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The effect of non-local convection on reaction-diffusion equations

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1. Introduction

We are concerned with the following reaction-diffusion-convection equation:

$$(1.1)\quad u_t = \left\{u_x - \int_I K(x-y)u(t, y)dy \cdot u\right\}_x + \varepsilon f(u),\ x \in I \equiv (-1/2, 1/2),\ t > 0$$

subject to the boundary conditions

$$(1.2)\qquad u_x - \int_I K(x-y)u(t, y)dy \cdot u = 0 \quad \text{at} \quad x = \pm 1/2$$

and the initial condition

$$(1.3)\qquad u(0, x) = u_0(x) \geqq 0, \quad x \in I.$$

This is a proto-type of spatially aggregating population models of biological individuals in a one dimensional finite habitat I, which was first proposed by Kawasaki [3] and discussed Nagai and Mimura [5] ($\varepsilon=0$) and Mimura and Ohara [4] ($\varepsilon>0$) in the whole interval $-\infty<x<\infty$. Here $u=u(t, x)$ represents the population density at time t and position x. From an ecological point of view, the convection velocity in the right hand side of (1.1) is specified as

$$\int_I K(x-y)\cdot u(y)dy = \int_x^{1/2} K(x-y)\cdot u(y)dy + \int_{-1/2}^x K(x-y)\cdot u(y)dy,$$

where $K(x)$ is an appropriate function satisfying $K(x)<0$ (resp. >0) for $x>0$ (resp. $x<0$). One knows that when

$$\int_x^{1/2} K(x-y)\cdot u(y)dy + \int_{-1/2}^x K(x-y)\cdot u(y)dy > 0 \quad (\text{resp.} <0).$$

the individuals move to the right (resp. left) direction. This indicates that the individuals move toward the region of higher distribution. For the growth term $\varepsilon f(u)$, we assume that ε is a sufficiently small constant, which implies that the dispersal process is very fast compared with the growth process of the species. For the ecological interpretation, see Shigesada [7]. The boundary conditions

(1.2) imply that the species is confined in the habitat I.

When the aggregative process is absent (i.e. $K(x)\equiv 0$), the problem (1.1), (1.2) is reduced to a usual reaction-diffusion equation with homogeneous Neumann boundary conditions, so that the asymptotic behavior of solutions can be completely analyzed. When $f(u)=u(1-u)(u-a)$ $(0<a<1)$ for instance, there are two stable equilibrium solutions $u\equiv 0$, 1 and an unstable one $u\equiv a$, and if $\max_{x\in I} u_0(x)<a$, $u(t, x)\to 0$ as $t\to +\infty$, while if $\min_{x\in I} u_0(x)>a$, $u(t, x)\to 1$ as $t\to +\infty$. That is, the unstable equilibrium solution $u\equiv a$ plays a role of a "separator" between $u\equiv 0$ and $u\equiv 1$. Therefore, in ecological terms, the parameter a is regarded as the index of extinction or existence of the species.

We are concerned with the asymptotic behavior of solutions to (1.1), (1.2), (1.3) in the presence of $K(x)$. In this paper, as the first step to study this problem, we specify $K(x)$ to be the simplest case;

$$K(x) \equiv \begin{cases} -r & (x>0) \\ +r & (x<0), \end{cases} \tag{1.4}$$

where r is a nonnegative constant (the case of more general $K(x)$ will be stated in the forthcoming paper [2]). When r is small or $K(x)$ is small for any $x \in \mathbf{R}$, we can imagine that the asymptotic behavior of solutions is qualitatively similar to that in the absense of $K(x)$. However, when r is large, the asymptotic behavior of the solutions is quite different from that in the absence of $K(x)$. Let us show two numerical experiments as follows:

i) Suppose that a is close to 1. Then, when r is large, $u(t, x)\to 0$ as $t\to +\infty$ even if $u_0(x)$ is large (Figure 1–1).

ii) Suppose that a is rather small. Then, when r is large, $u(t, x)$ goes to a solitary stationary state as $t\to +\infty$ even if $u_0(x)<a$ (Figure 1–2).

Ecologically, i) and ii) are interpreted as follows: When a is close to 1, the strongly aggregative effect, which corresponds to the case that r is large, never benefits

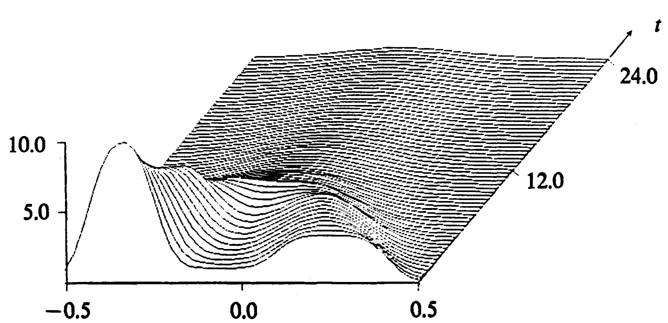

Fig. 1–1. Evolutional behavior of the solution $u(t, x)$, where $f(u)=u(1-u)(u-a)$, $r=28$, $a=0.8$, and $\varepsilon=3.0$.

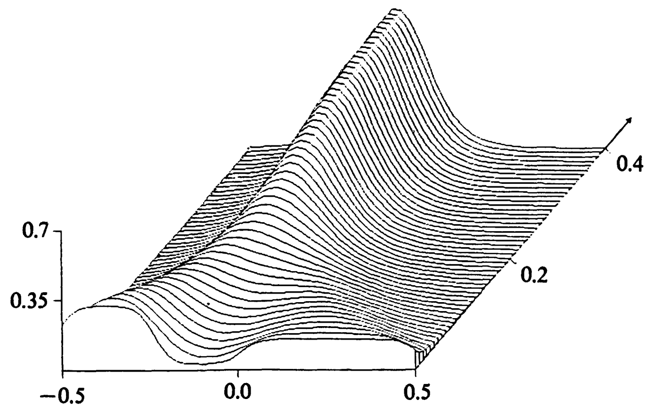

Fig. 1–2. Evolutional behavior of the solution $u(t, x)$ when $\int_I u_0(x)dx \simeq 0.29$, where $f(u)=u(1-u)(u-a)$, $r=28$, $a=0.35$ and $\varepsilon=3.0$.

to the survival of the species, while for the case ii), the species can survive by the strongly aggregative effect even if its initial population density is rather low. That is, aggregation has not always advantage to the survival of the species, which does essentially depend on the nonlinearity of the growth term $f(u)$.

To understand these situations, we study the dependency of the asymptotic behavior of solutions of (1.1), (1.2), (1.3) on the aggregative effect under the restriction that ε is sufficiently small.

Our results will be stated in Sections $2\sim5$. By applying the results to $f(u)=u(1-u)(u-a)$, the global picture of equilibrium solutions with respect to $r\in(0, \infty)$ can be shown for any fixed a. The solid lines correspond to stable equilibrium solutions and the broken line does to the unstable one. As being seen in Section 3, there is a critical value $a^*\in(0, 1)$ such that for $a\in(a^*, 1)$, there are

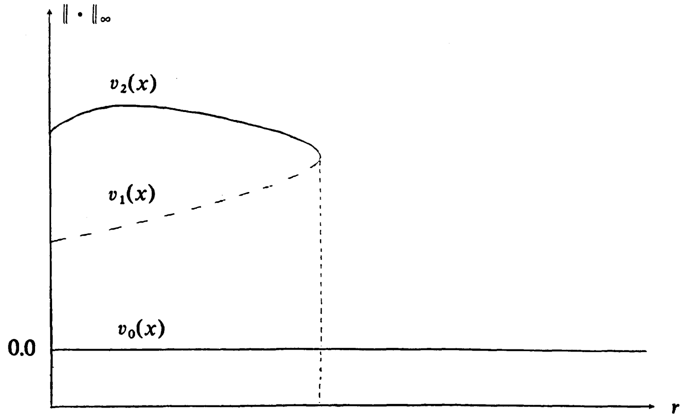

Fig. 2–1. a is fixed to satisfy $a\in(a^*, 1)$.

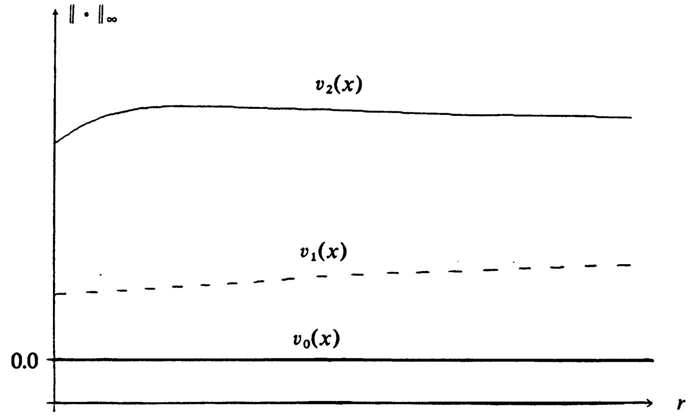

Fig. 2–2. a is fixed to satisfy $a \in (0, a^*)$.

Fig. 2. Global structure of equilibrium solutions with respect to $r \in [0, \infty)$ for fixed $a \in (0, 1)$. $v_0(x) \equiv 0$, $v_1(x)$ and $v_2(x)$ are single pulse solutions. —— represents a stable solution and ---- does an unstable one.

three equilibrium solutions $v_0(x) \equiv 0$, $v_1(x)$ and $v_2(x)$ and $v_0(x)$, $v_2(x)$ are asymptotically stable when r is small, while there is only one equilibrium solution $v_0(x) \equiv 0$ which is asymptotically stable when r is large (Figure 2–1). It is suggested that this situation explains the former case i) of the numerical experiments. On the other hand, for $a \in (0, a^*)$ there are three equilibrium solutions $v_0(x) \equiv 0$, $v_1(x)$ and $v_2(x)$ and $v_0(x)$, $v_2(x)$ are asymptotically stable for any $r > 0$ (Figure 2–2). In order to know attractive domains of $v_0(x)$ and $v_2(x)$, we apply 2-timing methods discussed by Shigesada [7] and Ei and Mimura [1] (this method will be stated in Section 6), and show that there is a critical value M^* such that when $\int_I u_0(x)dx > M^*$, $u(t, x) \to v_2(x)$, while when $\int_I u_0(x)dx < M^*$, $u(t, x) \to v_0(x) \equiv 0$. In fact, when $r = 28$ and $a = 0.35$ (which are specified in Figure 1–2), M^* is explicitly determined as $M^* \simeq 0.27$. Therefore, if $u_0(x)$ is chosen as $\int_I u_0(x)dx \simeq 0.29 > M^*$, we find $u(t, x) \to v_2(x)$, which explains the latter case ii) of the numerical experiments. Thus, the numerical experiments can be theoretically interpreted.

