text{Min-}\lhd^2$ since $A \in \mathfrak{A} \setminus \mathfrak{F}$.

(2) For each integer $n \geq 2$, set $S_n = \mathfrak{sl}(n, \mathfrak{k})$. We regard each matrix in S_n as a matrix in S_{n+1} with the $(n+1)$-th row and column consisting of 0, so that $S_n \leq S_{n+1}$ $(n \geq 2)$. Form the direct limit $L = \text{dir lim } S_n$, which may be considered as the Lie algebra L over $\mathfrak{k}$ satisfying $L = \bigcup_{n \geq 2} S_n$. Then it has been shown in [15, Example 3] that $L \in \mathrm{L}\mathfrak{F} \cap \mathfrak{S}$. We can further show that $L \in \mathfrak{S}^* \setminus \text{Min}$. In fact, let H ser L and $H \neq 0$. There is an integer $m \geq 2$ such that $H \cap S_n \neq 0$ for any integer $n \geq m$. To each integer $n \geq m$, there corresponds an integer $k(n) \geq n$ such that if $\mathfrak{k}$ is of characteristic $p > 0$ then $k(n)$ is not devided by p. Since $H \cap S_{k(n)}$ ser $S_{k(n)} \in \mathfrak{S} \cap \mathfrak{F}$, we have $S_n \leq S_{k(n)} = H \cap S_{k(n)} \leq H$ $(n \geq m)$. It follows that

$H=L$. Hence we have $L\in\mathfrak{S}^*$. For each integer $i\geq 2$, let A_i be the matrix in S_i with the (i, i)-component 1, the (1, 1)-component -1 and all the other components 0. Then it is easy to see that $\langle A_i\colon i\geq 2\rangle$ is an infinite-dimensional abelian subalgebra of L. Thus we have $L\notin$ Min.

REMARK. Stewart has proved that over any field $\mathfrak{k}$ of characteristic zero, $\mathrm{L}\mathfrak{F}\cap\mathrm{Min}=\mathfrak{F}$ (cf. [4, Corollary 10.2.2]). By making use of this result, Corollary 1.6 and Theorem 1.7, we can conclude that over any field $\mathfrak{k}$ of characteristic zero,

$$\begin{aligned}\mathrm{L}\mathfrak{F}\cap\text{Min-}\lhd > \mathrm{L}\mathfrak{F}\cap\text{Min-}\lhd^2 &= \mathrm{L}\mathfrak{F}\cap\text{Min-si} = \mathrm{L}\mathfrak{F}\cap\text{Min-asc}\\ &= \mathrm{L}\mathfrak{F}\cap\text{Min-ser} > \mathrm{L}\mathfrak{F}\cap\mathrm{Min} = \mathfrak{F}.\end{aligned}$$

2.

As a direct consequence of Theorem 1.7 (2), we see that there exists an infinite-dimensional, locally finite Lie algebra satisfying the minimal condition for serial subalgebras. So it seems to be interesting to consider Lie algebras for which minimal conditions imply finiteness of their dimensions. In this section we shall present relatively large classes $\mathfrak{X}$ (resp. $\mathfrak{Y}$) of Lie algebras such that $\mathfrak{X}\cap\text{Min-si}=\mathfrak{F}$ (resp. $\mathfrak{Y}\cap\text{Min-ser}=\mathfrak{F}$).

We begin with

LEMMA 2.1. $\acute{\mathrm{E}}(\lhd^2)\mathfrak{F}\cap\text{Min-}\lhd^2=\acute{\mathrm{E}}(\mathrm{si})\mathfrak{F}\cap\text{Min-si}=\mathfrak{F}$.

PROOF. Let Δ be either $\lhd^2$ or si. Assume that there exists an infinite-dimensional Lie algebra L belonging to the class $\acute{\mathrm{E}}(\Delta)\mathfrak{F}\cap\text{Min-}\Delta$. Then there are an infinite ordinal σ and a strictly ascending $\mathfrak{F}$-series $\{L_\alpha\colon\alpha\leq\sigma\}$ of L such that $L_\alpha\Delta L$ for any $\alpha\leq\sigma$. Set $\rho_{\mathfrak{F}}(L)=\langle H\colon H\in\mathfrak{F}\text{ and }H\lhd L\rangle$. Since $L_n\in\mathfrak{F}$ $(n<\omega)$, by using [4, Theorem 9.3.2(b)] we have $L_\omega=\bigcup_{n<\omega}L_n\leq\rho_{\mathfrak{F}}(L)\in\mathfrak{F}$. It follows that $L_n=L_\omega$ for some $n<\omega$. This is a contradiction.

