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Generic solvability of the equations of Navier-Stokes

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1. Introduction

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain in $\mathbf{R}^3$ with a smooth boundary $\partial\Omega$; $\partial\Omega$ is of class C^∞. We consider the equations of Navier-Stokes

$$(1.1) \qquad u' - \Delta u + u \cdot \nabla u + \nabla \pi = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0$$

on the cylindrical domain $\Omega \times (0, T) \subset \mathbf{R}^4$ with some $T > 0$, and we investigate strong solutions u of (1.1); these are solutions with $u \in L^p(0, T; H^{2,p}(\Omega)^3 \cap \mathring{H}^{1,p}(\Omega)^3)$ and $u' \in L^p(0, T; L^p(\Omega)^3)$ for some p with $2 \leqq p < \infty$.

Using the projection P_p: $L^p(\Omega)^3 \to H_p(\Omega)$ from $L^p(\Omega)^3$ onto the subspace $H_p(\Omega) \subset L^p(\Omega)^3$ of divergence free functions with zero normal component on $\partial\Omega$ (in the sense of [3]), we can write (1.1) in the following equivalent form as an evolution equation in $H_p(\Omega)$:

$$(1.2) \qquad u' + A_p u + P_p(u \cdot \nabla u) = P_p f, \quad u(0) = u_0, \quad 0 \leqq t \leqq T.$$

Here A_p: $v \to A_p v$: $= -P_p \Delta v$ denotes the Stokes operator with domain $D(A_p) := H^{2,p}(\Omega)^3 \cap \mathring{H}^{1,p}(\Omega)^3 \cap H_p(\Omega)$. We can define the fractional powers A_p^α of A_p with $0 \leqq \alpha \leqq 1$ and domain $D(A_p^\alpha) \supset D(A_p)$ as in [6]. Let $f \in L^p(0, T; L^p(\Omega)^3)$ and $u_0 \in D(A_p^{1-(1/p)+\delta})$ with some δ, $0 < \delta < 1/p$ (take $u_0 \in D(A_p)$ for example). Then a strong solution u of (1.1) or (1.2) is defined by the conditions $u \in L^p(0, T; D(A_p))$, $u' \in L^p(0, T; L^p(\Omega)^3)$ and (1.2).

The existence of strong solutions of (1.1) for arbitrary $T > 0$ is an important unsolved problem up to now. Therefore it is interesting to know properties of the set

$$R(u_0) := \{f \in L^p(0, T; L^p(\Omega)^3) \mid (1.2) \text{ has a unique strong solution } u \text{ with data } f, u_0\}$$

for a fixed initial value $u_0 \in D(A_p^{1-(1/p)+\delta})$. It is not known whether or not $R(u_0) = L^p(0, T; L^p(\Omega)^3)$; however we can prove some density properties of this set. This gives us some information how many f do exist such that (1.1) is strongly solvable.

Solonnikov's theory of local solvability [10; §10] tells us that $R(u_0) \subset L^p(0, T; L^p(\Omega)^3)$ is an open subset. In case $p = 2$ it has been shown that $R(u_0)$

is dense in the space $L^s(0, T; H^{-1,2}(\Omega)^3)$ with $1 \leqq s < 4/3$, where $H^{-1,2}(\Omega)^3$ is the dual space of $\mathring{H}^{1,2}(\Omega)^3$ ([4,12]). The aim of the present paper is to prove the following general density property.

1.3. THEOREM. *Let* $2 \leqq p < \infty$ *and* $u_0 \in D(A_p^{1-(1/p)+\delta})$ *with* $0 < \delta \leqq 1/p$. *Then the set* $R(u_0) \subset L^p(0, T; L^p(\Omega)^3)$ *is dense in the norm of* $L^s(0, T; L^q(\Omega)^3)$ *for all* s, $q \in (1, \infty)$ *with* $4 < 2/s + 3/q$. *Therefore, for every* $f \in L^p(0, T; L^p(\Omega)^3)$ *and every* $\varepsilon > 0$ *there exists some* $g \in L^p(0, T; L^p(\Omega)^3)$ *with* $\|g\|_{L^s(0,T;L^q(\Omega)^3)} \leqq \varepsilon$ *such that*

$$u' + A_p u + P_p(u \cdot \nabla u) = P_p f + P_p g, \quad u(0) = u_0$$

has a unique strong solution u.