2. Main results

In what follows, we specify $K(x)$ to be (1.4), that is,

$$k_r[u](x) \equiv \int_I K(x-y) \cdot u(y)dy = r\left(\int_x^{1/2} u(y)dy - \int_{-1/2}^{x} u(y)dy\right)$$

and assume that $f(u)$ satisfies $f(0) \geq 0$ and $f \in C^2(\mathbf{R})$. Let B denote the Banach space $C(\bar{I})$ with sup-norm $\|\cdot\|_\infty$. For $w \in C^1(\bar{I})$ and $r \in \mathbf{R}$, we define a mapping

R: $(w, r) \in B \times \mathbf{R} \to R(w, r)$ by $R(w, r)(x) \equiv u(x)$, where $u(x)$ is a solution of

$$(2.1)\qquad \begin{cases} w_x = u_x - k_r[u] \cdot u, \\ w(0) = u(0). \end{cases}$$

When $w(x)$ is a constant function, say $w(x) \equiv c$, R is explicitly represented as

$$R(c, r)(x) = \begin{cases} 4c \cdot exp\{-2(cr)^{1/2} \cdot x\}/(1+exp\{-2(cr)^{1/2} \cdot x\})^2 & (cr>0), \\ c \cdot sec^2\{(-cr)^{1/2} \cdot x\} & (cr \leqq 0). \end{cases}$$

When c and r are both positive, we find that $R(c, r)(x)$ is a symmetric nonnegative function in B with $\|R(c, r)(\cdot)\|_\infty = R(c, r)(0) = c$ (Figure 3). When cr is negative and sufficiently small, it is found that $R(c, r)(\cdot) \in B$.

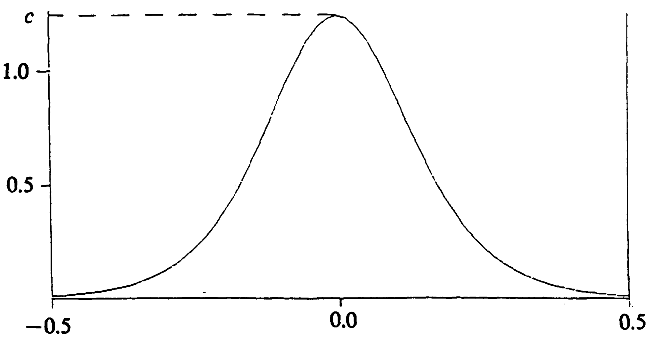

Fig. 3. Graph of $R(1.2, 4)(x)$.

THEOREM 1. *Fix $\varepsilon>0$ arbitrarily and $u_0(x) \in B$. Then a nonnegative solution $u(t, x; \varepsilon)$ of* (1.1), (1.2), (1.3) *uniquely exists in $C^1([0, T]; B)$ and $u(t, \cdot, \varepsilon) \in C^2(I)$ for some $T>0$. If* $\limsup_{u \to +\infty} f(u)/u^3 < 0$, *$u(t, x; \varepsilon)$ exists for any time and is globally bounded.*

Let us consider the stationary problem of (1.1), (1.2):

$$(2.2)\qquad \begin{cases} 0 = \{v_x - k_r[v] \cdot v\}_x + \varepsilon f(v) & \text{for} \quad x \in I. \\ v_x - k_r[v] \cdot v = 0 \quad \text{at} \quad x = \pm 1/2. \end{cases}$$

THEOREM 2. *Suppose that there exists $(c_0, r_0) \in \mathbf{R}_+ \times \mathbf{R}_+$ satisfying $\int_I f(R(c_0, r_0)(x))dx = 0$ and $\frac{\partial}{\partial c}\int_I f(R(c, r)(x))dx\,|_{(c_0, r_0)} \neq 0$, where $\mathbf{R}_+ \equiv [0, \infty)$. Then there uniquely exists a function $v(\varepsilon, r; c_0, r_0) \in C^2((-\varepsilon_0, \varepsilon_0) \times (r_0 - \varepsilon_0, r_0 + \varepsilon_0); B)$ for small $\varepsilon_0 > 0$ such that $v(\varepsilon, r; c_0, r_0)(x)$ is a solution of* (2.2) *satisfying $v(0, r_0; c_0, r_0)(x) = R(c_0, r_0)(x)$.*

REMARK 1. For fixed c_0 and r_0, $v(0, r; c_0, r_0)(x)$ satisfies the equation

(2.3) $$v_x - k_r[v]\cdot v = 0 \quad \text{in} \quad x \in I.$$

On the other hand, nonnegative solutions $v(x)$ of (2.3) are parametrized by $(c, r) \in \mathbf{R}_+ \times \mathbf{R}_+$ as $v(x) = R(c, r)(x)$ and $R(c, r)(x) = R(c', r')(x)$ implies $(c, r) = (c', r')$. Therefore, if $v(\varepsilon, r; c_0, r_0)(x)$ is a solution, there is a unique function $c(r) \in C^2((r_0 - \varepsilon_0, r_0 + \varepsilon_0); \mathbf{R})$ satisfying $v(0, r; c_0, r_0)(x) = R(c(r), r)(x)$, $c(r_0) = c_0$ and $\int_I f(R(c(r), r)(x))dx = 0$.

By Theorem 2 and Remark 1, we find that for small ε, $v(\varepsilon, r; c_0, r_0)(x)$ is approximated by $R(c(r), r)(x)$, especially $\|v(\varepsilon, r; c_0, r_0)\|_\infty$ is approximated by $R(c(r), r)(0) = c(r)$.

THEOREM 3. *In addition to the assumptions of Theorem 2, suppose that*

$$\frac{\partial}{\partial c}\int_I f(R(c, r)(x))dx\,|_{(c_0, r_0)} < 0 \quad (\textit{resp.} > 0).$$

Then there exists a positive constant ε_1 *such that for all* $(\varepsilon, r) \in (0, \varepsilon_1) \times (r_0 - \varepsilon_1, r_0 + \varepsilon_1) \cap \mathbf{R}_+$ $v(\varepsilon, r; c_0, r_0)(x)$ *is asymptotically stable (resp. unstable) in B.*

3. Applications

In this section, we will show two examples, which suggest that the number of equilibrium solutions and their stability crucially depend on the nonlinearities of $f(u)$.

EXAMPLE 1. We specify the growth term in (1.1) to be a cubic function of the form

$$f(u) = f_a(u) \equiv u(1-u)(u-a) \quad (0 < a < 1).$$

We study the dependency of equilibrium solutions of (2.2) and their stability on two parameters a in the growth term and r in the aggregative effect. Let $F_a(c; r) \equiv \int_I (R(c, r)(x))dx$. Then Theorems 2 and 3 show that zeros of $F_a(c; r)$ for fixed $a \in (0, 1)$ and $r \in \mathbf{R}_+$ correspond to equilibrium solutions of (2.2) and that the stability of each equilibrium solution is determined by the sign of $\partial F_a/\partial c$ at each zero. We first show the structure of zeros of $F_a(c; r)$, which is shown in Figure 4. Here, define "stable zero" (resp. "unstable zero") of $F_a(c; r)$ by c satisfying $F_a(c; r) = 0$ and $\partial F_a/\partial c < 0$ (resp. $\partial F_a/\partial c > 0$). We note that stable (resp. unstable) zeros correspond to asymptotically stable (resp. unstable) equilibrium solutions and that $c = 0$ is a zero of $F_a(c; r)$ for any (a, r), which corresponds to the trivial solution $u \equiv 0$ of (2.2). Then for arbitrarily fixed $r > 0$ we find that there exists $a(r) \in (0, 1)$ such that:

1) For $a(r)<a<1$, $F_a(c;\, r)$ has only one stable zero $c=0$ (Figures 5 and 6);
2) for $0<a<a(r)$, $F_a(c;\, r)$ has two stable zeros $c_0(r,\, a)=0$ and $c_2(r,\, a)$ and an unstable one $c_1(r,\, a)$ (Figures 5 and 7).

Here, we note that $\partial F_a/\partial c\,(c_i(r,\, a(r));\, r)=0\,(i=1,\, 2$; Figures 6 and 7) and that $a(r)$

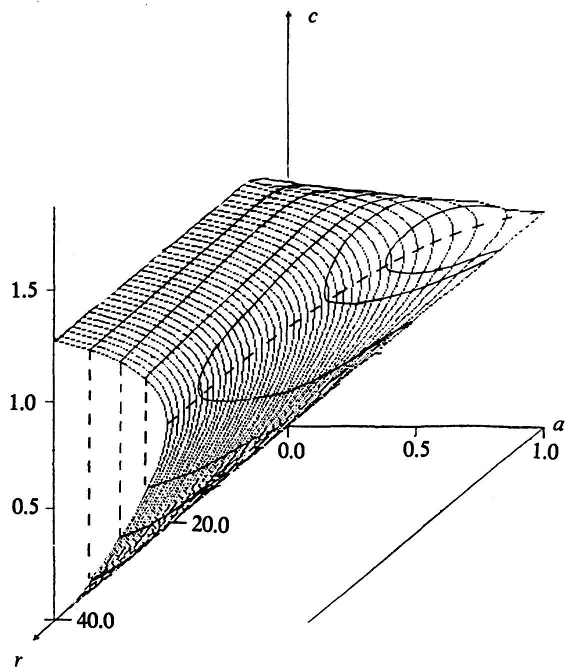

Fig. 4. The distribution of zeros of $F_a(c;\, r)$ except for $c=0$ in $(a,\, r,\, c)$-space.

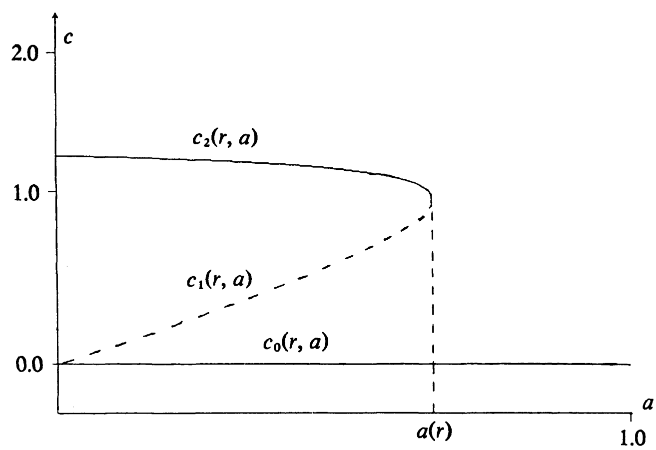

Fig. 5. A cross-section with any fixed $r>0$. —— represents a stable zero and ---- does an unstable one.

