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Error bounds for asymptotic expansions of scale mixtures of distributions

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1. Introduction

Asymptotic expansions of the distribution functions of various statistics have been obtained by many authors (see, e.g., the references in Pfanzagl [8]). However, the study on their error bounds is very restrictive. Usually, only rarely are explicit error bounds known. This paper is concerned with error bounds for some asymptotic expansions.

We consider the distribution of a scale mixture $X=\sigma Z$ of a random variable Z with the scale factor σ. It may be noted that many important statistics can be expressed as scale mixtures of distributions. Heyde [4] and Heyde and Leslie [5] have studied the errors in approximating the distribution function $F(x)$ of X by the distribution function $G(x)$ of Z. As it has been noted in Shimizu [9], Fujikoshi [1] obtained an asymptotic expansion of $F(x)$ and its error bound for the case where Z is distributed as the standard normal distribution N(0, 1) and $\sigma \geq 1$ with probability 1. Shimizu [9] obtained a similar result without assuming $\sigma \geq 1$, by inverting the characteristic function of X. In this paper we treat the case where Z has a general distribution. Assuming the smoothness of $G(x)$, we obtain an asymptotic expansion of $F(x)$ and its error bound. The results is obtaind by expanding the conditional distribution function of X given σ in a Taylor series in $\sigma^{-1}-1$. Based on this method, we give a further reduction for the case where Z is distributed as N(0, 1). The expansions and error bounds derived are different from the ones due to Fujikoshi [1] and Shimizu [9]. We examine the errors in approximating $F(x)$ by $G(x)$, especially for the case where Z is distributed as N(0, 1) and a chi-square distribution χ_b^2 with b degrees of freedom. We apply our general theory to the expansions of t-distribution and F-distribution.

2. Scale mixture of a general distribution

Let Z and σ be independent random variables and suppose that $\sigma > 0$ with probability 1. Then

$$X = \sigma Z \tag{2.1}$$

is said to be a scale mixture of Z with the scale factor σ. We denote the distribution functions of X and Z by $F(x)$ and $G(x)$, respectively. Our interest is to find the error bounds when we approximate $F(x)$ by $G(x)$ or expansions of $F(x)$ around $G(x)$. In this section we treat the case where Z has a general distribution. In Section 3 we treat the case where Z is distributed as N(0, 1). When Z is distributed as the exponential distribution, i.e., $G(x)=0$ if $x<0$, $G(x)=1-e^{-x}$, $x\geq 0$, Heyde and Leslie [5] defined $\rho_1(F, G)=E\{(\sigma-1)^2\}$ under the assumption of $E(\sigma)=1$ and $E(\sigma^2)<\infty$, as a convenient distance between $F(x)$ and $G(x)$, and proved

$$\sup_{x\geq 0} |F(x)-(1-e^{-x})| \leq 3.74\, E\{(\sigma-1)^2\}. \tag{2.2}$$

Hall [3] showed that 3.74 can be replaced by 2.77. Based on an expansion of $F(x)$, we shall see that an alternative convenient distance between $F(x)$ and $G(x)$ is

$$\rho_2(F, G) = E\{(\sigma^{-1}-1)^2\} \tag{2.3}$$

under the assumption of $E(\sigma^{-1})=1$ and $E(\sigma^{-2})<\infty$.

We can write

$$F(x) = E\left\{G\left(\frac{x}{\sigma}\right)\right\}. \tag{2.4}$$

Expanding $G(\sigma^{-1}x)=G(x+(\sigma^{-1}-1)x)$ by Taylor's theorem, we have

$$G(\sigma^{-1}x) = G_k(x;\sigma) + \Delta_k(x;\sigma), \tag{2.5}$$

where

$$G_k(x;\sigma) = G(x) + \textstyle\sum_{j=1}^{k-1} \frac{1}{j!}(\sigma^{-1}-1)^j x^j G^{(j)}(x),$$

$$\Delta_k(x;\sigma) = \frac{1}{k!}(\sigma^{-1}-1)^k x^k G^{(k)}(x+\theta(\sigma^{-1}-1)x)$$

and $0<\theta<1$. This suggests an asymptotic approximation to $F(x)$,

$$\begin{aligned} G_k(x) &= E_\sigma\{G_k(x;\sigma)\} \\ &= G(x) + \textstyle\sum_{j=1}^{k-1} \frac{1}{j!} E\{(\sigma^{-1}-1)^j\} x^j G^{(j)}(x). \end{aligned} \tag{2.6}$$

This approximation is defined if $G(x)$ has a $k-1$th derivative $G^{(k-1)}(x)$ and $E(\sigma^{-(k-1)})<\infty$. We find an error bound for this approximation under the following assumptions for some integer $k\geq 1$:

ASSUMPTION 1. $G(x)$ has a kth derivative $G^{(k)}(x)$, and

$$\sup_x |x^k G^{(k)}(x)| < \infty.$$

ASSUMPTION 2. $E(\sigma^{-k})<\infty$.

ASSUMPTION 3. $E(\sigma^{k})<\infty$.

THEOREM 2.1. *Suppose that $X=\sigma Z$ is a scale mixture of Z satisfying Assumptions* 1, 2 *and* 3. *Then*

$$\sup_x |F(x)-G_k(x)| \leq \frac{m_k}{k!} E\{(\sigma \vee \sigma^{-1}-1)^k\}, \tag{2.7}$$

where $\sigma \vee \sigma^{-1} = \mathrm{Max}(\sigma, \sigma^{-1})$..