REMARKS. a) The quantity $2/s + 3/q$ plays an important rôle in Serrin's regularity theory for the equation (1.1) ([8, 16]); a weak solution u is regular if $u \in L^s(0, T; L^q(\Omega)^3)$ holds for some s, $q \in (1, \infty)$ with $2/s + 3/q \leqq 1$.

b) It can be shown that Theorem 1.3 also holds for $\delta = 0$. This extension is not difficult to prove for $p = 2$; it would require the theory of Besov spaces for $2 < p < \infty$; however this detail does not seem to be very important.

c) Let u_0 be as in Theorem 1.3 and let $f \in L^p(0, T; L^p(\Omega)^3)$. Then from 1.3 it follows in particular that for every $\varepsilon > 0$ we can always find an additional external force $g \in L^p(0, T; L^p(\Omega)^3)$ with

$$\int_0^T \int_\Omega |g(x, t)| dx dt \leqq \varepsilon$$

such that the Navier-Stokes equation $u' - \Delta u + u \cdot \nabla u + \nabla \pi = f + g$ has a unique strong solution u with $u(0) = u_0$.

Our method to prove 1.3 rests on a regularization procedure for (1.1) using the Yosida approximation (given in [8, 9] in principle) and on an estimate of the nonlinear term $u \cdot \nabla u$ using the exponent $p = 5/4$ (given in [14, 15] in principle).

NOTATIONS. For $1 < p < \infty$ and $k = 1, 2, \ldots$ we need the usual spaces $L^p(\Omega)$, $H^{k,p}(\Omega)$, $\mathring{H}^{k,p}(\Omega)$, $C^k(\Omega)$ and $\mathring{C}^k(\Omega)$. For a Banach space H, $L^p(0, T; H)$ is the usual space with the norm $\|v\|_{L^p(0,T;H)} = \left(\int_0^T \|v\|_H^p dt\right)^{1/p}$, and $C(0, T; H)$ is the space of continuous functions v: $[0, T] \to H$ with norm $\|v\|_{C(0,T;H)} = \sup_{0 \leqq t \leqq T} \|v(t)\|_H$. In our proofs it is convenient to use the notations $\|v\|_{L^p(\Omega)} = \|v\|_p$ or $\|v\|_{L^p(\Omega)} = \|v\|_{1/p}$. Similarly, we use the notations $\|v\|_{L^p(0,T;L^q(\Omega))} = \|v\|_{q,p} = \|v\|_{1/q,1/p}$ and $\|v\|_{q,\infty} = \sup_{0 \leqq t \leqq T} \|v(t)\|_q$. The corresponding spaces of vector functions $v = (v_1, v_2, v_3)$ are denoted by $L^p(\Omega)^3$, $H^{k,p}(\Omega)^3, \ldots$, respectively.

We set $D_i := \partial/\partial x_i$ $(i = 1, 2, 3, x = (x_1, x_2, x_3) \in \Omega)$, $u' := \partial/\partial t$, $\nabla := (D_1, D_2,$

D_3), div $v := D_1v_1 + D_2v_2 + D_3v_3$ ($v = (v_1, v_2, v_3)$), $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$, $u \cdot \nabla u = (u \cdot \nabla)u = (u \cdot (\nabla u_1), u \cdot (\nabla u_2), u \cdot (\nabla u_3))$ and $\langle u, v \rangle := \int_\Omega u(x) \cdot v(x) dx$.

Let $H_p(\Omega)$ be the closure of $\{u \mid u \in \mathring{C}^\infty(\Omega)^3, \text{div } u = 0\}$ with respect to the $L^p(\Omega)^3$-norm. There exists a bounded linear projection operator P_p: $L^p(\Omega)^3 \to H_p(\Omega)$, and every $v \in L^p(\Omega)^3$ possesses a decomposition $v = P_pv + \nabla \pi$ with $\pi \in H^{1,p}(\Omega)$ ([3]).