REMARK. It is still an open problem whether the class Min coincides with the class $\mathfrak{F}$. However, in order to give the affirmative answer to this problem, it is sufficient to prove that $\mathfrak{S}^*\cap\mathrm{Min}\leq\mathfrak{F}$. In fact, if it is proved, then by using Corollary 1.3 and Lemma 2.1 we have

$$\mathrm{Min}\leq\acute{\mathrm{E}}(\lhd)((\mathfrak{S}^*\cap\mathrm{Min})\cup\mathfrak{F})\cap\mathrm{Min}\leq\acute{\mathrm{E}}(\lhd)\mathfrak{F}\cap\mathrm{Min}=\mathfrak{F}.$$

Let Δ be one of the relations $\lhd^\alpha$ (α is an ordinal), si, asc and ser. Let $\mathfrak{X}$ be a class of Lie algebras. We use $\hat{\mathrm{E}}(\Delta)\mathfrak{X}$ to denote the class of Lie algebras L having an $\mathfrak{X}$-series $\{\Lambda_\sigma, V_\sigma\colon\sigma\in\Sigma\}$ such that $\Lambda_\sigma\Delta L$ and $V_\sigma\Delta L$ for all $\sigma\in\Sigma$. We use $\mathrm{J}(\Delta)\mathfrak{X}$ to denote the class of Lie algebras L such that $L=\langle H\colon H\in\mathfrak{X}\text{ and }H\Delta L\rangle$. Moreover, we use $\mathrm{L}(\Delta)\mathfrak{X}$ to denote the class of Lie algebras L such that for any finite subset X of L there is an $\mathfrak{X}$-subalgebra H of L satisfying $X \subseteq H \Delta L$. Evidently $\mathrm{L}(\Delta)\mathfrak{X} \leq \mathrm{J}(\Delta)\mathfrak{X}$. It is not hard to see that $\hat{\mathrm{E}}$(si), $\hat{\mathrm{E}}$(asc), $\hat{\mathrm{E}}(\mathrm{ser})=\hat{\mathrm{E}}$, J(si), J(asc), J(ser), L(si), L(asc) and L(ser) are closure operations. In [4, p. 258] J(si) and J(asc) are respectively denoted by N and Ń.

LEMMA 2.2. (1) *If* $\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}$, *then* $\hat{\mathrm{E}}(\mathrm{asc})\mathfrak{X} \cap \text{Min-si} = \acute{\mathrm{E}}(\mathrm{si})\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}$ *and* $\mathrm{J}(\mathrm{asc})\mathfrak{X} \cap \text{Min-si} = \mathrm{J}(\mathrm{si})\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}$.

(2) *If* $\mathfrak{X} \cap \text{Min-ser} \leq \mathfrak{F}$, *then* $\hat{\mathrm{E}}\mathfrak{X} \cap \text{Min-ser} = \acute{\mathrm{E}}(\mathrm{si})\mathfrak{X} \cap \text{Min-ser} \leq \mathfrak{F}$ *and* $\mathrm{J}(\mathrm{ser})\mathfrak{X} \cap \text{Min-ser} = \mathrm{J}(\mathrm{si})\mathfrak{X} \cap \text{Min-ser} \leq \mathfrak{F}$.

PROOF. (1) Let $L \in \hat{\mathrm{E}}(\mathrm{asc})\mathfrak{X} \cap \text{Min-si}$. By [13, Theorem] L has an $\mathfrak{X}$-series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ consisting of subideals. We may assume that $V_\sigma < \Lambda_\sigma$ for all $\sigma \in \Sigma$. Since $\Lambda_\sigma < \Lambda_\tau$ iff $\sigma < \tau$, Σ must be a well-ordered set. Hence $L \in \acute{\mathrm{E}}(\mathrm{si})\mathfrak{X}$. Using Lemma 2.1 we have

$$\begin{aligned} &\hat{\mathrm{E}}(\mathrm{asc})\mathfrak{X} \cap \text{Min-si} = \acute{\mathrm{E}}(\mathrm{si})\mathfrak{X} \cap \text{Min-si} \\ &= \acute{\mathrm{E}}(\mathrm{si})(\mathfrak{X} \cap \text{Min-si}) \cap \text{Min-si} \leq \acute{\mathrm{E}}(\mathrm{si})\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}. \end{aligned}$$

Owing to [4, Theorem 9.3.2 (b)], we have $\mathrm{J}(\mathrm{si})\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}$. Therefore by [13, Theorem] we obtain

$$\begin{aligned} &\mathrm{J}(\mathrm{asc})\mathfrak{X} \cap \text{Min-si} = \mathrm{J}(\mathrm{si})\mathfrak{X} \cap \text{Min-si} \\ &= \mathrm{J}(\mathrm{si})(\mathfrak{X} \cap \text{Min-si}) \cap \text{Min-si} \leq \mathrm{J}(\mathrm{si})\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}. \end{aligned}$$