PROOF. Letting $x+\theta(\sigma^{-1}-1)x=t$, we have

$$\Delta_k(x;\sigma) = \frac{1}{k!} t^k G^{(k)}(t)(\sigma^{-1}-1)^k\{1+\theta(\sigma^{-1}-1)\}^{-k}. \tag{2.8}$$

Noting that $|1+\theta(\sigma^{-1}-1)|^{-k}\leq 1$ if $\sigma<1$, and $|1+\theta(\sigma^{-1}-1)|^{-k}\leq\sigma^k$ if $\sigma\geq 1$, we obtain

$$|\Delta_k(x;\sigma)| \leq \frac{1}{k!} m_k(\sigma\vee\sigma^{-1}-1)^k.$$

Taking the expectation of the both sides, we obtain

$$\begin{aligned} |F(x)-G(x)| &= |E\{\Delta_k(x;\sigma)\}| \\ &\leq E\{|\Delta_k(x;\sigma)|\} \leq \frac{1}{k!} m_k E\{(\sigma\vee\sigma^{-1}-1)^k\}. \end{aligned}$$

We note that in the case of k is even, we can replace the error bound in (2.7) by

$$(m_k/k!)[E\{(\sigma^{-1}-1)^k\} + E\{(\sigma-1)^k\}], \tag{2.9}$$

which will be more computable.

It may be noted that the expansion formula (2.6) involves the moments of σ^{-1}, but does not involve the moments of σ. In our applications, the scale factor is defined by $\sigma=(\chi_n^2/n)^{-1}$, where χ_n^2 is a chi-square variate with n degrees of freedom. In this case, all the moments of σ^{-1} exist, and hence Assumption 2 is automatically satisfied. On the other hand, the kth moment of σ exists only for the case of $n>2k$. So, it is interesting to find an alternative error bound, which depends only on the moments of σ^{-1}. Such an error bound can be obtained by using the method in Shimizu [9]. Let $u>1$ be a given constant, and define

$$d_k(u) = \sup_x \sup_{s>u} \left| G\left(\frac{x}{s}\right) - G_k(x;s) \right|, \tag{2.10}$$

which is a decreasing function of u for $u>1$.

THEOREM 2.2. *Let $X=\sigma Z$ be a scale mixture of Z satisfying Assumptions 1 and 2. Then*

$$\sup_x |F(x)-G_k(x)| \le A_k E\{|\sigma^{-1}-1|^k\}, \tag{2.11}$$

where

$$A_k = \inf_{u>1} \left\{\frac{u^k}{k!} m_k + \left(\frac{u}{u-1}\right)^k d_k(u)\right\}. \tag{2.11}$$

PROOF. The proof pararelles that of Theorem 2 in Shimizu [9]. From Chebyshev inequality and (2.8) it follows that

$$\begin{aligned} P(\sigma>u) &\le P(|\sigma^{-1}-1|\ge 1-u^{-1}) \\ &\le E\{|\sigma^{-1}-1|^k\}/(1-u^{-1})^k. \end{aligned}$$

and

$$|\Delta_k(x;s)| \le \frac{1}{k!} m_k|s^{-1}-1|^k u^k, \quad \text{if} \quad s<u.$$

Using these two inequalities, we obtain

$$\begin{aligned} E_\sigma\{|\Delta_k(x;\sigma)|\} &= \int_0^\infty |\Delta_k(x;s)|dP(\sigma\le s) \\ &\le \frac{1}{k!} m_k u^k \int_0^u |s^{-1}-1|^k dP(\sigma\le s) + d_k(u)P(\sigma\ge u) \\ &\le \left\{\frac{u^k}{k!} m_k + \left(\frac{u}{u-1}\right)^k d_k(u)\right\} E\{|\sigma^{-1}-1|^k\} \end{aligned}$$

which concludes the theorem.

We consider the problem of approximating $F(x)$ by $G(x)$. By putting $k=2$ in Theorems 2.1 and 2.2 we can give the following two error bounds under Assumption 1 and $E(\sigma^{-1})=1$:

(i) if $E(\sigma^{-2})<\infty$ and $E(\sigma^2)<\infty$,

$$\sup_x |F(x)-G(x)| \le \frac{1}{2} m_2[E\{(\sigma^{-1}-1)^2\}+E\{(\sigma^2-1)^2\}]. \tag{2.13}$$

(ii) if $E(\sigma^{-2})<\infty$,

$$\sup_x |F(x)-G(x)| \le A_2 E\{(\sigma^{-1}-1)^2\}, \tag{2.14}$$

where $A_2=\inf_{u>1}\left\{\frac{1}{2} m_2 u^2 + u^2(u-1)^{-2} d_2(u)\right\}$ and

$$d_2(u) = \sup_x \sup_{s>u} \left| G\left(\frac{x}{s}\right) - G(x) - (s^{-1}-1)xG^{(1)}(x) \right| .$$