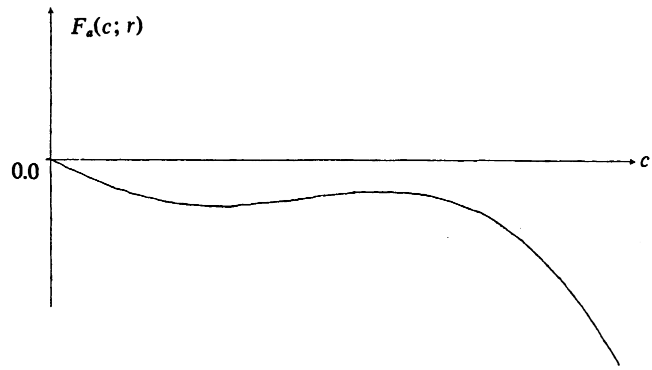

Fig. 6. Graph of $F_a(c; r)$ with $a(r)<a<1$.

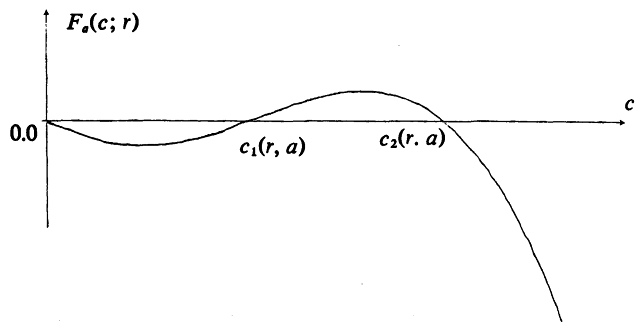

Fig. 7. Graph of $F_a(c; r)$ with $0<a<a(r)$.

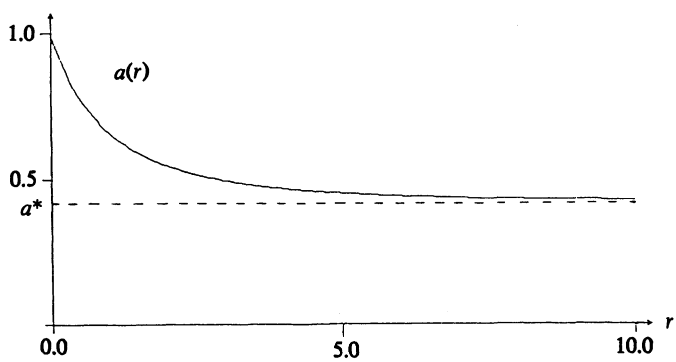

Fig. 8. Graph of $a(r)$.

is strictly monotone decreasing with r (Figure 8) and $\lim_{r\to+\infty} a(r)=a^*\equiv(7-2\sqrt{6})$ $\simeq 0.42$. Then for arbitrarily fixed a, we find that:

3) when $a^*<a<1$, there is a constant $r(a)$ (inverse function of $a(r)$ defined in $(a^*, 1])$ such that there exist three zeros $c_i(r, a)$ $(i=0, 1, 2)$ for $0<r<r(a)$ and there exists only one zero $c_0(r, a)=0$ for $r>r(a)$ (Figure 9–1);
4) when $0<a<a^*$, there exist three zeros $c_i(r, a)$ $(i=0, 1, 2)$ for any $r>0$ (Figure 9–2).

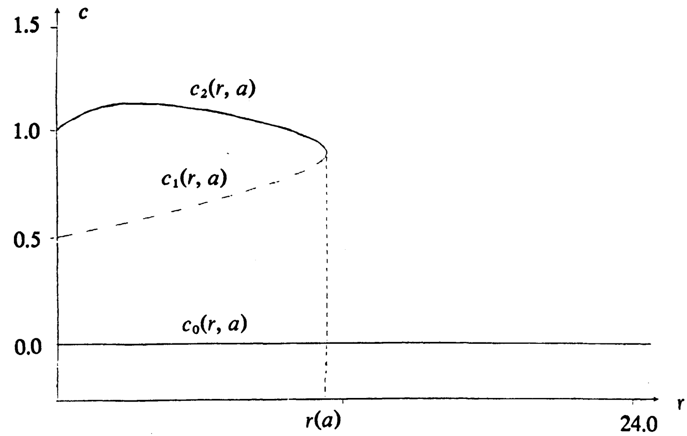

Fig. 9–1. $a=0.5$ $(a^*<a<1)$.

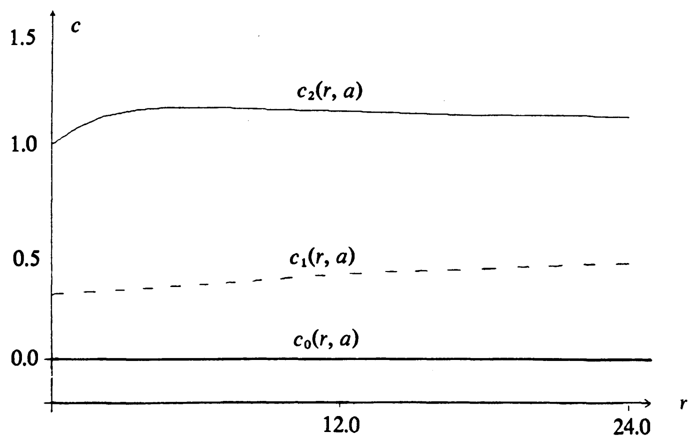

Fig. 9–2. $a=0.3$ $(0<a<a^*)$.

Fig. 9. A cross-section with fixed a. —— represents a stable zero and ---- does an unstable one.

We come back to the original problem (2.2). Define $L(r_0, \delta)\equiv\{(r, a)|0\leqq r\leqq r_0, 0\leqq a\leqq a(r)\}\setminus\{(r, a)|r(a)-\delta\leqq r\leqq r(a), a^*<a\leqq 1\}$ for $r_0>0$ and $\delta>0$. Then Theorems 2 and 3 imply that for r_0 and δ there exists $\varepsilon_0>0$ such that $v_i(\varepsilon; r, a)(x)$

$\equiv v(\varepsilon, r; c_i(r, a), r)(x)$ $(i=1, 2)$ exist and $v_2(\varepsilon; r, a)(x)$ (resp. $v_1(\varepsilon; r, a)(x)$) is asymptotically stable (resp. unstable) for $0<\varepsilon<\varepsilon_0$ and $(r, a)\in L(r_0, \delta)$ and that $v_0(\varepsilon; r, a)(x)\equiv v(\varepsilon, r; c_0(r, a), r)(x)$ exists and is asymptotically stable for $0<\varepsilon<\varepsilon_0$ and $(r, a)\in\{(r, a)|0\leqq r\leqq r_0, 0\leqq a\leqq 1\}$. With these results in mind, consider Case 3) (a is fixed satisfying $a\in(a^*, 1)$). Then, for large $r_0>0$ and small $\delta>0$ satisfying $r_0>r(a)$ and $r(a)-\delta>0$, there is $\varepsilon_0>0$ such that for $0<r<r(a)-\delta$, three equilibrium solutions $v_i(\varepsilon; r, a)$ $(i=0, 1, 2)$ exist for $0<\varepsilon<\varepsilon_0$ while only $v_0(\varepsilon; r, a)\equiv 0$ exists for $r(a)<r<r_0$. This means ecologically that when the parameter a is fixed in the range of $a^*<a<1$, the species necessarily becomes extinct as the aggregation constant r increases, that is, the aggregative effect hinders the survival of the species. Consider Case 4) (a is in $(0, a^*)$). Then for large $r_0>0$, there is $\varepsilon_0>0$ such that three equilibrium solutions $v_i(\varepsilon; r, a)$ $(i=0, 1, 2)$ exist for $0<\varepsilon<\varepsilon_0$ and $0<r<r_0$. Since there are two stable equilibrium solutions $v_0(\varepsilon: r, a)$ and $v_2(\varepsilon; r, a)$ which correspond to the states of extinction and existence respectively, whether the species can exist or not depends on the initial distribution $u_0(x)$.

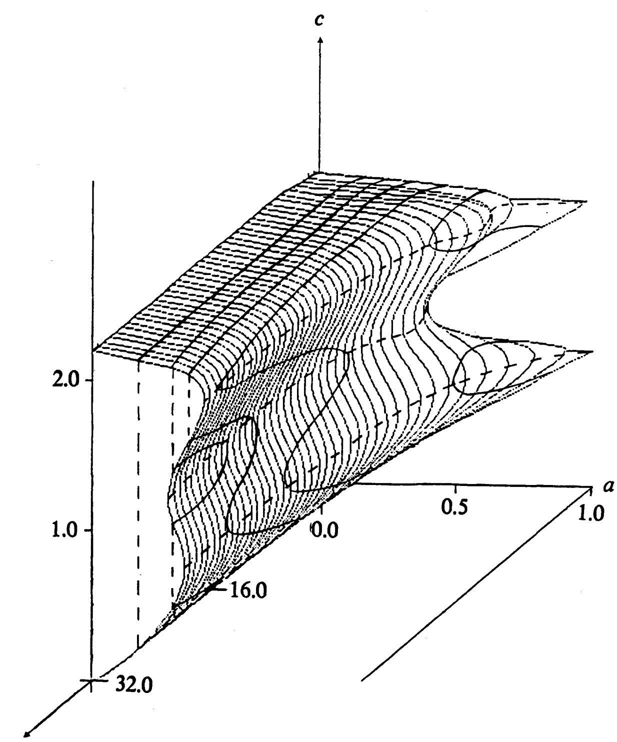

Fig. 10. The distribution of zeros of $F_a(c; r)$ except for $c=0$ in (a, r, c)-space.

EXAMPLE 2. We next consider the case when $f(u)=f_a(u)\equiv u(1-u)(u-a)(u-a-1)(u-2)$ $(0<a<1)$. In a similar way to Example 1, we consider zeros of $F_a(c; r)\equiv \int_I f(R(c, r)(x))dx$. The distribution of zeros of $F_a(c; r)$ except for $c=0$ in (a, r, c)-space is drawn in Figure 10, For arbitrarily fixed $r>0$, a cross-section is shown in Figure 11. By simple calculations, we know that $F_a(c; r)$ has at most five zeros, say, $c_i(r, a)$ $(i=0, 1, 2, 3, 4)$ $(c_0(r, a)=0<c_1(r, a)<c_2(r, a)<c_3(r, a)<c_4(r, a))$ and that there exist $a_i(r)\in(0, 1)$ $(i=1, 2, 3)$ such that:

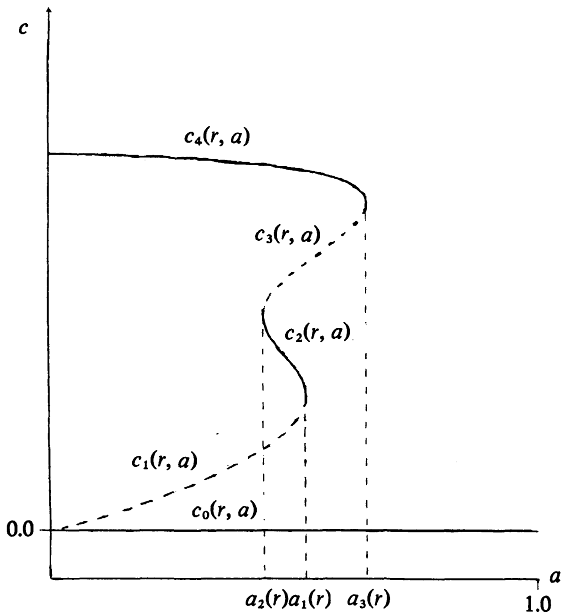

Fig. 11. A cross-section with arbitrarily fixed $r>0$. —— represents a stable zero and ---- does an unstable one.

i) $c_2(r, a)$ exists if and only if $a_2(r)<a<a_1(r)$;
ii) $c_3(r, a)$ exists if and only if $a_2(r)<a<a_3(r)$;
iii) Neither $c_2(r, a)$ nor $c_3(r, a)$ exists if and only if $0<a<a_2(r)$.