Let Δ_p: $D(\Delta_p) \to L^p(\Omega)^3$ be the usual Laplace operator in $L^p(\Omega)^3$ with $D(\Delta_p) = H^{2,p}(\Omega)^3 \cap \mathring{H}^{1,p}(\Omega)^3$ and $\Delta_p u = D_1^2u + D_2^2u + D_3^2u$. $P_p\Delta_p$: $D(P_p\Delta_p) \to H_p(\Omega)$ is the usual Stokes operator with $D(P_p\Delta_p) = D(\Delta_p) \cap H_p(\Omega)$. We set

$$A_p := -P_p\Delta_p \quad \text{and} \quad B_p := -\Delta_p.$$

In our proofs we need some well known embedding properties which follow from the ellipticity of the Laplace operator ([13]):

Suppose $1 < p \leqq q < \infty$, $0 \leqq \beta \leqq \alpha \leqq 1$, $2\alpha - 3/p \geqq 2\beta - 3/q$. Then we have

$$\|B_q^\beta v\|_q \leqq c\|B_p^\alpha v\|_p, \quad v \in D(B_p^\alpha), \tag{1.4}$$

where $c = c(p, q, \alpha, \beta, \Omega) > 0$ does not depend on v.

Using Giga's characterization $D(A_p^\alpha) = D(B_p^\alpha) \cap H_p(\Omega)$ ([6]), we see that the following holds too:

$$\|A_q^\beta v\|_q \leqq c\|A_p^\alpha v\|_p \qquad \text{for all} \quad v \in D(A_p^\alpha), \tag{1.5}$$

where q, p, β, α, c are as above.

In case $\beta = 0$, $q = \infty$, $2\alpha - 3/p > -3/q = 0$, these estimates remain valid; we get in particular $\|v\|_\infty \leqq c\|A_p^\alpha v\|_p$ in this case.

The operator $-A_p$ generates for p, $1 < p < \infty$, an analytic semigroup e^{-tA_p}, $t \geqq 0$, in $H_p(\Omega)$ ([14, 5]). Therefore, we get for every $v \in L^p(0, T; D(A_p))$ with $v' \in L^p(0, T; L^p(\Omega)^3)$ the representation

$$v(t) = e^{-tA_p}v(0) + \int_0^t e^{-(t-s)A_p}(v' + A_pv)ds \tag{1.6}$$

for almost all $t \in [0, T]$. Using (1.5) and the well known property $\|A_p^\alpha e^{-tA_p}\| \leqq ct^{-\alpha}$ ([2]), we can derive from (1.6) the following imbedding properties:

Suppose $v \in L^p(0, T; D(A_p))$, $v' \in L^p(0, T; L^p(\Omega)^3)$, $v(0) \in D(A_p^{1-1/p})$, $1 < p \leqq q < \infty$. Then we have (after redefinition on a set of measure zero)

$$v \in C(0, T; L^q(\Omega)^3), \quad \|v\|_{q,\infty} \leqq c(\|A_p^{1-1/p}v(0)\|_p + \|v'\|_{p,p} + \|A_pv\|_{p,p}) \quad \text{for} \quad 2 - 5/p > -3/q, \tag{1.7}$$

and moreover

$$D_iv \in C(0, T; L^q(\Omega)^3), \quad \|D_iv\|_{q,\infty} \leqq c(\|A_p^{1-1/p}v(0)\|_p + \|v'\|_{p,p} + \|A_pv\|_{p,p}) \tag{1.8}$$

$$\text{for} \quad 2-5/p>1-3/q, \quad i=1,2,3,$$

where $c=c(p, q, \Omega)$ does not depend on T since Ω is bounded and $\|e^{-tA_p}\|$ decays exponentially. In case $p=2$, it can be shown by using the scalar product that (1.8) also holds in case $2-5/p=1-3/q$, i.e. $q=2$. The continuity assertion on v and $D_i v$ follows from the continuity of $J_k v$ resp. $D_i J_k v$ by letting $k \to \infty$ and using the estimates above with $J_k v$ instead of v; $J_k v$ is the Yosida approximation to be introduced later.