Noting that $G(x) - G(x/s)$ and $(1-s^{-1})xG^{(1)}(x)$ are non-negative for $s>1$ and $x>0$, and are non-positive for $s>1$ and $x<0$, we obtain a simple uniform bound for $d_2(u)$,

$$\text{(2.15)} \quad d_2(u) \le [\sup_x \sup_{s>u} |G(x)-G(x/s)|] \vee [\sup_x \sup_{s>u} |(1-s^{-1})xG^{(1)}(x)|]$$
$$\le 1 \vee m_1$$

for $u>1$. Therefore we obtain

$$\text{(2.16)} \qquad A_2 \le \inf_{u>1} \left\{ \frac{1}{2} m_2 u^2 + (1 \vee m_1) u^2 (u-1)^{-2} \right\}$$
$$= \left\{ \frac{1}{2} m_2 + (1 \vee m_1)(u^* - 1)^{-2} \right\} u^{*2},$$

where $u^* = \sqrt[3]{2(1 \vee m_1)/m_2} + 1$. When Z is distributed as the exponential distribution, we have $m_1 = \sup_{x>0} xe^{-x} = e^{-1}$, $m_2 = \sup_{x>0} x^2 e^{-x} = 4e^{-1}$, and $A_2 \le 4.47$. This implies that

$$\text{(2.17)} \qquad \sup_{x>0} |F(x) - (1-e^{-x})| \le 4.47 E\{(\sigma^{-1}-1)^2\} .$$

We make no attempt in the proof of (2.16) to estimate the best (smallest) values of A_2. The value given may be improved by replacing (2.15) by a strong inequality.

3. Scale mixtures of the normal distribution

In this section we treat the case where Z is distributed as N(0, 1). Let $\Phi(x)$ and $\phi(x)$ be the distribution function of Z and its probability density function, respectively, i.e.,

$$\text{(3.1)} \qquad \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt = \int_{-\infty}^{x} \phi(t) dt.$$

Let $H_j(x)$ be the Hermit polynomials defined by

$$\text{(3.2)} \qquad \Phi^{(j+1)}(x) = \phi^{(j)}(x) = (-1)^j H_j(x) \phi(x) .$$

Since $\Phi(x)$ satisfies Assumption 1, we can apply Theorems 2.1 and 2.2, and obtain an asymptotic expansion of the distribution function of X and its error bounds. However, the results do not reflect the property of Z or X being symmetric about 0. We shall derive an alternative expansion and its error bounds, by considering the symmetry of Z about 0 and using Theorem 2.1.

For the mixtures of the normal distribution, some results have been obtained. Heyde and Leslie [5] defined $\rho_1(F(x), \Phi(x)) = E\{(\sigma^2-1)^2\}$ as a convenient distance between $F(x)$ and $\Phi(x)$ under the assumption of $E(\sigma^2)=1$ and $E(\sigma^4)<\infty$, and showed

$$\sup_x |F(x)-\Phi(x)| \leq 2.55E\{(\sigma^2-1)^2\}. \tag{3.3}$$

Hall [3] showed that 2.55 can be replaced by 1.944. Fujikoshi [1] obtained an error bound of an asymptotic expansion of the distribution of the MLE in a multivariate linear model. The result can be expressed in terms of mixtures of the normal distribution as follows: If $\sigma \geq 1$ with probability 1 and $E(\sigma^{2k})<\infty$,

$$\sup_x |F(x)-Q_k(x)| \leq \frac{l_{2k}}{2^k k!} E\{(\sigma^2-1)^2\}, \tag{3.4}$$

where

$$Q_k(x) = \Phi(x) - \textstyle\sum_{j=1}^{k-1} \frac{1}{2^j j!} E\{(\sigma^2-1)^j\} H_{2j-1}(x)\phi(x), \tag{3.5}$$

$$l_j = \sup_x |\Phi^{(j)}(x)| = \sup_x |H_{j-1}(x)\phi(x)|. \tag{3.6}$$

The numerical values of $l_{2k}/\{2^k k!\}$ for $k=1, 2, 3, 4$ are given as follows:

k	1	2	3	4
$l_{2k}/\{2^k k!\}$	0.1210	0.0688	0.0481	0.0369

It is known (Fujikoshi [2]) that

$$l_{2k}/\{2^k k!\} \leq \frac{1}{2k\pi}. \tag{3.7}$$

Shimizu [9] extended (3.4) to the case of $\sigma>0$ with probability 1, and showed that

$$\sup_x |F(x)-Q_k(x)| \leq \frac{1}{2k\pi} E\{(\sigma^2 \vee \sigma^{-2}-1)^k\}, \tag{3.8}$$

by expanding the characteristic function of X and inverting it. Further, he gave an alternative bound,

$$\sup_x |F(x)-Q_k(x)| \leq \tilde{B}_k E\{(\sigma^2-1)^k\}, \tag{3.9}$$

where $\tilde{B}_k = \inf_{0<v<1} \{(2\pi k v^{2k})^{-1} + (1-v^2)^{-k} \tilde{\delta}_k(v)\}$and,

$$\tilde{\delta}_k(v) = \sup_x \sup_{0<s<v} \left| \Phi\left(\frac{x}{s}\right) - \Phi(x) + \textstyle\sum_{j=1}^{k-1} \frac{1}{2^j j!} (s^2-1)^j H_{2j-1}(x)\phi(x) \right|.$$