It is shown that $c_0(r, a)$, $c_2(r, a)$ and $c_4(r, a)$ are stable zeros while $c_1(r, a)$ and $c_3(r, a)$ are unstable ones (Figures 10 and 11). From Figure 12 we see that $a_1(r)$ is monotone decreasing and $a_2(r)$ is unimodal with r and that there exists $r_0>0$ such that:

i) $a_1(r)>a_3(r)$ when $0<r<r_0$ (Figure 13–1);
ii) $a_1(r)<a_3(r)$ when $r>r_0$ (Figure 13–2).

We define $a^{\#}$, a_i^* and $a_{\#}$ by $a^{\#}\equiv a_1(r_0)=a_3(r_0)$, $a_i^*\equiv\lim_{r\to\infty} a_i(r)$ $(i=1, 2, 3)$ and $a_{\#}\equiv\sup_{r>0} a_2(r)$, respectively (note that $a_2^*<a_1^*<a_3^*<a_{\#}<a^{\#}$). In order to study the r-dependency of the distribution of zeros of $F_a(c; r)$ for fixed $a\in(0, 1)$,

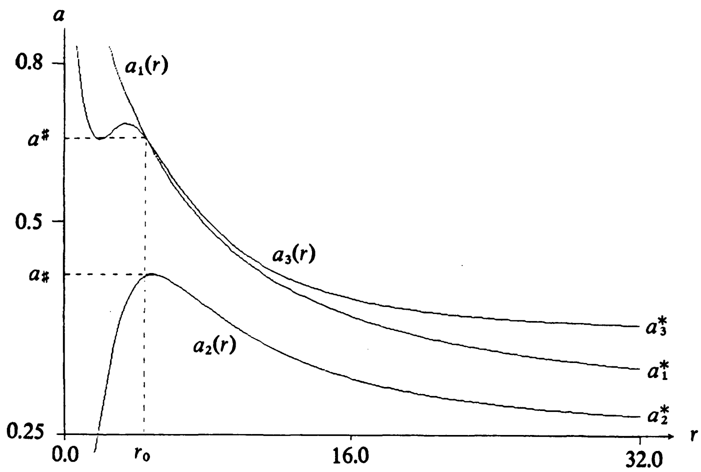

Fig. 12. Graphs of $a_1(r)$, $a_2(r)$ and $a_3(r)$.

we must consider the following 6 cases (Figures 14–1 ~ 14–6):

1) $a^{\#}<a<1$; 2) $a_{\#}<a<a^{\#}$; 3) $a_3^*<a<a_{\#}$;

4) $a_1^*<a<a_3^*$; 5) $a_2^*<a<a_1^*$: 6) $0<a<a_2^*$.

Equilibrium solutions of (2.2) correspond to zeros $c_i(r, a)$ ($i=0, 1, 2, 3, 4$). In Cases 1)–3), the species necessarily becomes extinct as r increases to infinity. In these cases, the aggregative effect hinders the survival of the species. In Case 4), stable zero $c_4(r, a)$ exists for all $r>0$ but another stable zero $c_2(r, a)$ can exist only for r in the ranges of $0<r<r_1$ and $r_2<r<r_3$ (Figure 14–4). In Cases 5) and 6), all zeros exist for sufficiently large $r>0$. These situations suggest that the distribution of equilibrium solutions of (2.2) is much sensitively influenced by the aggregative effect (r) as well as the growth term (a). Especially for fixed $a\in(a_2^*, a_1^*)$, we can see the appearance of 4 limit points.

4. Proofs

PROOF OF THEOREM 1. Since Theorem 1 is valid for any fixed $\varepsilon>0$, we simply write $\varepsilon f(u)$ as $f(u)$. The local existence and the uniqueness of nonnegative solutions of (1.1), (1.2), (1.3) are shown in a standard manner. We only show the uniform boundedness of solutions. Let $M(t)$ be the solution of the following problem:

$$\text{(4.1)}\qquad \begin{cases} \dfrac{dM}{dt} = 2M^2 + f(M) \\ M(0) = \|u_0\|_\infty + 1. \end{cases}$$

Setting $U(t, x)\equiv u(t, x)-M(t)$ and substituting U into (1.1), we obtain

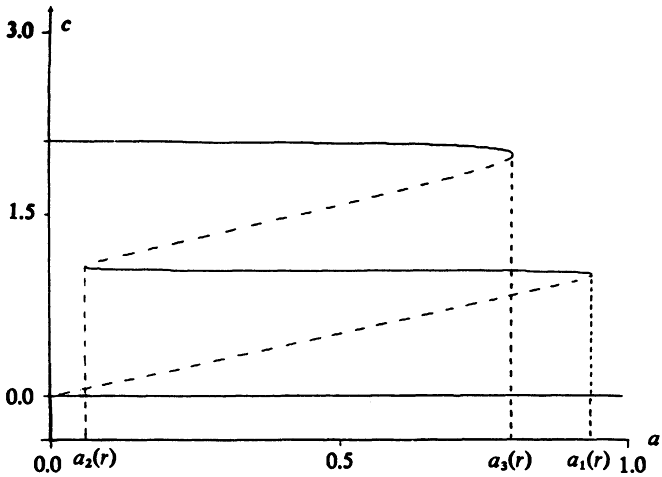

Fig. 13–1. $r=0.4$ ($r<r_0$).

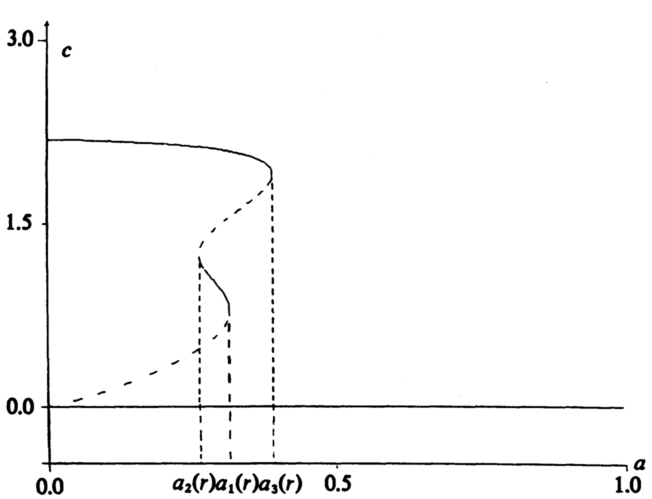

Fig. 13–2. $r=28$ ($r>r_0$).

Fig. 13. A cross-section with fixed r. —— represents a stable zero and ---- does an unstable one.

(4.2) $$U_t \equiv U_{xx} + 2ru \cdot U - k_r[u] \cdot U_x + h(U, M) \cdot U$$

with the boundary conditions

$$U_x - k_r[u] \cdot u = 0 \quad \text{at} \quad x = \pm 1/2$$

and the initial condition

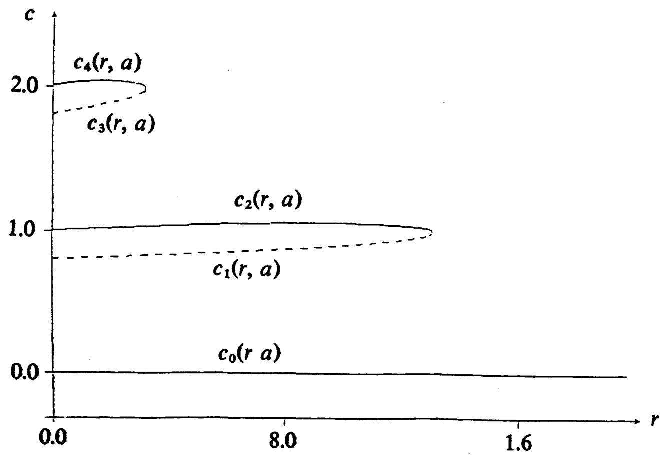

Fig. 14–1. $a=0.8$ ($a^{\#}<a<1$).

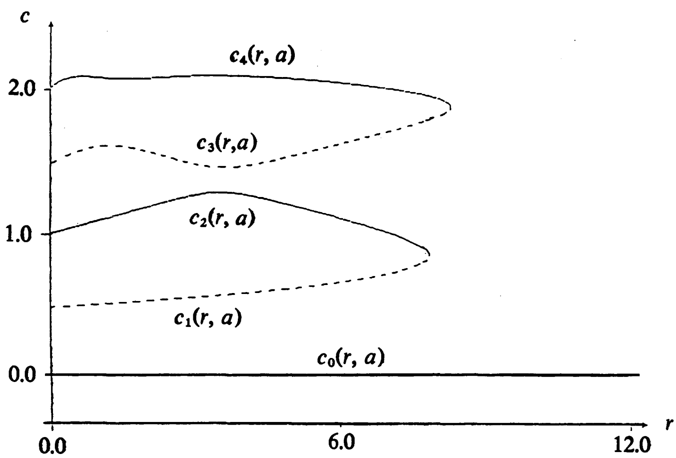

Fig. 14–2. $a=0.52$ ($a_{\#}<a<a^{\#}$).

$$U(0, x) = u_0(x) - M(0) < 0,$$

where $h(U, M)\cdot U \equiv f(U+M)-f(M)$. By maximum principle, we get $U(t, x) \leqq 0$, that is, $u(t, x) \leqq M(t)$. Since the assumption of Theorem 1 implies that $M(t)$ is bounded, it is shown that $u(t, x)$ is L^∞-uniformly bounded. ■

PROOF OF THEOREM 2. It is shown in the Appendix that R can be extended to

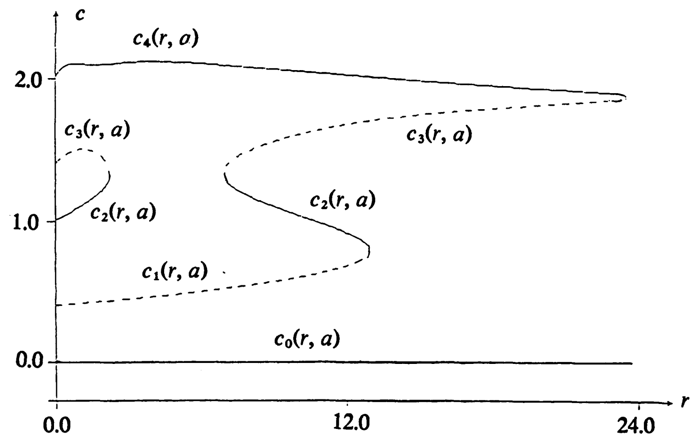

Fig. 14–3. $a=0.4$ ($a_3^*<a<a_\#$).