(2) By using Lemma 1.1, as in the proof of (1), we have

$$\begin{aligned} &\hat{\mathrm{E}}\mathfrak{X} \cap \text{Min-ser} = \acute{\mathrm{E}}(\mathrm{si})\mathfrak{X} \cap \text{Min-ser} \\ &= \acute{\mathrm{E}}(\mathrm{si})(\mathfrak{X} \cap \text{Min-ser}) \cap \text{Min-ser} \leq \acute{\mathrm{E}}(\mathrm{si})\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}, \\ &\mathrm{J}(\mathrm{ser})\mathfrak{X} \cap \text{Min-ser} = \mathrm{J}(\mathrm{si})\mathfrak{X} \cap \text{Min-ser} \\ &= \mathrm{J}(\mathrm{si})(\mathfrak{X} \cap \text{Min-ser}) \cap \text{Min-ser} \leq \mathrm{J}(\mathrm{si})\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}. \end{aligned}$$

Let $\mathfrak{X}$ be a class of Lie algebras and (A, B) be one of the following pairs of closure operations:

$$(\mathrm{J}(\mathrm{si}), \acute{\mathrm{E}}(\mathrm{si})), \quad (\mathrm{J}(\mathrm{asc}), \hat{\mathrm{E}}(\mathrm{asc})), \quad (\mathrm{J}(\mathrm{ser}), \hat{\mathrm{E}}).$$

We recall ([4, p. 20]) that {A, B} is defined to be the closure operation such that $\{\mathrm{A}, \mathrm{B}\}\mathfrak{X}$ is the smallest class containing $\mathfrak{X}$ which is A-closed and B-closed. For any ordinal α, we inductively define the class $(\mathrm{AB})^\alpha\mathfrak{X}$ as follows: $(\mathrm{AB})^0\mathfrak{X} = \mathfrak{X}$; $(\mathrm{AB})^{\alpha+1}\mathfrak{X} = \mathrm{AB}((\mathrm{AB})^\alpha\mathfrak{X})$ for each ordinal α; $(\mathrm{AB})^\lambda\mathfrak{X} = \bigcup_{\alpha<\lambda}(\mathrm{AB})^\alpha\mathfrak{X}$ for each limit ordinal λ. Furthermore, we denote by $(\mathrm{AB})^*\mathfrak{X}$ the class of Lie algebras L such that $L \in (\mathrm{AB})^\alpha\mathfrak{X}$ for some ordinal α. Then it is not hard to verify that $(\mathrm{AB})^*\mathfrak{X}$ is an

$\{\mathrm{A}, \mathrm{B}\}$-closed class such that $\mathfrak{X} \leq (\mathrm{AB})^*\mathfrak{X} \leq \{\mathrm{A}, \mathrm{B}\}\mathfrak{X}$. Thus we have $\{\mathrm{A}, \mathrm{B}\}\mathfrak{X} = (\mathrm{AB})^*\mathfrak{X}$.

PROPOSITION 2.3. (1) *If* $\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}$, *then*

$$\{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathfrak{X} \cap \text{Min-si} = \{\mathrm{J(si)}, \acute{\mathrm{E}}(\mathrm{si})\}\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}.$$

(2) *If* $\mathfrak{X} \cap \text{Min-ser} \leq \mathfrak{F}$, *then*

$$\{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{X} \cap \text{Min-ser} = \{\mathrm{J(si)}, \acute{\mathrm{E}}(\mathrm{si})\}\mathfrak{X} \cap \text{Min-ser} \leq \mathfrak{F}.$$

PROOF. We here prove (1) only, since (2) can be proved similarly. Let α be an ordinal and assume that

$$(\mathrm{J(si)}\acute{\mathrm{E}}(\mathrm{si}))^{\alpha}\mathfrak{X} \cap \text{Min-si} = (\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha}\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}.$$

Then by using Lemma 2.2 (1), we have

$$\begin{aligned}
&(\mathrm{J(si)}\acute{\mathrm{E}}(\mathrm{si}))^{\alpha+1}\mathfrak{X} \cap \text{Min-si} \leq (\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha+1}\mathfrak{X} \cap \text{Min-si} \\
&= (\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha+1}\mathfrak{X} \cap \mathfrak{F} = \mathrm{J(si)E}((\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha}\mathfrak{X}) \cap \mathfrak{F} \\
&= \mathrm{J(si)E}((\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha}\mathfrak{X} \cap \mathfrak{F}) \cap \mathfrak{F} \leq (\mathrm{J(si)}\acute{\mathrm{E}}(\mathrm{si}))^{\alpha+1}\mathfrak{X} \cap \mathfrak{F}.
\end{aligned}$$