It may be noted that the formula (3.5) is based on the moments of σ^2. Now we shall derive an alternative expansion in terms of the moments of σ^{-2}. Using the symmetry of X about 0, we have

$$(3.10)\qquad F(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(x)P(\sigma^2 Z^2 \le x^2)$$
$$= E_\sigma\left\{\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(x)\tilde{\Phi}(x^2/\sigma^2)\right\}$$

where $\tilde{\Phi}(x)$ is the distribution function of Z^2, i.e., $\tilde{\Phi}(x)=0$ if $x \le 0$,

$$(3.11)\qquad \tilde{\Phi}(x) = \int_0^x (2\pi)^{-1/2} e^{-t/2} t^{-1/2} dt,$$

if $x>0$, and $\operatorname{sgn}(x)=1$ if $x>0$, $=0$ if $x=0$ and $=-1$ if $x<0$. Considering a Taylor expansion as in the proof of Theorem 2.1, we have

$$(3.12)\qquad F(x) = E_\sigma\{\Phi_k(x;\sigma) + \tilde{\Delta}_k(x;\sigma)\},$$

where

$$\Phi_k(x;\sigma) = \Phi(x) + \frac{1}{2}\operatorname{sgn}(x)\textstyle\sum_{j=1}^{k-1}\frac{1}{j!}(\sigma^{-2}-1)^j \tilde{\Phi}^{(j)}(x^2),$$

$$\tilde{\Delta}_k(x;\sigma) = \frac{1}{2\cdot k!}(\sigma^{-2}-1)^k \operatorname{sgn}(x) x^{2k} \tilde{\Phi}^{(k)}(x^2 + \theta(\sigma^{-2}-1)x^2)$$

and $0<\theta<1$. From (3.11) we obtain that for $x>0$,

$$(3.13)\qquad x^j \tilde{\Phi}^{(j)}(x) = (-1)^{j-1}\left(\frac{1}{2}\right)^{j-1}\frac{1}{\sqrt{2\pi}} e^{-x/2} x^{-1/2} L_j(x),$$

where $L_j(x)$ is given by

$$(3.14)\qquad L_j(x) = x^j + \textstyle\sum_{i=1}^{j-1}(2i-1)!!\binom{j-1}{i}x^{j-i}$$

with $(2i-1)!! = 1\cdot 3\cdots(2i-1)$. Therefore we have

$$(3.15)\qquad \operatorname{sgn}(x) x^{2j}\tilde{\Phi}^{(j)}(x^2) = (-1)^{j-1}\left(\frac{1}{2}\right)^{j-1} x^{-1}L_j(x^2)\phi(x).$$

For $j=1, 2, 3, 4$,

$$x^{-1}L_1(x^2) = x, \quad x^{-1}L_2(x^2) = x^3 + x,$$
$$x^{-1}L_3(x^2) = x^5 + 2x^3 + 3x,$$
$$x^{-1}L_4(x^2) = x^7 + 3x^5 + 9x^3 + 15x.$$

(3.16) $\Phi_k(x;\sigma) = \Phi(x) + \sum_{j=1}^{k-1} \frac{1}{2^j j!}(-1)^{j-1}(\sigma^{-2}-1)^j x^{-1} L_j(x^2)\phi(x)$.

We make the following Assumptions 4 and 5 for some integer $k \geq 1$, so that we can define an asymptotic approximation $E_\sigma(\Phi(x;\sigma))$, and derive its error bounds:

ASSUMPTION 4. $E(\sigma^{-2k}) < \infty$.

ASSUMPTION 5. $E(\sigma^{2k}) < \infty$.

THEOREM 3.1. *Suppose that $X = \sigma Z$ is a scale mixture of the standard normal distribution satisfying Assumptions 4 and 5. Then*

$$\sup_x |F(x) - \Phi_k(x)| \leq \frac{m_k}{2 \cdot k!} E\{(\sigma^2 \vee \sigma^{-2} - 1)^k\}, \tag{3.17}$$

where

$$\begin{aligned} \Phi_k(x) &= E_\sigma(\Phi_k(x;\sigma)) \\ &= \Phi(x) + \sum_{j=1}^{k-1} \frac{1}{2^j j!}(-1)^{j-1} E\{(\sigma^{-2}-1)^j\} x^{-1} L_j(x^2)\phi(x), \end{aligned} \tag{3.18}$$

$$m_k = \sup_x |x^k \tilde{\Phi}^{(k)}(x)| = \sup_x \left| \frac{1}{2^{j-1}} x^{-1} L_j(x^2)\phi(x) \right|. \tag{3.19}$$

PROOF. From (3.12) and (3.16) we can write

$$F(x) - \Phi_k(x) = E_\sigma(F(x) - \Phi_k(x;\sigma)) = E_\sigma(\tilde{\Delta}_k(x;\sigma)). \tag{3.20}$$

Letting $x^2 + \theta(\sigma^{-2}-1)x^2 = t^2$, we can express $\tilde{\Delta}_k(x;\sigma)$ as

$$\tilde{\Delta}_k(x;\sigma) = \frac{1}{2 \cdot k!} \operatorname{sgn}(x) t^{2k} \tilde{\Phi}^{(k)}(t^2)(\sigma^{-2}-1)^k \{1 + \theta(\sigma^{-2}-1)\}^{-k}. \tag{3.21}$$

These show that (3.17) can be proved by the same method as in the proof of Theorem 2.1.