The linearized equation for (1.2) is given by

$$u' + A_p u = P_p f, \quad u(0) = u_0, \quad 0 \leqq t \leqq T$$

in the space $H_p(\Omega)$. Let $f \in L^p(0, T; L^p(\Omega)^3)$ and $v(t) := \int_0^t e^{-(t-s)A_p} P_p f ds$. Then the estimate

$$\|v'\|_{p,p} + \|A_p v\|_{p,p} \leqq c\|f\|_{p,p}$$

with $c=c(p, \Omega)>0$ has been developed by Solonnikov ([10]). Using the property $\|A_p^{1-(1/p)+\delta} e^{-tA_p}\| \leqq c t^{-(1-(1/p)+\delta)}$, we get easily the estimate $\left(\int_0^T \|A_p e^{-tA_p} u_0\|_p^p dt\right)^{1/p} \leqq c\|A_p^{1-(1/p)+\delta} u_0\|_p$ with $0<\delta \leqq 1/p$.

Therefore, for all $f \in L^p(0, T; L^p(\Omega)^3)$ and $u_0 \in D(A_p^{1-(1/p)+\delta})$ with $0<\delta \leqq 1/p$, we obtain a unique solution

$$u: t \longrightarrow u(t) = e^{-tA_p} u_0 + \int_0^t e^{-(t-s)A_p} P_p f ds$$

of $u' + A_p u = P_p f$, $u(0) = u_0$, and it holds

$$\|u'\|_{p,p} + \|A_p u\|_{p,p} \leqq c(\|A_p^{1-(1/p)+\delta} u_0\|_p + \|f\|_{p,p}) \tag{1.9}$$

with $c=c(p, \Omega)>0$.

In fact, (1.9) holds for $2 \leqq p < \infty$ also with $\delta=0$. This follows for $p=2$ rather elementary using the scalar product and the self-adjointness of A_2, and for $2<p<\infty$ it follows from the imbedding property $D(A_p^{1-1/p}) \subset B^{1-1/p,p}$ where $B^{1-1/p,p}$ is a certain Besov space (a similar argument has been used in [8; p. 362]). However, we omit the details.

For $p=2$, we get instead of (1.8) the estimate

$$\|A_2^{1/2} u(t)\|_2 \leqq c(\|A_2^{1/2} u_0\|_2 + \|u'\|_{2,2} + \|A_2 u\|_{2,2}) \tag{1.10}$$

with some $c>0$.

In the following $c, c_1, c_2, \ldots$ are always positive constants whose values may change.

2. Proof of the main theorem

The proof of Theorem 1.3 rests on the regularization of (1.1) by the Yosida approximation similar as in [8] and [9]. From well known semigroup properties of e^{-tA_p} $(t \geqq 0)$ we get easily that the operators

$$J_k := (I + k^{-1}A_p)^{-1}, \quad k = 1, 2, \dots$$

fulfill the following conditions: $\|J_k\| \leqq c$ where $c = c(p, \Omega) > 0$ does not depend on k, and $\lim_{k \to \infty} J_k v = v$ for all $v \in H_p(\Omega)$. J_k approximates the identity operator I in the strong sense.

An important property is the estimate

$$\|A_p^\alpha J_k\| \leqq ck^\alpha \tag{2.1}$$

where $c = c(p, \Omega) > 0$ and $0 \leqq \alpha \leqq 1$ ([2, 17]).

The idea of the proof is to solve in the strong sense the regularized Navier-Stokes equation

$$u' + A_p u + P_p[(J_k u) \cdot \nabla u] = P_p f, \quad u(0) = u_0 \tag{2.2}$$

instead of (1.2). Then we write (2.2) in the form

$$u' + A_p u + P_p[u \cdot \nabla u] = P_p f + P_p[(I - J_k)u \cdot \nabla u]$$

and show that the term $P_p[(I - J_k)u \cdot \nabla u]$ tends to zero as $k \to \infty$ in the space $L^s(0, T; L^q(\Omega)^3)$ with $4 < 2/s + 3/q$; this will prove the theorem.