It follows that $(\mathrm{J(si)}\acute{\mathrm{E}}(\mathrm{si}))^{\alpha+1}\mathfrak{X} \cap \text{Min-si} = (\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha+1}\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}$. Therefore by transfinite induction on α we can easily see that for any ordinal α,

$$(\mathrm{J(si)}\acute{\mathrm{E}}(\mathrm{si}))^{\alpha}\mathfrak{X} \cap \text{Min-si} = (\mathrm{J(asc)}\hat{\mathrm{E}}(\mathrm{asc}))^{\alpha}\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}.$$

Thus $\{\mathrm{J(si)}, \acute{\mathrm{E}}(\mathrm{si})\}\mathfrak{X} \cap \text{Min-si} = \{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}$.

Before showing the main theorem of this section, we introduce and investigate new classes $\mathfrak{I}$ and $\mathfrak{I}^*$. We define the class $\mathfrak{I}$ (resp. $\mathfrak{I}^*$) to be the class of Lie algebras having no infinite-dimensional, simple (resp. absolutely simple) factors of ideals. Equivalently, $L \in \mathfrak{I}$ (resp. $\mathfrak{I}^*$) iff $H/K \in \mathfrak{F}$ whenever $H, K \lhd L$, $K \leq H$ and $H/K \in \mathfrak{S}$ (resp. $\mathfrak{S}^*$). Both $\mathfrak{I}$ and $\mathfrak{I}^*$ are Q-closed. Obviously $\mathfrak{I} \leq \mathfrak{I}^*$. By the proof of Theorem 1.7 (2) we have $\mathrm{L}\mathfrak{F} \nleq \mathfrak{I}^*$. However, we have

PROPOSITION 2.4. (1) *Over any field* $\mathfrak{k}$, $\mathrm{L(asc)}\mathfrak{F} \leq \mathfrak{I}$ *and* $\mathrm{L(ser)}\mathfrak{F} \leq \mathfrak{I}^*$.
(2) *Over any field* $\mathfrak{k}$ *of characteristic zero*, $\mathrm{L(ser)}\mathfrak{F} \leq \mathfrak{I}$.

PROOF. (1) It is clear that $\{\mathrm{S}, \mathrm{Q}\}\mathrm{L(asc)}\mathfrak{F} = \mathrm{L(asc)}\mathfrak{F}$ and $\mathrm{SL(ser)}\mathfrak{F} = \mathrm{L(ser)}\mathfrak{F}$. By using [4, Proposition 13.2.4], we have $\mathrm{QL(ser)}\mathfrak{F} = \mathrm{L(ser)}\mathfrak{F}$. Let $L \in \mathrm{L(asc)}\mathfrak{F} \cap \mathfrak{S}$ and $0 \neq x \in L$. There exists an ascendant $\mathfrak{F}$-subalgebra F of L containing x. Since $0 \neq F \operatorname{asc} L \in \mathfrak{S}$, by [2, Theorem 3.8] we have $L = F \in \mathfrak{F}$. Hence we have $\mathrm{L(asc)}\mathfrak{F} \cap \mathfrak{S} \leq \mathfrak{F}$. Thus $\mathrm{L(asc)}\mathfrak{F} \leq \mathfrak{I}$ since $\{\mathrm{S}, \mathrm{Q}\}\mathrm{L(asc)}\mathfrak{F} = \mathrm{L(asc)}\mathfrak{F}$. We can similarly prove that $\mathrm{L(ser)}\mathfrak{F} \leq \mathfrak{I}^*$.

(2) Assume that $\mathfrak{k}$ is of characteristic zero. Then by (1) and [14, Theorem 8] we have $\mathrm{L(ser)}\mathfrak{F} \cap \mathfrak{S} \leq \mathfrak{I}^* \cap \mathfrak{S}^* \leq \mathfrak{F}$. Therefore $\mathrm{L(ser)}\mathfrak{F} \leq \mathfrak{I}$ since $\{\mathrm{S}, \mathrm{Q}\}\mathrm{L(ser)}\mathfrak{F} = \mathrm{L(ser)}\mathfrak{F}$.

We now have the main theorem of this section, in which we present a relatively large class of Lie algebras L such that $L \in$ Min-si (resp. Min-ser) implies $L \in \mathfrak{F}$.