For the case of mixtures of the normal distribution, we have two asymptotic expansions and their error bounds given by (3.4) and (3.17), respectively. The numerical values of $m_k/(2 \cdot k!)$ and $(2k\pi)^{-1}$ involved in their error bounds are given for $k=2, 4, 6$ as follows:

k	2	4	6
$m_k/(2 \cdot k!)$	0.0791	0.0501	0.380
$(2k\pi)^{-1}$	0.0796	0.0398	0.0265

This shows that the error bound in (3.17) is smaller than the one in (3.8) in the case of $k=2$, but the result is reverse in the case of $k=4$, 6. However, it may be noted that the two asymptotic expansions have their own merits because one is based on the moments of σ^{-2}, and the other is based on the moments of σ^2.

The result (3.9) shows that we can give an asymptotic expansion and its error bound under Assumption 5 only. On the other hand, the asymptotic expansion (3.18) involves the moments of σ^{-2}, but does not involve the moments of σ^2. So, it is interesting to derive an error bound for (3.18), which depends only on the moments of σ^{-2}. Let $u>1$ be a given constant, and define

$$\delta_k(u) = \sup_x \sup_{s>u} \left| \Phi\left(\frac{x}{s}\right) - \Phi_k(x; s) \right| , \tag{3.22}$$

where $\Phi_k(x; s)$ is defined by (3.16). Then our result is given in the following theorem.

THEOREM 3.2. *Suppose that $X=\sigma Z$ is a scale mixture of the standard normal distribution satisfying Assumption* 4. *Then*

$$\sup_x |F(x)-\Phi_k(x)| \leq B_k E\{|\sigma^{-2}-1|^k\} , \tag{3.23}$$

where

$$B_k = \inf_{u>1} \left\{ \frac{u^{2k}}{2\cdot k!} m_k + \left(\frac{u^2}{u^2-1}\right)^k \delta_k(u) \right\} . \tag{3.24}$$

PROOF. Let $F(x)-\Phi_k(x)=E_\sigma(\tilde{\Delta}_k(x; \sigma))$. Then we have two expressions for $\tilde{\Delta}_k(x; \sigma)$. One is given by (3.21). The other is $\Phi(x/\sigma)-\Phi_k(x; \sigma)$ since $F(x)=E_\sigma(\Phi(x/\sigma))$. Using the two expressions we obtain

$$|\tilde{\Delta}_k(x; \sigma)| \leq \begin{cases} (2\cdot k!)^{-1} m_k |\sigma^{-2}-1|^{2k} u^k, & 0<\sigma\leq u, \\ \delta_k(u), \quad u<\sigma, \end{cases}$$

where $u>1$. Therefore we can prove (3.23) by the same method as in the proof of Theorem 2.2.

From Theorems 3.1 and 3.2 we have the following two error bounds for the approximation of $F(x)$ by $\Phi(x)$ under the assumption of $E(\sigma^{-2})=1$:

(i) if $E(\sigma^{-4})<\infty$ and $E(\sigma^4)<\infty$,

$$\sup_x |F(x)-\Phi(x)| \leq \frac{1}{4} m_2 [E\{(\sigma^{-2}-1)^2\} + E\{(\sigma^2-1)^1\}] , \tag{3.25}$$

(ii) if $E(\sigma^{-4})<\infty$,

$$\sup_x |F(x)-\Phi(x)| \leq B_2 E\{(\sigma^{-2}-1)^2\} , \tag{3.26}$$

where $B_2 = \inf_{u>1}\left\{\frac{1}{4}m_2u^4 + u^4(u^2-1)^{-2}\delta_2(u)\right\}$ and

$$\delta_2(u) = \sup_x \sup_{s>u}\left|\Phi\left(\frac{x}{s}\right) - \Phi(x) - \frac{1}{2}(s^{-2}-1)x\phi(x)\right|.$$

We note that $(1/4)m_2 \le 0.08$. By the same way as in (2.15) and noting $\Phi(0) = \frac{1}{2}$, we obtain

$$\delta_2(u) \le \frac{1}{2} \vee \frac{1}{2}(\sqrt{\pi}\, e)^{-1} = \frac{1}{2}.$$

Hence $B_2 \le 1.84$. This implies that

$$\sup_x |F(x) - \Phi(x)| \le 1.84E\{(\sigma^{-2}-1)^2\}. \tag{3.27}$$

4. *t*-distribution

Let Z and χ_n^2 be independently distributed as the standard normal distribution N(0, 1) and a chi-square distribution with n degrees of freedom, respectively. Then the distribution of

$$T_n = \left(\frac{\chi_n^2}{n}\right)^{-1/2} Z = \sigma Z$$

is called t-distribution with n degrees of freedom. Our interest is to find error bounds for asymptotic expansions of the distribution function $F(x)$ of T_n. It is well known (see, e.g., Johson and Kotz [7]) that

$$F(x) = \Phi(x) - \phi(x)\left\{\frac{1}{n}a_1(x) + \frac{1}{n^2}a_2(x)\right\} + 0(n^{-3}), \tag{4.1}$$

where $a_1(x) = x(x^2+1)/4$ and $a_2(x) = x(3x^6 - 7x^4 - 5x^2 - 3)/96$. Shimizu [9] gave an expansion and its error bound by using (3.8). The expansion is not the same as (4.1). We examine alternative expansions, based on Theorem 3.1.