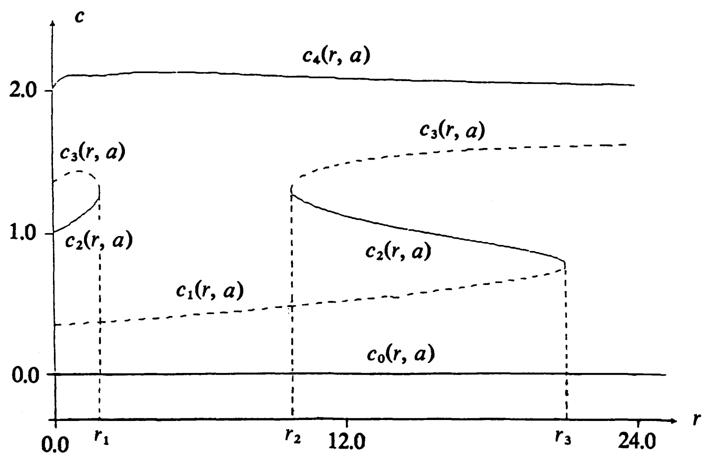

Fig. 14–4. $a=0.35$ ($a_1^*<a<a_3^*$).

an analytic ampping from D into B, where D is a neighborhood of $\{(c, r)|c, r \in \mathbf{R}_+\}$ in $B \times \mathbf{R}$. Transforming the stationary problem of (2.2) by $v=R(w, r)$ and regarding it as an equation in the Banach space B, we obtain

$$Aw \equiv -w_{xx} = \varepsilon F(w, r), \tag{4.3}$$

where A is an operator in B and $F(\cdot, r)$ is a mapping defined by $F(w, r)(x)=f(R(w, r)(x))$. From (2.1), (2.2) we find that the domain of A is $D(A)=\{w \in C^2(\bar{I})|w_x=0$

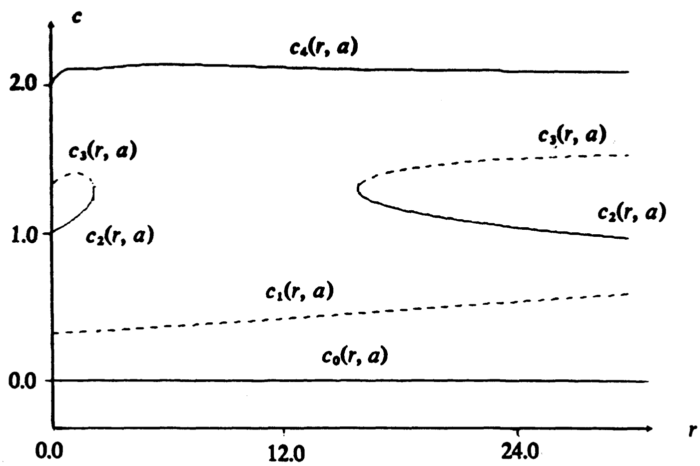

Fig. 14–5. $a=0.32$ ($a_2^*<a<a_1^*$).

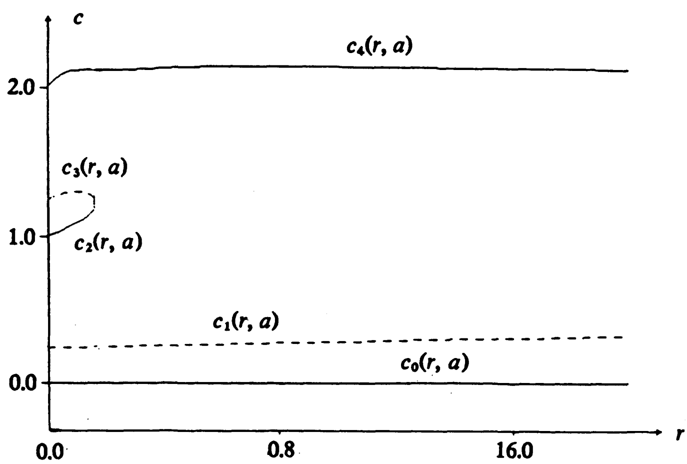

Fig. 14–6. $a=0.24$ ($0<a<a_2^*$).

Fig. 14. A cross-section with fixed a. —— represents a stable zero and ---- does an unstable one.

at $x=\pm 1/2\}$. Then the spectrum of A, $\sigma(A)$, consists only of eigenvalues and is represented by $\sigma(A)=\{0=\lambda_0<\lambda_1<\lambda_2<\cdots\}$. Note that $\mathrm{Ker}\,A=span\{1\}=\mathbf{R}$. Let Q be the projection to $\mathrm{Ker}\,A$. We observe that $Qw=\int_I w(x)dx$ for $w\in B$. Then the assumptions of Theorem 2 imply that there exists $(c_0, r_0)\in \mathrm{Ker}\,A\times\mathbf{R}_+$ such that $QF(c_0, r_0)=0$ and $QF_w(c_0, r_0)|_{\mathrm{Ker}A}\neq 0$, where $F_w=(\partial/\partial w)F$. Therefore, in quite a similar manner to the proof of Theorem 3 in Ei and Mimura [1], we can prove that there uniquely exists a function $w(\varepsilon, r)\in C^2((-\varepsilon_0, \varepsilon_0)\times(r_0-\varepsilon_0, r_0+\varepsilon_0); D(A))$ for small $\varepsilon_0>0$ such that $w(\varepsilon, r)$ is a solution of (4.3) and $w(0, r_0)(x)$

$=c_0$. Thus, we know that $v(\varepsilon, r)(x) \equiv R(w(\varepsilon, r), r)(x)$ is a solution of (2.2). ■

REMARK 2. Since $v(0, r)(x) = R(c(r), r)(x)$ (see Remark 1), the relation $v(\varepsilon, r)(x) = R(w(\varepsilon, r), r)(x)$ implies $w(0, r)(x) = c(r)$.

PROOF OF THEOREM 3. In this proof, we write $w(\varepsilon, r)$ and $R(w, r)$ simply as $w(\varepsilon)$ and $R(w)$, respectively, where $w(\varepsilon, r)(x)$ is defined in (4.3), and denote the norm and the operator norm of B by the same symbol $\|\cdot\|$. Noting that $R(\cdot, r)$ is an isomorphism of $D_r \equiv \{y \in B | (y, r) \in D \cap (B \times \{r\})\}$ to $R(D_r, r)$ for any fixed $r \geqq 0$, we transform (1.1), (1.2), (1.3) by $w(t, x; \varepsilon, r) = R^{-1}(u(t, \cdot\,; \varepsilon, r), r)(x)$. Then we have

$$
(4.4)\quad \begin{cases} w_t = w_{xx} + 2r \int_0^x u ds \cdot w_x - E_r u \cdot (w_x - w_x(0)) + \\ \varepsilon \left\{ f(u) + 2r \int_0^x u ds \cdot \int_0^x f(u) ds - E_r f(u) \cdot \int_0^x u ds - E_r u \cdot \int_0^x f(u) ds \right\} \\ w(0, x; \varepsilon, r) = R^{-1}(u_0, r)(x), \\ w_x(t, \pm 1/2; \varepsilon, r) = 0, \end{cases}
$$

where $u = u(t, x; \varepsilon, r) = R(w(t, \cdot\,; \varepsilon, r), r)(x)$, $w = w(t, x; \varepsilon, r)$ and $E_r \in B^*$ (the dual space of B) is defined by $E_r y \equiv k_r[y](0) = r\left(\int_0^{1/2} y(s)ds - \int_{-1/2}^0 y(s)ds\right)$ for $y \in B$. By the continuity of $R(\cdot, r)$ from D_r to B, the asymptotic stability (or instability) of $v(\varepsilon, r; \varepsilon_0, r_0)$ in B is reduced to that of $w(\varepsilon)$ in B ($v(\varepsilon, r; \varepsilon_0, r_0) = R(w(\varepsilon), r)$) for (4.4). To show the asymptotic stability (or instability) of $w(\varepsilon)$ in B, it suffices to investigate spectrum of a linearized operator at $w(\varepsilon)$ since (4.4) is a semilinear equation of w as equation in the Banach space B. Hereafter, "′" represents Fréchet derivative and "$y\cdot$" for $y \in B$ denotes the product operator. Let A be the operator defined in the proof of Theorem 2 and the mapping v_r: $y \in D_r \to v_r(y)$ be $v_r \equiv r \int_0^x R(y, r)(s)ds$, which is analytic.

LEMMA 1. *The linearized operator $\tilde{L}_{\varepsilon,r}$ at $w(\varepsilon)$ is represented by*

$$
(4.5)\quad \begin{aligned} \tilde{L}_{\varepsilon,r} V = AV - 2v_r(w(\varepsilon)) \cdot V_x - \varepsilon \Big\{ & f'(R(w(\varepsilon))) \cdot R'(w(\varepsilon))V \\ & + 2\int_0^x f'(R(w(\varepsilon))) \cdot (R'(w(\varepsilon))V)(s)ds \cdot v_r(w(\varepsilon)) \\ & + v_r(w(\varepsilon)) \cdot E_r[f'(R(w(\varepsilon))) \cdot R'(w(\varepsilon))V] \Big\} \end{aligned}
$$

for $V \in D(A)$.