THEOREM 2.5. *Over any field* $\mathfrak{k}$,
(1) $\{\mathrm{J(asc)}, \hat{\mathrm{E}}\mathrm{(asc)}\}\mathfrak{I} \cap \text{Min-si} = \mathfrak{F}$;
(2) $\{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{I}^* \cap \text{Min-ser} = \mathfrak{F}$.

PROOF. Using Lemmas 1.2 and 2.1, we have

$$(\mathfrak{I} \cap \text{Min-si}) \cup (\mathfrak{I}^* \cap \text{Min-ser}) \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}.$$

Therefore the results follow from Proposition 2.3.

By making use of Proposition 2.4, Theorem 2.5 and [4, Theorem 8.1.4], we have

COROLLARY 2.6. (1) *Over any field* $\mathfrak{k}$,

$$\{\mathrm{J(asc)}, \hat{\mathrm{E}}\mathrm{(asc)}\}\mathfrak{F} \cap \text{Min-si} = \{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{F} \cap \text{Min-ser} = \mathfrak{F}.$$

(2) *Over any field* $\mathfrak{k}$ *of characteristic zero,*

$$\mathrm{L(ser)}\mathfrak{F} \cap \text{Min-}\lhd^2 = \mathfrak{F}.$$

REMARK. Over any field $\mathfrak{k}$, $\mathrm{L(asc)}\mathfrak{F} \cap \text{Min-}\lhd > \mathfrak{F}$. In fact, let L be the Lie algebra constructed in the proof of Theorem 1.7 (1). Then $L \in \mathfrak{Z} \cap \text{Min-}\lhd \leq \mathrm{L(asc)}\mathfrak{F} \cap \text{Min-}\lhd$ but $L \notin \mathfrak{F}$.

3.

Let $\mathfrak{X}$ be a class of Lie algebras. $\mathfrak{X}$ is said to be a class of generalized soluble Lie algebras if $\mathfrak{X} \cap \mathfrak{F} \leq \mathrm{E}\mathfrak{A} \leq \mathfrak{X}$. In this section we shall study generalized soluble Lie algebras satisfying minimal conditions, and present relatively large classes $\mathfrak{X}$ (resp. $\mathfrak{Y}$) of generalized soluble Lie algebras such that $\mathfrak{X} \cap \text{Min-si} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ (resp. $\mathfrak{Y} \cap \text{Min-ser} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$).

PROPOSITION 3.1. (1) *If* $\mathfrak{X} \cap \text{Min-si} \leq \mathrm{E}\mathfrak{A} \leq \mathfrak{X}$, *then*

$$\{\mathrm{J(asc)}, \hat{\mathrm{E}}\mathrm{(asc)}\}\mathfrak{X} \cap \text{Min-si} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}.$$

(2) *If* $\mathfrak{X} \cap \text{Min-ser} \leq \mathrm{E}\mathfrak{A} \leq \mathfrak{X}$, *then*

$$\{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{X} \cap \text{Min-ser} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}.$$

PROOF. We here prove (1) only, since (2) can be proved similarly. Clearly we have $\mathfrak{X} \cap \text{Min-si} \leq \text{E}\mathfrak{A} \cap \text{Min-si} \leq \mathfrak{F}$. It follows from Proposition 2.3 (1) that

$$\text{E}\mathfrak{A} \cap \mathfrak{F} \leq \{\text{J(asc)}, \hat{\text{E}}\text{(asc)}\}\mathfrak{X} \cap \text{Min-si} = \{\text{J(si)}, \acute{\text{E}}\text{(si)}\}\mathfrak{X} \cap \text{Min-si} \leq \mathfrak{F}.$$

Therefore in order to obtain the result, it suffices to show that

$$\{\text{J(si)}, \acute{\text{E}}\text{(si)}\}\mathfrak{X} \cap \mathfrak{F} \leq \text{E}\mathfrak{A}.$$

Let α be an ordinal and let $\mathfrak{Y} = (\text{J(si)}\acute{\text{E}}\text{(si)})^{\alpha}\mathfrak{X}$. Suppose that $\mathfrak{Y} \cap \mathfrak{F} \leq \text{E}\mathfrak{A}$. Then $\acute{\text{E}}\text{(si)}\mathfrak{Y} \cap \mathfrak{F} = \text{E}(\mathfrak{Y} \cap \mathfrak{F}) \leq \text{E}\mathfrak{A}$. Hence by using [4, Corollary 2.2.11], we have

$$(\text{J(si)}\acute{\text{E}}\text{(si)})^{\alpha+1}\mathfrak{X} \cap \mathfrak{F} = \text{J(si)}(\acute{\text{E}}\text{(si)}\mathfrak{Y}) \cap \mathfrak{F} \leq \text{J(si)}\text{E}\mathfrak{A} \cap \mathfrak{F} \leq \text{E}\mathfrak{A}.$$