The next lemma yields the solvability of (2.2) in the strong sense for each $k = 1, 2, \dots$.

2.3. Lemma. *Let $2 \leqq p < \infty$, $f \in L^p(0, T; L^p(\Omega)^3)$, and $u_0 \in D(A_p^{1-(1/p)+\delta})$ with $0 < \delta \leqq 1/p$. Then for each fixed $k = 1, 2, \dots$, there exists a unique $u \in L^p(0, T; D(A_p))$ which fulfills $u' \in L^p(0, T; L^p(\Omega)^3)$ and (2.2). It holds the energy equality*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \|u_0\|_2^2 + 2\int_0^t \langle f(\tau), u(\tau)\rangle d\tau \tag{2.4}$$

and therefore the inequality

$$\|u(t)\|_2^2 + c_1\|\nabla u\|_{2,2}^2 \leqq \|u_0\|_2^2 + c_2\|f\|_{2,2}^2 \tag{2.5}$$

where $c_1 = c_1(\Omega) > 0$ and $c_2 = c_2(\Omega) > 0$ depend only on Ω.

Proof. We solve (2.2) by Banach's fixed point theorem; however for technical reasons we start with regularized initial values $J_m u_0$ instead of u_0. Thus we solve the equations

$$(2.6) \qquad u' + A_p u + P_p[(J_k u)\cdot \nabla u] = P_p f, \quad u(0) = J_m u_0$$

for fixed k, $m=1, 2,\dots$ in the strong sense (i.e. $u \in L^p(0, T; D(A_p))$ and $u' \in L^p(0, T; L^p(\Omega)^3)$). The solution u depends on k, m; later on we get the desired solution of (2.2) by letting $m\to\infty$.

Instead of (2.6) we can solve the equivalent integral equation

$$(2.7) \qquad u(t) = e^{-tA_p}J_m u_0 + \int_0^t e^{-(t-\tau)A_p}(P_p f - P_p[(J_k u)\cdot \nabla u])d\tau, \quad 0 \leqq t \leqq T.$$

This equation can be solved using Banach's fixed point theorem. To show this, we have first to estimate the nonlinear term $P_p[(J_k u)\cdot \nabla u]$; in particular from this estimate it will follow that $P_p[(J_k u)\cdot \nabla u] \in L^p(0, T; L^p(\Omega)^3)$ is well defined for strong solutions u.

For $2<p<\infty$ we can choose some r with $2-5/p>1-3/r$ and $2<p<r<\infty$, and for $p=2$ we choose $r=2$. Then we obtain from (1.5), (1.7), (1.8), and (2.1) the following estimates for the nonlinear term:

$$\|P_p[(J_k u)\cdot \nabla u]\|_p \leqq c_1\|(J_k u)\cdot \nabla u\|_p \leqq c_2\|J_k u\|_{1/p-1/r}\|\nabla u\|_{1/r},$$

$$\|\nabla u\|_{r,\infty} \leqq c_3(\|A_p^{1-1/p}J_m u_0\|_p + \|u'\|_{p,p} + \|A_p u\|_{p,p}),$$

$$\|J_k u\|_{1/p-1/r} \leqq c_4\|A_p^{3/2r}J_k u\|_{1/p} \leqq c_5 k^{3/2r}\|u\|_{1/p},$$

$$\|u\|_{p,\infty} \leqq c_6(\|A_p^{1-1/p}J_m u_0\|_p + \|u'\|_{p,p} + \|A_p u\|_{p,p}),$$

$$\begin{aligned}\|P_p[(J_k u)\cdot \nabla u]\|_{p,p} &\leqq c_2\left(\int_0^T \|J_k u\|^p_{1/p-1/r}\|\nabla u\|^p_{1/r}dt\right)^{1/p}\\ &\leqq c_7 T^{1/p}\|u\|_{p,\infty}\|\nabla u\|_{r,\infty}\\ &\leqq c_8 T^{1/p}(\|A_p^{1-1/p}J_m u_0\|_p + \|u'\|_{p,p} + \|A_p u\|_{p,p})^2.\end{aligned}$$

Thus we obtain

$$(2.8) \quad \|(J_k u)\cdot \nabla u\|_{L^p(0,T;L^p(\Omega)^3)} \leqq c T^{1/p}(\|A_p^{1-1/p}J_m u_0\| + \|u'\|_{p,p} + \|A_p u\|_{p,p}),$$

where $c=c(p, k, \Omega)>0$ still depends on k but not on T.