PROOF. The linearized operator $\tilde{L}_{\varepsilon,r}$ at $w(\varepsilon)$ is

(4.6) $$\begin{aligned}\tilde{L}_{\varepsilon,r}V = {} & AV - w_x(\varepsilon)\cdot 2v_r'(w(\varepsilon))V - 2v_r(w(\varepsilon))\cdot V_x \\ & + (w_x(\varepsilon)-w_x(\varepsilon)(0))\cdot E_rR'(w(\varepsilon))V + E_rR(w(\varepsilon))\cdot(V_x - V_x(0)) \\ & - \varepsilon\Big\{f'(R(w(\varepsilon)))\cdot R'(w(\varepsilon))V \\ & + 2v_r(w(\varepsilon))\cdot\int_0^x f'(R(w(\varepsilon))\cdot(R'(w(\varepsilon))\cdot(R'(w(\varepsilon))V)(s)ds \\ & - v_r(w(\varepsilon))\cdot E_r[f'(R(w(\varepsilon)))\cdot R'(w(\varepsilon))V] - E_rf(R(w(\varepsilon)))\cdot v_r'(w(\varepsilon))V \\ & - \int_0^x f(R(w(\varepsilon))(s))ds\cdot E_rR'(w(\varepsilon))V \\ & - E_rR(w(\varepsilon))\cdot\int_0^x f'(R(w(\varepsilon)(s))\cdot(R'(w(\varepsilon))V)(s)ds \\ & + 2\int_0^x f(R(w(\varepsilon))(s))ds\cdot v_r'(w(\varepsilon))V\Big\}.\end{aligned}$$

for $V\in D(A)$. Let S be the subspace of B consisting of all even functions. Since $R(\cdot, r)$ maps S into itself and $w(\varepsilon)\in S$ (see the Appendix), it follows that $R(w(\varepsilon))$, $f(R(w(\varepsilon)))$ and $f'(R(w(\varepsilon)))\in S$ and $w_x(\varepsilon)(0)=0$. Noting that $E_r\equiv 0$ on S and

$$w_x(\varepsilon)(x) = -\varepsilon\int_0^x f(R(w(\varepsilon))(s))ds$$

from (4.3) and $w_x(\varepsilon)(0)=0$, we have (4.5). ■

By Theorem 2 and Remark 2, we can write $w(\varepsilon)$ as

$$w(\varepsilon) = w(\varepsilon, r) = c(r) + O(\varepsilon).$$

Here $y(\varepsilon, r)=O(\varepsilon^n)$ for $n\in\mathbf{N}$ means that $\|y(\varepsilon, r)\|\leqq C\varepsilon^n$ uniformly for small $\varepsilon>0$ and r in a neighborhood of r_0. Then we can write $\tilde{L}_{\varepsilon,r}$ as

$$\tilde{L}_{\varepsilon,r}V\equiv\tilde{A}_{\varepsilon,r}V-\varepsilon\tilde{F}_rV-\varepsilon^2\tilde{G}_r(\varepsilon)V,$$

where

$$\begin{aligned}\tilde{A}_{\varepsilon,r}V &\equiv -V_{xx}-2v_r(w(\varepsilon))\cdot V_x, \\ \tilde{F}_rV &\equiv f'(R(c(r)))\cdot R'(c(r))V + 2v_r(c(r))\cdot\int_0^x f'(R(c(r))(s))\cdot(R'(c(r))V)(s)ds \\ &\qquad v_r(c(r))\cdot E_r[f'(R(c(r)))\cdot R'(c(r))V]\end{aligned}$$

and

$$\varepsilon^2\tilde{G}_r(\varepsilon)V\equiv-\tilde{L}_{\varepsilon,r}V+\tilde{A}_{\varepsilon,r}V-\varepsilon\tilde{F}_rV.$$

Note that $\tilde{F}_r$ and $\tilde{G}_r(\varepsilon)$ are bounded linear operators in B and that $\varepsilon\tilde{F}_r=O(\varepsilon)$,

$\varepsilon^2 G_r(\varepsilon)=O(\varepsilon^2)$. Define $z_r(\varepsilon)$ and $\phi_{\varepsilon,r}$ by $z_r(\varepsilon)(x)\equiv v_r(w(\varepsilon))(x)$ and

$$\phi_{\varepsilon,r}(x) \equiv exp\left\{2\int_0^x z_r(\varepsilon)(s)ds\right\}\Big/\int_I exp\left\{2\int_0^x z_r(\varepsilon)(s)ds\right\}dx,$$

respectively. Then the transformation $U=\phi_{\varepsilon,r}\cdot V$ yields

$$L_{\varepsilon,r}U \equiv \phi_{\varepsilon,r}\cdot \tilde{L}_{\varepsilon,r}(\phi_{\varepsilon,r}^{-1}\cdot U) = A_{\varepsilon,r}U - \varepsilon F_r U - \varepsilon^2 G_r(\varepsilon)U,$$

where $A_{\varepsilon,r}U\equiv -(U_x-2z_r(\varepsilon)\cdot U)_x$ for all U in the domain $D(A_{\varepsilon,r})=\{U\in C^2(I)|U_x-2z_r(\varepsilon)\cdot U=0$ at $x=\pm 1/2\}$ and $F_rU\equiv\phi_{\varepsilon,r}\cdot\tilde{F}_r(\phi_{\varepsilon,r}^{-1}\cdot U)$, $G_r(\varepsilon)U\equiv\phi_{\varepsilon,r}\cdot\tilde{G}_r(\varepsilon)(\phi_{\varepsilon,r}^{-1}\cdot U)$. Since the transformation $V\to U$ is a homeomorphism on B, the spectrum of $L_{\varepsilon,r}$ is the same as that of $\tilde{L}_{\varepsilon,r}$. Let $J(\eta)\equiv(-\eta,\eta)\times(r_0-\eta, r_0+\eta)$ and $J_+(\eta)\equiv(0,\eta)\times(r_0-\eta, r_0+\eta)\cap\mathbf{R}_+$ for $\eta>0$. Since $L_{\varepsilon,r}$ has a compact resolvent depending smoothly on $(\varepsilon, r)\in J(\varepsilon_0)$ for some $\varepsilon_0>0$, the spectrum of $L_{\varepsilon,r}$ consists only of point spectrum $\sigma_p(L_{\varepsilon,r})$ and every element of $\sigma_p(L_{\varepsilon,r})$, $\lambda_i(\varepsilon, r)$, is a function of $C^2(J(\varepsilon_0);\mathbf{C})$ such that $\lambda_i(0,r)=\lambda_i(r)\,(i=0,1,2,\ldots)$, where $\lambda_i(r)\in\sigma_p(L_{0,r})=\sigma_p(A_{0,r})=\{0=\lambda_0(r)<\lambda_1(r)<\lambda_2(r)<\cdots\}$. So, for sufficiently small $\varepsilon_0>0$, Re $\lambda_0(\varepsilon, r)<$ Re $\lambda_1(\varepsilon, r)<$ Re $\lambda_2(\varepsilon, r)<\cdots$ for $(\varepsilon, r)\in J(\varepsilon_0)$. Noting that $\lambda_0(0, r)=0$, we will show

$$\frac{\partial}{\partial\varepsilon}\mathrm{Re}\,\lambda_0(\varepsilon, r)|_{(0,r_0)} > 0 \quad (\text{resp.} <0), \tag{4.7}$$

which implies that $w(\varepsilon, r)$ is stable (resp. unstable) for $(\varepsilon, r)\in J_+(\varepsilon_0)$. Let $\psi(\varepsilon, r)$ be an eigenfunction of $\lambda_0(\varepsilon, r)$, which we can take as the function $\psi(\varepsilon, r)\in C^2(J(\varepsilon_0); B)$ satisfying $\psi(0, r)=\phi_{0,r}$. Define $B^1_{\varepsilon,r}\equiv\mathrm{Ker}\,A_{\varepsilon,r}$ and the projection from B to $B^1_{\varepsilon,r}$ by $Q_{\varepsilon,r}$. The we find that $B^1_{\varepsilon,r}=span\{\phi_{\varepsilon,r}\}$ and $(Q_{\varepsilon,r}y)(x)=\int_I y(s)ds\cdot\phi_{\varepsilon,r}(x)$ for $y\in B$. Operating $Q_{\varepsilon,r}$ to the both side of $L_{\varepsilon,r}\psi(\varepsilon, r)=\lambda_0(\varepsilon, r)\psi(\varepsilon, r)$, we have

$$-\varepsilon Q_{\varepsilon,r}(F_r+\varepsilon G_r(\varepsilon))\psi(\varepsilon, r) = \lambda_0(\varepsilon, r)Q_{\varepsilon,r}\psi(\varepsilon, r). \tag{4.8}$$

By differentiating (4.8) by ε and putting $\varepsilon=0$ and $r=r_0$, we obtain

$$-Q_{0,r_0}F_{r_0}\phi_{0,r_0} = \frac{\partial}{\partial\varepsilon}\lambda_0(\varepsilon, r)|_{(0,r_0)}\cdot Q_{0,r_0}\phi_{0,r_0}.$$

Since $Q_{0,r_0}\phi_{0,r_0}=\phi_{0,r_0}$ and $Q_{0,r_0}F_{r_0}\phi_{0,r_0}=\int_I(F_{r_0}\phi_{0,r_0})(x)dx\cdot\phi_{0,r_0}$, Theorem 3 can be proved by using the following lemma.

LEMMA 2. *Under the assumptions of Theorem* 3, *we have*

$$\int_I (F_{r_0}\phi_{0,r_0})(x)dx < 0 \quad (\text{resp.} >0). \tag{4.9}$$

PROOF. We simply write Q_{0,r_0}, F_{r_0} and ϕ_{0,r_0}, B^1_{0,r_0}, $R(\cdot, r_0)$ as Q_0, F_0 and ϕ_0, B^1_0, $R_0(\cdot)$ respectively. Then, the left hand side of (4.9) is

$$\int_I (F_0\phi_0)(x)dx = \int_I \phi_0 \cdot \tilde{F}_0(\phi_0^{-1}\cdot\phi_0)(x)dx$$
$$= \int_I \phi_0 \cdot (\tilde{F}_0 1)(x)dx,$$

where $\tilde{F}_0 \equiv \tilde{F}_{r_0}$ and $\tilde{F}_0 1$ denotes the operation of $\tilde{F}_0$ on the constant function 1. Since $\phi_0(x) = exp\left\{2\int_0^x z_0(s)ds\right\} \Big/ \int_I exp\left\{2\int_0^x z_0(s)ds\right\} dx$ and $\int_I exp\left\{2\int_0^x z_0(s)ds\right\} dx > 0$, it suffices to show

$$\text{(4.10)} \qquad \int_I exp\left\{2\int_0^x z_0(\varepsilon)(s)ds\right\} \cdot (\tilde{F}_0 1)(x)dx < 0 \quad (\text{resp.} > 0),$$

where $z_0(s) \equiv z_{r_0}(0)(s) = v_{r_0}(w(0, r_0))(s) = v_{r_0}(c_0)(s)$. Note that $z_0(x)$ and $exp\left(2\int_0^x z_0(s)ds\right) \cdot z_0(x)$ are odd. Then by the definition of $\tilde{F}_0$, the left hand side of (4.10) is

$$\int_I exp\left(2\int_0^x z_0(s)ds\right) \cdot \Big\{f'(R_0(c_0)(x) \cdot (R_0'(c_0)1)(x)$$
$$+2\int_0^x f'(R_0(c_0)(s)) \cdot (R_0'(c_0)1)(s)ds \cdot z_0(x)$$
$$+E_{r_0}[f'(R_0(c_0)) \cdot R_0'(c_0)1] \cdot z_0(x)\Big\} dx$$
$$= \int_I exp\left(2\int_0^x z_0(s)\, ds\right) \cdot \Big\{f'(R_0(c_0)(x)) \cdot (R_0'(c_0)1)(x)$$
$$+2\int_0^x f'(R_0(c_0)(s)) \cdot (R_0'(c_0)1)(s)ds \cdot z_0(x)\Big\} dx$$
$$= \int_I exp\left(2\int_0^x z_0(s)ds\right) \cdot f'(R_0(c_0)(x)) \cdot (R_0'(c_0)1)(x)dx$$
$$+ \int_I \left\{exp\left(2\int_0^x z_0(s)ds\right)\right\}_x \cdot \int_0^x f'(R_0(c_0)(s)) \cdot (R_0'(c_0)1)(s)dsdx$$
$$\equiv K_1 + K_2.$$