For a positive integer j, let

$$\begin{aligned} q_j &= E\{(\sigma^{-2}-1)^j\} = E\{(\chi_n^2/n-1)^j\}, \\ \tilde{q}_j &= E\{(\sigma^2-1)^j\} = E\{(n/\chi_n^2-1)^j\}, \\ N^{(j)} &= \textstyle\prod_{i=1}^j \{n+2(i-1)\}, \quad N_{(j)} = \prod_{i=1}^j (n-2i). \end{aligned} \tag{4.2}$$

Then, using $E\{(\chi_n^2)^j\} = N^{(j)}$ and $E\{(\chi_n^2)^{-j}\} = N_{(j)}^{-1}$ (if $n-2j>0$), we can write q_j and $\tilde{q}_j$ for $j = 1, 2, \ldots, 6$ as follows:

$$
\begin{aligned}
&q_1 = 0, \quad q_2 = 2n^{-1}, \quad q_3 = 8n^{-2}, \\
&q_4 = 12n^{-2}(1+4n^{-1}), \quad q_5 = 32n^{-3}(5+12n^{-1}), \\
&q_6 = 20n^{-3}(1+12n^{-1}+32n^{-2}), \quad \tilde{q}_1 = 2N_{(1)}^{-1}, \\
&\tilde{q}_2 = 2(n+4)N_{(2)}^{-1}, \quad \tilde{q}_3 = 4(7n+12)N_{(3)}^{-1}, \\
&\tilde{q}_4 = 4(3n^2+92n+96)N_{(4)}^{-1}, \\
&\tilde{q}_5 = 8(55n^2+652n+480)N_{(5)}^{-1}, \\
&\tilde{q}_6 = 8(15n^3+1520n^2+10224n+5760)N_{(6)}^{-1}.
\end{aligned} \tag{4.3}
$$

Therefore, setting

$$
\begin{aligned}
\Phi_2(x) &= \Phi(x), \\
\Phi_4(x) &= \Phi_2(x) + \phi(x)x^{-1}\left\{-\frac{1}{4n}L_2(x^2) + \frac{1}{6n^2}L_3(x^2)\right\}, \\
\Phi_6(x) &= \Phi_4(x) + \phi(x)x^{-1}\left\{-\frac{1}{32n^2}\left(1+\frac{4}{n}\right)L_4(x^2)\right. \\
&\qquad \left. + \frac{1}{120n^3}\left(5+\frac{12}{n}\right)L_5(x^2)\right\},
\end{aligned} \tag{4.4}
$$

we obtain the following inequalities for the error $\Delta_k = \sup|F(x)-\Phi_k(x)|$:

$$
\begin{aligned}
\Delta_2 &\le \beta_2 = 0.0792(q_2+\tilde{q}_2), \\
\Delta_4 &\le \beta_4 = 0.0502(q_4+\tilde{q}_4), \\
\Delta_6 &\le \beta_6 = 0.0381(q_6+\tilde{q}_6).
\end{aligned} \tag{4.5}
$$

We note that (4.5) can be used to obtain an error bound for (4.1). In fact, noting $a_1(x) = (1/4)x^{-1}L_2(x^2)$, we obtain

$$
\begin{aligned}
&\sup_x \left| F(x) - \Phi(x) - \frac{1}{n}\phi(x)a_1(x) \right| \\
&\qquad \le \beta_2 + \frac{1}{6}n^{-2}\sup_x |x^{-1}L_3(x^2)\phi(x)| \le \beta_2 + 0.487n^{-2}.
\end{aligned} \tag{4.6}
$$

Here $\sup|x^{-1}L_j(x^2)\phi(x)| = 2^{j-1}\sup|x^j\tilde{\Phi}^{(j)}(x)| = 2^{j-1}m_j$, and the numerical values of m_j for $j=2(1)6$ are given as the ones of $m_j(1)$ in Table 1 in Section 6. Similarly, noting $a_2(x) = (1/32)x^{-1}L_4(x^2) - (1/6)x^{-1}L_3(x^2)$, we obtain

$$
\begin{aligned}
&\sup_x \left| F(x) - \Phi(x) - \phi(x)\left\{\frac{1}{n}a_1(x) + \frac{1}{n^2}a_2(x)\right\} \right| \\
&\qquad \le \beta_6 + \frac{1}{8}n^{-3}\sup_x |x^{-1}L_4(x^2)\phi(x)| \\
&\qquad\quad + \frac{1}{120}(5+12n^{-1})n^{-3}\sup_x |x^{-1}L_5(x)\phi(x)|
\end{aligned} \tag{4.7}
$$

$$\leq \beta_6 + 2.405n^{-3} + 1.378(5+12n^{-1})n^{-3}.$$

5. *F*-distribution

Let χ_b^2 and χ_n^2 be mutually independent chi-square variables with b and n degrees of freedom, respectively. Put $\sigma=(\chi_n^2/n)^{-1}$, $Z=\chi_b^2$ and $X=\sigma Z$. Then $b^{-1}X$ is distributed as the F-distribution with b and n degrees of freedom. Our interest is to find an error bound for asymptotic expansions of the distribution function $F(x)$ of X when b is fixed and n is large. The limiting distribution of X is a chi-square distribution of b degrees of freedom. Let $G(x; b)$ and $g(x; b)$ be the distribution function and the probability density function of χ_b^2, respectively. The probability density function is given by $g(x; b)=0$ if $x\leq 0$ and $g(x; b)=\{\Gamma(b/2)2^{b/2}\}^{-1}e^{-x/2}x^{b/2-1}$ if $x>0$. An expansion for $F(x)$ is given as a special case of Hotelling T_0^2 statistic. The result (see, e.g., Ito [6], Siotani [10]) is given by