Therefore by transfinite induction on α we can easily see that for any ordinal α,

$$(\text{J(si)}\acute{\text{E}}\text{(si)})^{\alpha}\mathfrak{X} \cap \mathfrak{F} \leq \text{E}\mathfrak{A}.$$

It follows that $\{\text{J(si)}, \acute{\text{E}}\text{(si)}\}\mathfrak{X} \cap \mathfrak{F} = (\text{J(si)}\acute{\text{E}}\text{(si)})^{*}\mathfrak{X} \cap \mathfrak{F} \leq \text{E}\mathfrak{A}$.

We now have the first main result of this section.

THEOREM 3.2. *Let $\mathfrak{X}$ be an* {I, Q}*-closed class of Lie algebras over any field* $\mathfrak{k}$. *Then*:

(1) *If* $\mathfrak{X} \cap \mathfrak{S} \leq \mathfrak{F}_1 \leq \mathfrak{X}$, *then*

$$\{\text{J(asc)}, \hat{\text{E}}\text{(asc)}\}\mathfrak{X} \cap \text{Min-si} = \text{E}\mathfrak{A} \cap \mathfrak{F}.$$

(2) *If* $\mathfrak{X} \cap \mathfrak{S}^* \leq \mathfrak{F}_1 \leq \mathfrak{X}$, *then*

$$\{\text{J(ser)}, \hat{\text{E}}\}\mathfrak{X} \cap \text{Min-ser} = \text{E}\mathfrak{A} \cap \mathfrak{F}.$$

PROOF. We here prove (1) only, since (2) can be proved similarly. It is easy to see that $\text{E}\mathfrak{A} \leq \acute{\text{E}}\mathfrak{F}_1 \leq \acute{\text{E}}\mathfrak{X}$. Let $L \in \mathfrak{X} \cap \mathfrak{F}$. Then L has a descending series $\{L_i: 0 \leq i \leq n\}$ such that L_{i+1} is a maximal ideal of L_i $(0 \leq i < n)$. Since $\mathfrak{X}$ is {I, Q}-closed, $L_i/L_{i+1} \in \mathfrak{X} \cap \mathfrak{S} \leq \mathfrak{F}_1$. It follows that $L \in \text{E}\mathfrak{F}_1 \leq \text{E}\mathfrak{A}$. Hence we have $\mathfrak{X} \cap \mathfrak{F} \leq \text{E}\mathfrak{A}$. Clearly $\mathfrak{X} \leq \mathfrak{I}$. Using Theorem 2.5 (1), we have

$$\acute{\text{E}}\mathfrak{X} \cap \text{Min-si} = \acute{\text{E}}\mathfrak{X} \cap \mathfrak{F} = \text{E}(\mathfrak{X} \cap \mathfrak{F}) \leq \text{E}\mathfrak{A} \leq \acute{\text{E}}\mathfrak{X}.$$

Therefore by Proposition 3.1 (1)

$$\text{E}\mathfrak{A} \cap \mathfrak{F} \leq \{\text{J(asc)}, \hat{\text{E}}\text{(asc)}\}\mathfrak{X} \cap \text{Min-si} \leq \{\text{J(asc)}, \hat{\text{E}}\text{(asc)}\}\acute{\text{E}}\mathfrak{X} \cap \text{Min-si} = \text{E}\mathfrak{A} \cap \mathfrak{F}.$$

This completes the proof.

Owing to [4, Corollaries 8.5.5 and 9.3.6], we have

$$\text{ELE}\mathfrak{A} \cap \text{Min-si} = \text{J(si)}\text{E}\mathfrak{A} \cap \text{Min-si} = \acute{\text{E}}\mathfrak{A} \cap \text{Min-asc} = \text{E}\mathfrak{A} \cap \mathfrak{F}.$$

On the other hand, in [7, Corollary 4.4] Ikeda has proved that

$$\acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N} \cap \text{Min-}\lhd^2 = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}.$$

By [4, Lemma 8.5.4] $\mathrm{LE}\mathfrak{A} \cap \mathfrak{S} \leq \mathfrak{F}_1$. Therefore, as a special case of Theorem 3.2, we further have the following result which generalizes the above results.