By integrating K_2 by parts, it follows that

$$K_1 + K_2 = \left[exp\left(2\int_0^x z_0(s)ds\right) \cdot \int_0^x f'(R_0(c_0)(s)) \cdot (R_0'(c_0)1)(s)\right]_{-1/2}^{1/2},$$

which is

$$exp\Big(2\int_0^{1/2} z_0(s)ds\Big)\cdot\int_I f'(R_0(c_0)(s))\cdot(R_0'(c_0)1)(s)ds$$
$$= exp\Big(2\int_0^{1/2} z_0(s)ds\Big)\cdot\frac{\partial}{\partial c}\int_I f(R(c,\, r)(s))ds\,|_{(c_0, r_0)},$$

since $z_0(x)$ is odd and $f'(R_0(c_0)(x))\cdot(R_0'(c_0)1)(x)$ is even. Thus, the assumptions of Theorem 3 imply (4.9). ■

REMARK 3. In the proof of Theorem 3, we used the symmetry of $w(\varepsilon)$. But in the case that $f(u)$ depends on x, $w(\varepsilon)$ is in general not symmetric. In this case, the proof of Theorem 3 can be modified by using the following mapping $T(r, w)$ instead of $R(r, w)$: $T(r, w)(x) \equiv u(x)$ for $w \in C^1(\bar{I})$ and $r \in \mathbf{R}_+$, where $u(x)$ is a solution of

$$\begin{cases} w_x = u_x + k_r[u]\cdot u, \\ w(-1) = u(-1). \end{cases}$$

Then the results in Section 2 similarly hold.

5. Appendix

LEMMA A1. *R of (2.1) can be extended to an analytic mapping from D to B. Here D is an open neighborhood of $\mathbf{R}_+\times\mathbf{R}_+$ in $B\times\mathbf{R}$. Moreover, R maps $(S\times\mathbf{R}_+)\cap D$ into S, where S is the subspace of B consisting of all even functions, that is,*

$$S = \{w \in B \mid w(-x) = w(x)\}.$$

PROOF. In (2.1), putting $v(x) \equiv \int_0^x u(s)ds$, $K \equiv \int_{-1/2}^0 u(s)ds - \int_0^{1/2} u(s)ds$ and integrating (2.1) from 0 to x, we have

$$\begin{cases} w = v_x + r(v^2 + Kv), \\ v(0) = 0, \end{cases} \tag{5.1}$$

$$v(-1/2) + v(1/2) + K = 0. \tag{5.2}$$

If we find v, K satisfying (5.1) and (5.2) for $w \in B$ and $r \in \mathbf{R}_+$, we obtain $u = R(w, r)$ by putting $u \equiv v_x$. For arbitrarily fixed $w \in B$, r and $K \in \mathbf{R}$, (5.1) is an initial value problem of ordinary differential equations of v. So the solution $v \in C^1$ exists and we write it as $v(x; K, w, r)$. Here we note that if $v(\cdot\,; K_0, w_0, r_0) \in B$ for some $(K_0, w_0, r_0) \in \mathbf{R}\times B\times\mathbf{R}$, there exists an open neighborhood $U(K_0, w_0, r_0)$ of (K_0, w_0, r_0) in $\mathbf{R}\times B\times\mathbf{R}$ such that for all $(K, w, r) \in U(K_0, w_0, r_0)$, $v(\cdot\,; K, w, r)$

is an element of B and the correspondence of $U(K_0, w_0, r_0) \in (K, w, r) \to v(\cdot\,; K, w, r) \in B$ is analytic. Hence, defining $G\colon U(K_0, w_0, r_0) \to \mathbf{R}$ by

$$G(k, w, r) \equiv v(-1/2; K, w, r) + v(1/2; K, w, r) + K,$$

we see that G is analytic. On the other hand, for any $(c_0, r_0) \in \mathbf{R}_+ \times \mathbf{R}_+$, (5.1) and (5.2) is explicitly solvable as

(5.3) $\quad K = 0 \quad$ and

$$v(x) = v(x; 0, c_0, r_0) = \left(\frac{c_0}{r_0}\right)^{1/2} \cdot \frac{1 - exp\{-2\cdot(c_0\cdot r_0)^{1/2}\cdot x\}}{1 + exp\{-2\cdot(c_0\cdot r_0)^{1/2}\cdot x\}} \in B$$

(when $r_0 = 0$, $K = 0$ and $v(x) = c_0$). Let us consider the mapping

$$G(k, w, r) \equiv v(-1/2; K, w, r) + v(1/2; K, w, r) + K \quad \text{in} \quad U(0, c_0, r_0).$$

Suppose

$$\frac{\partial G}{\partial K}(0, c_0, r_0) \neq 0. \tag{5.4}$$

Then by the implicit function theorem, there uniquely exists $K(w, r; c_0, r_0) \in C^\omega(U(c_0, r_0); \mathbf{R})$ such that $G(K(w, r; c_0, r_0), w, r) = 0$ and $K(c_0, r_0; c_0, r_0) = 0$. Here $U(c_0, r_0)$ is an open neighborhood of (c_0, r_0) in $B \times \mathbf{R}$. Then $v(x; K(w, r; c_0, r_0), w, r)$ satisfies (5.1), (5.2) and we obtain $R(w, r; c_0, r_0)(x) \equiv u(x)$ by $u(x) \equiv v_x(x) = w(x) - r\cdot\{v^2(x; K(w, r; c_0, r_0), w, r) + K(w, r; c_0, r_0)\cdot v(x; K(w, r; c_0, r_0), w, r)\}$ for $(w, r) \in U(c_0, r_0)$. Let us show that $R(w, r; c_0, r_0) = R(w, r; c_1, r_1)$ for $(w, r) \in U(c_0, r_0) \cap U(c_1, r_1)$. (5.3) implies that $K(c, r; c_i, r_i) = 0$ for any $(c, r) \in \mathbf{R}_+ \times \mathbf{R}_+ \cap U(c_i, r_i)$ $(i = 0, 1)$. Therefore, by the uniqueness of the mapping $K(w, r; c_i, r_i)$ in a neighborhood of (c_i, r_i), it follows that $K(w, r; c_0, r_0) = K(w, r; c_1, r_1)$ for $(w, r) \in U(c_0, r_0) \cap U(c_1, r_1)$. This implies that $R(w, r; c_0, r_0) = R(w, r; c_1, r_1)$ in $U(c_0, r_0) \cap U(c_1, r_1)$. Thus, we can define $R(w, r)$ by $R(w, r) \equiv R(w, r; c, r)$ for $(w, r) \in U(c, r)$ with the domain $D = \bigcup_{(c,r) \in \mathbf{R}_+ \times \mathbf{R}_+} U(c, r)$.

Let us show (5.4). Using (5.1) and (5.3), we obtain by simple calculations

$$\frac{\partial v}{\partial K}(x; 0, c_0, r_0) = -\frac{1}{2} \cdot \left(\frac{1 - exp\{-2\cdot(c_0\cdot r_0)^{1/2}\cdot x\}}{1 + exp\{-2\cdot(c_0\cdot r_0)^{1/2}\cdot x\}}\right)^{1/2}.$$

Hence, we have

$$\begin{aligned}\frac{\partial G}{\partial K}(0, c_0, r_0) &= -\frac{1}{2} \cdot \left(\frac{1 - exp\{-(c_0\cdot r_0)^{1/2}\}}{1 + exp\{-(c_0\cdot r_0)^{1/2}\}}\right)^{1/2} \\ &\quad -\frac{1}{2} \cdot \left(\frac{1 - exp\{-(c_0\cdot r_0)^{1/2}\}}{1 + exp\{-(c_0\cdot r_0)^{1/2}\}}\right)^{1/2} + 1 \\ &= -\left(\frac{1 - exp\{-(c_0\cdot r_0)^{1/2}\}}{1 + exp\{-(c_0\cdot r_0)^{1/2}\}}\right)^{1/2} + 1 \neq 0,\end{aligned}$$

which implies (5.4).

Finally, we shall show that R maps $D_S \equiv (S \times \mathbf{R}_+) \cap D$ into S. For $(w, r) \in D_S \cap [C^1(\bar{I}) \times \mathbf{R}_+]$, we define a mapping $X: (w, r) \to u \in S$ by $X(w, r)(x) \equiv u(x)$ and $u(x) \equiv v_x(x)$, where v is a solution of

$$(5.5) \qquad \begin{cases} w = v_x + r \cdot v^2 \\ v(0) = 0. \end{cases}$$

Then it is easily shown that X can be extended to an analytic mapping from $S \times \mathbf{R}_+$ into S. On the other hand, R and X can be regarded as mappings from D_S into B satisfying (5.1) and (5.2). Note that X satisfies (5.1) and (5.2) as $K=0$. Moreover, we observe that for $(c, r) \in \mathbf{R}_+ \times \mathbf{R}_+$ $X(c, r)(x) = R(c, r)(x)$. The uniqueness of a mapping satisfying (5.1), (5.2) in $C^\omega(D_S; B)$ can be shown by using the implicit function theorem in quite a similar way to the proof of the uniqueness in $C^\omega(D; B)$. Thus, we have $X(w, r) = R(w, r)$ in a neighborhood of $\mathbf{R}_+ \times \mathbf{R}_+ \subset S \times \mathbf{R}_+$, which completes the proof. ■

LEMMA A2. *The solution of* (4.3), $w(\varepsilon, r)(x)$, *with* $w(0, r_0)(x) = c_0$ *is an even function.*

PROOF. Since by Lemma A1 we see that the mapping R maps $S \times \mathbf{R}_+$ into S, we can restrict the problem (4.3) to the problem with respect to the Banach space S. Quite similarly to the proof of Theorem 2, we can show the unique existence of the function $\tilde{w}(\varepsilon, r) \in C^2((-\varepsilon_0, \varepsilon_0) \times (r_0 - \varepsilon_0, r_0 + \varepsilon_0); S)$ for some $\varepsilon_0 > 0$ such that $\tilde{w}(\varepsilon, r)(\cdot)$ is a solution of (4.3) and $w(0, r_0)(x) = c_0$. $\tilde{w}(\varepsilon, r)$ can also be regarded as

$$(5.6) \qquad \tilde{w}(\varepsilon, r) \in C^2((-\varepsilon_0, \varepsilon_0) \times (r_0 - \varepsilon_0, r_0 + \varepsilon_0); B)$$

with

$$(5.7) \qquad \tilde{w}(0, r_0)(x) = c_0 .$$

Since the solution of (4.3) satisfying (5.6), (5.7) is unique, we have $\tilde{w}(\varepsilon, r) = w(\varepsilon, r)$, which implies $w(\varepsilon, r) \in S$. ■

6. Concluding remarks

In this section, we briefly explain 2-timing methods to obtain attractive domains of the equilibrium solutions of (1.1), (1.2) and (1.3) (see also Nayfeh [6], Shigesada [7]).