$$(5.1)\quad F(x) = G(x; b) + \frac{b}{4n}\{(b-2)G(x; b)-2bG(x; b+2)$$
$$+(b-2)G(x; b+4)\} + \frac{b}{96n^2}\sum_{j=0}^{4}\gamma_j G(x; b+2j) + O(n^{-3}),$$

where $\gamma_0=(b-2)(b-4)(3b-2)$, $\gamma_1=-12b^2(b-2)$, $\gamma_2=6b(b+2)(3b+2)$, $\gamma_3=-4(b+2)(b+4)(3b+4)$ and $\gamma_4=3(b+2)(b+4)(b+6)$. On the other hand, we can given an alternative expansion and its error bound, based on Theorem 2.1. For a positive integer j, let

$$(5.2)\qquad x^jG^{(j)}(x; b) = (-1)^{j-1}2^{-(j-1)}L_j(x; b)g(x; b).$$

Here $L_j(x; 1)$ is the same one as $L_j(x)$ in (3.13) or (3.14). We can see that $L_j(x; b)$ is a polynomial of degree j, and is given by

$$(5.3)\qquad L_j(x; b) = x^j + \sum_{i=1}^{j-1}(2-b)\cdots(2i-b)\binom{j-1}{i}x^{j-i}.$$

For $j=1, 2, 3, 4$,

$$L_1(x; b) = x,\quad L_2(x; b) = x\{x+(2-b)\},$$
$$L_3(x; b) = x\{x^2+2(2-b)x+(2-b)(4-b)\},$$
$$L_4(x; b) = x\{x^3+3(2-b)x^2+3(2-b)(4-b)x+(2-b)(4-b)(6-b)\}.$$

Using (4.3) we can write the asymptotic expansions $G_k(x)=G_k(x; b)$ $(k=2, 4, 6)$ in (2.6) as follows:

$$G_2(x;\,b) = G(x;\,b),$$

$$G_4(x;\,b) = G_2(x;\,b) + g(x;\,b)\left\{-\frac{1}{2n}L_2(x;\,b) + \frac{1}{3n^2}L_3(x;\,b)\right\},$$

$$G_6(x;\,b) = G_4(x;\,b) + g(x;\,b)\left\{-\frac{1}{160^2}\left(1+\frac{4}{n}\right)L_4(x;\,b) + \frac{1}{60n^3}\left(1+\frac{12}{n}\right)L_5(x;\,b)\right\}.$$

From Theorem 2.1 and (2.9) we have

$$\Delta_k(b) = \sup_x |F(x) - G_k(x;\,b)| \leq \beta_k(b) = c_k(b)(q_k + \tilde{q}_k),$$

for $k=2$, 4, 6, where q_k and $\tilde{q}_k$ are given by (4.2), and

$$c_k(b) = \frac{1}{k!}m_k(b) = \frac{1}{k!}\sup_x |x^j G^{(j)}(x;\,b)|. \tag{5.5}$$

The numerical values of $c_k(b)$ and $m_k(b)$ are given in Tables 1 and 2 in Section 6 for $k=2(1)6$ and $b=1(1)20$.

The inequalities (5.4) can be used to obtain an error bound for the asymptotic expansion (5.1). Using $g(x;\,b+2)=b^{-1}xg(x;\,b)$ and $G(x;\,b+2) = -2g(x;\,b+2) + G(x;\,b)$, we can simplify (5.1) as

TABLE 1. The values of $m_k(b)$ for $k=2, 3,\ldots, 6$ and $b=1, 2,\ldots, 20$.

b	$m_2(b)$	$m_3(b)$	$m_4(b)$	$m_5(b)$	$m_6(b)$
1	0.3165	0.7290	2.4048	10.3296	54.668
2	0.5413	1.3443	4.6888	21.0561	115.649
3	0.7403	1.9456	7.0888	32.9888	157.423
4	0.9259	2.5500	9.6374	46.2324	268.640
5	1.1034	3.1627	12.3408	60.8087	361.630
6	1.2750	3.7858	15.1992	76.7208	456.961
7	1.4423	4.4203	18.2104	93.9665	532.542
8	1.6062	5.0664	21.3718	112.543	709.416
9	1.7674	5.7241	24.6811	132.450	848.966
10	1.9263	6.3933	28.1355	153.683	1001.15
11	2.0833	7.0740	31.7326	176.246	1265.80
12	2.2386	7.7657	35.4701	200.227	1566.32
13	2.3925	8.4685	39.3463	239.595	1904.19
14	2.5451	9.1819	43.3592	282.681	2280.73
15	2.6965	9.9059	47.5061	329.538	2697.47
16	2.8470	10.6401	51.7872	380.223	3155.30
17	2.9965	11.3845	57.4452	434.759	3656.21
18	3.1451	12.1387	64.2609	493.298	3937.11
19	3.2930	12.9028	71.4232	555.771	4792.32
20	3.4401	13.6761	78.9380	622.265	5429.66

TABLE 2. The values of $c_k(b)$ for $k=2, 3,\dots, 6$ and $b=1, 2,\dots, 20$.