COROLLARY 3.3. *Over any field* $\mathfrak{k}$,

$$\{\mathrm{J(asc)}, \acute{\mathrm{E}}\}\mathrm{LE}\mathfrak{A} \cap \text{Min-si} = \{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathfrak{A} \cap \text{Min-si}$$
$$= \{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{A} \cap \text{Min-ser} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}.$$

REMARK. By [6, Corollary 2.3] we have $\mathrm{LE}\mathfrak{A} \leq \hat{\mathrm{E}}(\lhd)\mathfrak{A}$. Hence $\{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathrm{LE}\mathfrak{A} = \{\mathrm{J(asc)}, \hat{\mathrm{E}}(\mathrm{asc})\}\mathfrak{A}$ and $\{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathrm{LE}\mathfrak{A} = \{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{A}$.

Next we consider another type of relatively large classes of generalized soluble Lie algebras.

Let L be a Lie algebra. As in group theory, L is said to be residually commutable if either $a \notin [a, b]^L$ or $b \notin [a, b]^L$ whenever $a, b \in L \setminus 0$. We use $\mathfrak{Rc}$ to denote the class of residually commutable Lie algebras. It has been proved in [12, Theorems 4.5 and 4.8] that $\hat{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{Rc}$ and $\mathfrak{Rc} \cap \text{Min} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$. Hence $\mathfrak{Rc}$ is a relatively large class of generalized soluble Lie algebras. In the recent paper [6] we have introduced and investigated the class $\mathfrak{R}^{(*)}_{(*)}$, which is also relatively large, of generalized soluble Lie algebras. $L \in \mathfrak{R}^{(*)}_{(*)}$ iff $x \in L \setminus 0$ implies $x \notin ([x, L^{(*)}]^L)^{(*)}$, where $L^{(*)}$ denotes the intersection of all the terms in the transfinite derived series of L. Then we have proved in [6, Proposition 4.1 (2) and Theorem 4.6] that $\hat{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{R}^{(*)}_{(*)}$ and $\mathfrak{R}^{(*)}_{(*)} \cap \text{Min-}\lhd \leq \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

In [1] L is said to be quasi-artinian if for any descending chain $I_1 \geq I_2 \geq \cdots$ of ideals of L there exists an integer $n > 0$ such that $[I_n, L^{(n)}] \leq \bigcap_{i \geq 1} I_i$. In [9] we have used qmin-$\lhd$ to denote the class of quasi-artinian Lie algebras. Note that Min-$\lhd <$ qmin-$\lhd$. Moreover, by using [7, Corollary 2.9], we can easily prove the following result as in the proof of [11, Theorem 3.7].

LEMMA 3.4. *Over any field* $\mathfrak{k}$, $\mathfrak{Rc} \cap \text{qmin-}\lhd \leq \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

By making use of Lemma 3.4, [4, Lemma 9.2.1] and [6, Theorem 4.6], we have

PROPOSITION 3.5. *Over any field* $\mathfrak{k}$,

(1) $\mathfrak{Rc} \cap \text{Min-}\lhd^2 = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$;

(2) $\mathfrak{R}^{(*)}_{(*)} \cap \text{Min-}\lhd^2 = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$.

We here define the closure operation M by specifying the M-closed classes as follows: A class $\mathfrak{X}$ of Lie algebras is M-closed iff $L/I \in \mathfrak{X}$ whenever I is a minimal ideal of an $\mathfrak{X}$-algebra L. It is not hard to show that both of the union and the intersection of all the classes of any non-empty collection of M-closed classes are also M-closed. Therefore M is indeed a closure operation (cf. [4, p. 19]), and every class $\mathfrak{X}$ has the largest M-closed subclass, which we denote by $\mathfrak{X}^{\mathrm{M}}$. Furthermore, we introduce the class $\mathfrak{M}$ of Lie algebras having no non-abelian minimal ideals. Then by [6, Proposition 4.5] we have $\mathfrak{R}^{(*)}_{(*)} \leq \mathfrak{M}^{\mathrm{M}}$, and by modifying the proof of [6, Proposition 4.5] we can easily see that $\mathfrak{R}\mathfrak{c} \leq \mathfrak{M}^{\mathrm{M}}$. It follows that $\mathrm{E}\mathfrak{A} \leq \mathfrak{M}^{\mathrm{M}}$.

We now have the second main result of this section.

THEOREM 3.6. *Over any field* $\mathfrak{k}$,

(1) $\{\mathrm{J}(\mathrm{asc}), \acute{\mathrm{E}}(\mathrm{asc})\}\mathfrak{M}^{\mathrm{M}} \cap \text{Min-si} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$;

(2) $\{\mathrm{J}(\mathrm{ser}), \acute{\mathrm{E}}\}\mathfrak{M}^{\mathrm{M}} \cap \text{Min-ser} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$.