Let us introduce two different time scales, t and τ $(=\varepsilon t)$ and consider solutions of (1.1), (1.2) and (1.3) as functions of these two time scales, that is, $u(t, x; \varepsilon) = u(t, \tau, x; \varepsilon)$. We look for solutions of the form

(6.1) $$u(t, \tau, x; \varepsilon) = u^0(t, \tau, x) + \varepsilon u^1(t, \tau, x) + O(\varepsilon^2),$$

where u^1 is bounded for all $t>0$. By noting

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}$$

and inserting (6.1) into (1.1) and equating coefficients of like powers of ε^0, we obtain

(6.2) $$u_t^0 = \{u_x^0 - k[u^0]\cdot u^0\}_x, \quad -1/2 < x < 1/2, \quad t > 0,$$

with the boundary conditions

(6.3) $$u_x^0 - k[u]\cdot u^0 = 0 \quad \text{at} \quad x = \pm 1/2$$

and the initial condition

(6.4) $$u^0(0, 0, x) = u_0(x),$$

where $k[u](x) = \int_I K(x-y)u(y)dy$. Similarly, the equation for $u^1(t, \tau, x)$ can be described by

(6.5) $$u_t^1 + u_\tau^0 = \{u_x^1 - k[u^1]\cdot u^0 - k[u^0]\cdot u^1\}_x + f(u^0)$$

with the boundary conditions

(6.6) $$u_x^1 - k[u^1]\cdot u^0 - k[u^0]\cdot u^1 = 0 \quad \text{at} \quad x = \pm 1/2$$

and the initial condition

(6.7) $$u^1(0, 0, x) = 0.$$

Since (6.2) with (6.3) preserves the total mass, the solution of (6.2) can be expressed as $u^0(t, M(\tau), x)$ with $M(\tau) = \int_I u^0(t, \tau, x)dx$. Here the function $M(\tau)$ is determined as follows: Let us integrate (6.5) over I. We then have

(6.8) $$\frac{\partial}{\partial t}\int_I u^1(t, \tau, x)\, dx + \frac{dM}{d\tau} = \int_I f(u^0(t, M(\tau), x))dx$$

due to the boundary conditions (6.6). We put formally $t=\infty$ in (6.8) and suppose that $u(t, x; \varepsilon)$ converges to a steady state as $t \to \infty$. Then we may assume that

$$\frac{\partial}{\partial t}\int_I u^1(t, \tau, x)dx \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

Thus, (6.8) reduces to

$$\frac{dM}{d\tau} = \int_I f(u^0(\infty, M(\tau), x)dx, \tag{6.9}$$

with

$$M(0) = \int_I u^0(0, 0, x)dx = \int_I u_0(x)dx. \tag{6.10}$$

If $M(\tau)$ of (6.9), (6.10) is solved, we obtain u^0 as $u^0(t, M(\tau), x)$. It thus can be expected that $u(t, x; \varepsilon)$ is approximated by $u^0(t, M(\varepsilon t), x)$.

Let us apply this approach to our problem for the case that $f(u)=u(1-u)\cdot(u-a)$ $(0<a<1)$ and

$$k[u] = r\left(\int_x^{1/2} u(s)ds - \int_{-1/2}^x u(s)ds\right)$$

for arbitrarily fixed $r>0$ and $0<a<1$. $u^0(\infty, M, x)$ in the right hand side of (6.9) can be regarded as an equilibrium solution of (6.2) with total mass M. Thus, it turns out that nonnegative solutions of (2.3) can be parametrized by the total mass M. On the other hand, solutions of (2.3) can also be parametrized by c ($v(x) = R(c, r)(x)$, see Remark 1). Then the relation between c and M is given by

$$M = \tilde{M}(c) \equiv 2\left(\frac{c}{r}\right)^{1/2} \cdot \frac{1-exp(c\cdot r)^{1/2}}{1+exp(c\cdot r)^{1/2}} = \int_I R(c, r)(x)dx$$

and the correspondence $\tilde{M}$: $c \to \tilde{M}(c)$ is bijective from $\mathbf{R}_+$ into itself satisfying $\tilde{M}(0)=0$ and $(d/dc)\tilde{M}(c)>0$ for all $c\in\mathbf{R}_+$. Then we find that $v^0(x, \tilde{M}(c))= R(c, r)(x)$, where $v^0(x, M)$ is the solution of (2.3) parametrized by M and therefore

$$\int_I f(u^0(\infty, \tilde{M}(c), x)dx = \int_I f(v^0(x, \tilde{M}(c)))dx = \int_I f(R(c, r)(x))dx.$$

Moreover, we see that if $\int_I f(v^0(x, M_0))dx=0$ for some M_0, $c_0\equiv\tilde{M}^{-1}(M_0)$ satisfies $\int_I f(R(c_0, r)(x))dx=0$ and that conversely if $\int_I f(R(c_0, r)(x))dx=0$ for some c_0, $M_0\equiv\tilde{M}(c_0)$ satisfies $\int_I f(v^0(x, M_0))dx=0$. In other words, the distribution of zeros of $F_S(M)\equiv\int_I f(v^0(x, M))dx$ is given by that of $F_R(c)\equiv \int_I f(R(c, r)(x))dx$. Especially if $a<a^*$ (see Example 1 in Section 3), $F_S(M)$ has three zeros (Figure 15). We write those zeros as $M_0=0$, M_1, M_2 $(0=M_0< M_1<M_2)$. We consider this situation for fixed $a\in(0, a^*)$. By (6.9), (6.10) and the graph of $F_S(M)$, we know that

A) $M(\tau) \longrightarrow M_0 = 0$ as $\tau \longrightarrow \infty$ if $M(0) < M_1$,

B) $M(\tau) \longrightarrow M_2$ as $\tau \longrightarrow \infty$ if $M(0) > M_1$,

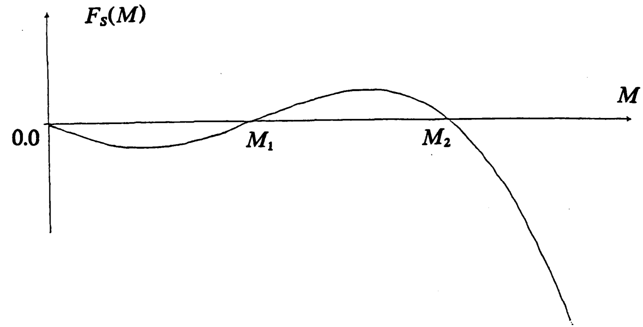

Fig. 15. Graph of $F_S(M)$ with $a<a^*$.

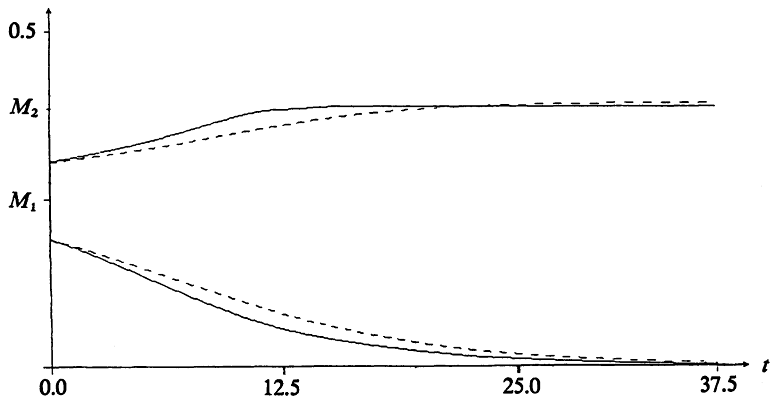

Fig. 16. —— represents the total volume $\int_I u(t, x; \varepsilon)dx$ of the solution of (1.1), (1.2), (1.3), where $r=28$, $a=0.35$ and $\varepsilon=0.1$. ---- represents the solution $M(\varepsilon t)$ of (6.9), (6.10) with $r=28$, $a=0.35$ and $\varepsilon=0.1$.

C) $M(\tau) \equiv M_1$ for all $\tau > 0$ if $M(0) = M_1$.

Consequently, we find the approximate solution $u^0(t, M(\varepsilon t), x)$ converges to $u^0(\infty, M(\infty), x)=v^0(x, M_0)=0$ in the case A) and converges to $v^0(x, M_2)$ in the case B). On the other hand, defining $c_i=\tilde{M}^{-1}(M_i)$ $(i=0, 1, 2)$, we see that there exist $v_i(\varepsilon)(x) \equiv v(\varepsilon, r; c_i, r)(x)$ $(i=0, 1, 2)$, solutions of (2.2), for small ε such that $v_i(0)(x)=R(c_i, r)(x)$. Thus, when ε is sufficiently small, we expect that A) implies $u(t, x; \varepsilon) \to v_0(\varepsilon)(x)=0$ and that B) implies $u(t, x; \varepsilon) \to v_2(\varepsilon)$. Here in the critical case C), we expect nothing for the behavior of $u(t, x; \varepsilon)$ only from the approximate function $u^0(t, M(\varepsilon t), x)$. Figure 16 shows that the approximate function $u^0(t, M(\varepsilon t), x)$ agrees well for all $t>0$ with numerical solutions of (1.1), (1.2), (1.3) which are solved by a finite difference scheme. Solid curves represent $\int_I u(t, x; \varepsilon)dx$ derived from the numerical solutions with $\varepsilon=0.1$ while the broken curves are the solutions of (6.9), (6.10). Though we claim that 2-timing methods presented here are applicable to obtain attractive domains of coexisting stable equilibrium solutions, the complete justification has not been yet accomplished.

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