b	$c_2(b)$	$c_3(b)$	$c_4(b)$	$c_5(b)$	$c_6(b)$
1	0.1582	0.1215	0.1002	0.0861	0.0759
2	0.2707	0.2240	0.1954	0.1755	0.1606
3	0.3701	0.3243	0.2954	0.2749	0.2186
4	0.4630	0.4250	0.4016	0.3853	0.3731
5	0.5517	0.5271	0.5142	0.5067	0.5023
6	0.6375	0.6333	0.6393	0.6393	0.6472
7	0.7211	0.7367	0.7588	0.7831	0.7396
8	0.8031	0.8444	0.8905	0.9379	0.9853
9	0.8837	0.9540	1.0284	1.1038	1.1791
10	0.9632	1.0556	1.1723	1.2807	1.3905
11	1.0417	1.1790	1.3222	1.4687	1.7581
12	1.1193	1.2943	1.4779	1.6686	2.1754
13	1.1962	1.4114	1.6394	1.9966	2.6447
14	1.2726	1.5303	1.8066	2.3557	3.1677
15	1.3482	1.6510	1.9794	2.7462	3.7465
16	1.4235	1.7734	2.1578	3.1685	4.3824
17	1.4982	1.8974	2.3936	3.6233	5.0781
18	1.5725	2.0231	2.6775	4.1108	5.4682
19	1.6465	2.1505	2.9760	4.6314	6.6560
20	1.7205	2.2794	3.2888	5.1855	7.5412

$$F(x) = F(x;\, b) + g(x;\, b)\left\{\frac{1}{n}a_1(x) + \frac{1}{n^2}a_2(x)\right\} + 0(n^{-3}), \tag{5.6}$$

where

$$a_1(x) = -\frac{1}{2}L_2(x;\, b), \quad a_2(x) = \frac{1}{3}L_3(x;\, b) - \frac{1}{16}L_4(x;\, b). \tag{5.7}$$

Therefore, from the inequalities (5.4) in the case of $k=2, 4$ we obtain

$$\sup_x \left| F(x) - G(x;\, b) - \frac{1}{n}g(x;\, b)a_1(x)\right| \le \beta_4(b) + 8n^{-2}c_3(b), \tag{5.8}$$

and

$$\sup_x \left| F(x) - G(x;\, b) - g(x;\, b)\left\{\frac{1}{n}a_1(x) + \frac{1}{n^2}a_2(x)\right\}\right| \le \beta_6(b) + 48n^{-3}c_3(b) + 32n^{-3}(1+12n^{-1})c_5(b).$$

In the special case of $b=2$,

$$G(x;\, 2) = \begin{cases} 0, & x \le 0, \\ 1-e^{-x/2}, & x > 0. \end{cases}$$

It is easily seen that

$$L_j(x;\, 2) = x^j, \quad c_k(2) = k^k e^{-k}/k!.$$

Therefore, we obtain

$$\begin{aligned}(5.10)\qquad &\left|F(x)-G(x;2)-e^{-x}\textstyle\sum_{j=1}^{k-1}\frac{1}{2^j j!}(-1)^{j-1}x^j q_j\right| \\ &\leq \frac{k^k e^{-k}}{k!}(q_k+\tilde{q}_k)\end{aligned}$$

if $n-2k>0$ and k is even. We note that q_k exists for any positive integer k, but $\tilde{q}_k$ exists only for the case of $n-2k>0$.

6. The numerical values of $m_k(b)$ and $c_k(b)$

The error bounds obtained in this paper depend on the quantity $m_k=\sup_x |x^k G^{(k)}(x)|$ or $c_k=m_k/k!$, where $G^{(k)}(x)$ is the kth derivative of the distribution function $G(x)$ of Z. In the case of mixtures of the standard normal distribution or a chi-square distribution, we need the values of

$$(6.1)\qquad m_k(b) = \sup_x |x^k G^{(k)}(x;b)|,$$

or $c_k(b)=m_k(b)/k!$, where $G^{(k)}(x;b)$ is a kth derivative of the distribution function $G(x;b)$ of a chi-square distribution χ_b^2 with b degrees of freedom. The explicit expression for $m_k(b)$ is available only for special values of k and b. For example,

$$\begin{aligned}&m_1(b) = bg(b;b),\\ (6.2)\qquad &m_2(b) = \frac{1}{2}g(b+\sqrt{2b};b)(b+\sqrt{2b})(\sqrt{2b}+2),\\ &m_k(2) = k^k e^{-k},\end{aligned}$$

where $g(x;b)$ is the probability density function of χ_b^2. To find the numerical values of $m_k(b)$ for various values of k and b, we use

$$\begin{aligned}(6.3)\qquad \frac{d}{dx}\{x^k G^{(k)}(x;b)\} &= \frac{d}{dx}\{(-1)^{k-1}2^{-(k-1)}L_k(x;b)g(x;b)\}\\ &= (-1)^k 2^{-k} g(x;b)D_k(x;b),\end{aligned}$$

where $L_k(x;b)$ is given by (5.2), and

$$(6.4)\qquad D_k(x;b) = x^k + \textstyle\sum_{j=1}^k(-b)(2-b)\cdots(2(j-1)-b)\binom{k}{j}x^{k-j}.$$

Let Ω be the set of positive roots of $D_k(x;b)=0$. Then we can find the values of $m_k(b)$ by

$$(6.5) \qquad m_k(b) = \sup_{\xi \in \Omega} |2^{-(k-1)} L_k(\xi;\, b) g(\xi;\, b)|\,.$$

The numerical values of $m_k(b)$ and $c_k(b)$ for $k=2(1)6$ and $b=1(1)20$ are given in Tables 1 and 2.

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