PROOF. Assume that $\mathfrak{M}^{\mathrm{M}} \cap \text{Min-si} \not\leq \mathrm{E}\mathfrak{A}$. There exists a non-soluble Lie algebra L belonging to $\mathfrak{M}^{\mathrm{M}} \cap \text{Min-si}$. By Lemma 1.2 L has an ascending chief series $\{L_\alpha \colon \alpha \leq \sigma\}$ such that $L_{\alpha+1}/L_\alpha \in \mathfrak{S} \cup \mathfrak{F}$ for any $\alpha < \sigma$. By induction on n we see that $L/L_n \in \mathfrak{M}^{\mathrm{M}}$ for any $n < \omega$. Since L_{n+1}/L_n is a minimal ideal of L/L_n, we have

$$L_{n+1}/L_n \in (\mathfrak{S} \cup \mathfrak{F}) \cap \mathfrak{A} = \mathfrak{F} \cap \mathfrak{A},$$

so that $L_n \in \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ for any $n < \omega$. Hence $\omega \leq \sigma$ since $L \notin \mathrm{E}\mathfrak{A}$. By Lemma 2.1 we have

$$L_\omega \in \acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \text{Min-si} = \mathfrak{F}.$$

It follows that $L_n = L_\omega$ for some $n < \omega$. This is a contradiction. Hence we have $\mathfrak{M}^{\mathrm{M}} \cap \text{Min-si} \leq \mathrm{E}\mathfrak{A} \leq \mathfrak{M}^{\mathrm{M}}$. Therefore the results are immediately deduced from Proposition 3.1.

REMARK. By the above theorem we see that the class $\mathfrak{M}^{\mathrm{M}}$ is a class of generalized soluble Lie algebras, and that the classes $\{\mathrm{J}(\mathrm{asc}), \acute{\mathrm{E}}(\mathrm{asc})\}\mathfrak{M}^{\mathrm{M}}$ and $\{\mathrm{J}(\mathrm{ser}), \acute{\mathrm{E}}\}\mathfrak{M}^{\mathrm{M}}$, which seem to be considerably large, are also classes of generalized soluble Lie algebras. On the other hand, the class $\mathfrak{M}$ is not a class of generalized soluble Lie algebras. In fact, let S be a 3-dimensional simple Lie algebra over a field $\mathfrak{k}$ with basis $\{x, y, z\}$ such that

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y.$$

Let V be the underlying vector space S and regard V as an abelian Lie algebra. By making V into an S-module under the adjoint action of S, we form the split extension $L = V \dotplus S$ of V by S. Then by [5, Proposition 12] every proper ideal of L is contained in V. It follows that $L \in \mathfrak{M} \cap \mathfrak{F}$. However, L is not soluble.

References

[1] F. A. M. Aldosray: On Lie algebras with finiteness conditions, Hiroshima Math. J. **13** (1983), 665–674.

[2] R. K. Amayo: Ascendant subalgebras of Lie algebras, preprint, Universität Bonn, 1975.

[3] R. K. Amayo and I. Stewart: Descending chain conditions for Lie algebras of prime characteristic, J. Algebra **35** (1975), 86–98.

[4] R. K. Amayo and I. Stewart: Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

[5] M. Honda: Lie algebras in which the join of any set of subideals is a subideal, Hiroshima Math. J. **13** (1983), 349–355.

[6] M. Honda: Classes of generalized soluble Lie algebras, Hiroshima Math. J. **16** (1986), 367–386.

[7] T. Ikeda: Hyperabelian Lie algebras, Hiroshima Math. J. **15** (1985), 601–617.

[8] Y. Kashiwagi: Lie algebras which have an ascending series with simple factors, Hiroshima Math. J. **11** (1981), 215–227.

[9] F. Kubo and M. Honda: Quasi-artinian Lie algebras, Hiroshima Math. J. **14** (1984), 563–570.

[10] E. M. Levič: On simple and strictly simple rings, Latvijas PSR Zinātņ. Akad. Vēstis Fiz. Tehn. Zinātņ. Sēr. **6** (1965), 53–58 (Russian).

[11] T. Sakamoto: Quasi-artinian groups, Hiroshima Math. J. **15** (1985), 221–233.

[12] G. E. Stevens: Topics in the theory of infinite dimensional Lie algebras, Ph. D. thesis, Univ. of Michigan, 1974.

[13] I. Stewart: The minimal condition for subideals of Lie algebras implies that every ascendant subalgebra is a subideal, Hiroshima Math. J. **9** (1979), 35–36.

[14] I. Stewart: Subideals and serial subalgebras of Lie algebras, Hiroshima Math. J. **11** (1981), 493–498.

[15] S. Tôgô: Locally finite simple Lie algebras, Hiroshima Math. J. **14** (1984), 407–413.

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