In particular we get $P_p[(J_k u)\cdot \nabla u] \in L^p(0, T; L^p(\Omega)^3)$ whenever $u \in L^p(0, T; D(A_p))$, $u' \in L^p(0, T; L^p(\Omega)^3)$.

At first we solve (2.7) with $p=2$ and fixed m, k by Banach's fixed point theorem. For this purpose we set

$$(Fu)(t) := e^{-tA_2}J_m u_0 + \int_0^t e^{-(t-\tau)A_2}(P_2 f - P_2[(J_k u)\cdot \nabla u])d\tau,$$

write (2.7) in the form $u=Fu$, and we apply the fixed point theorem to the mapping $F\colon u\to Fu$ defined on the set

$$\mathscr{C}_R(u_0, T_1) := \{u \in L^2(0, T_1; D(A_2)) \mid u' \in L^2(0, T_1; L^2(\Omega)^3), \quad u(0) = J_m u_0, \\ \|u'\|_{2,2} + \|A_2 u\|_{2,2} \leqq R\}.$$

We show that the conditions of this theorem are fulfilled for some $R>0$ and some sufficiently small $T_1>0$ with $T_1 \leqq T$; the metric on $\mathscr{C}_R(u_0, T_1)$ is given by $\|u-\tilde{u}\|^* := \|u'-\tilde{u}'\|_{2,2} + \|A_2 u - A_2 \tilde{u}\|_{2,2}$.

The applicability of the fixed point theorem can be derived from the following inequalities

$$(2.9) \quad \begin{cases} \|Fu\|^* = \|Fu-0\|^* \\ \quad \leqq c_1(\|A_2^{1/2} J_m u_0\|_2 + \|f\|_{2,2}) + c_2 T_1^{1/2} \\ \qquad \cdot (\|A_2^{1/2} J_m u_0\|_2 + \|u'\|_{2,2} + \|A_2 u\|_{2,2})^2, \\ \|Fu - F\tilde{u}\|^* \\ \quad \leqq c_3(\|u'-\tilde{u}'\|_{2,2} + \|A_2 u - A_2 \tilde{u}\|_{2,2}) \\ \qquad \cdot (\|A_2^{1/2} J_m u_0\|_2 + \|u'\|_{2,2} + \|\tilde{u}'\|_{2,2} + \|A_2 u\|_{2,2} + \|A_2 \tilde{u}\|_{2,2}) T_1^{1/2} \end{cases}$$

where $c_\nu = c_\nu(k, m, \Omega) > 0$ ($\nu = 1, 2, 3$) depends on k and m.

We obtain (2.9) by applying (1.9) and (2.8) to (2.7) in the following way ($\delta = 0$ for $p=2$):

$$\begin{aligned} \|Fu\|^* &\leqq c_5(\|A_2^{1/2} J_m u_0\|_2 + \|f\|_{2,2} + \|(J_k u) \cdot \nabla u\|_{2,2}) \\ &\leqq c_6(\|A_2^{1/2} J_m u_0\|_2 + \|f\|_{2,2}) \\ &\quad + c_7 T_1^{1/2} (\|A_2^{1/2} J_m u_0\|_2 + \|u'\|_{2,2} + \|A_2 u\|_{2,2})^2, \\ \|Fu - F\tilde{u}\|^* &\leqq c_8 \|(J_k u) \cdot \nabla u - (J_k \tilde{u}) \cdot \nabla \tilde{u}\|_{2,2} \\ &\leqq c_9(\|(J_k(u-\tilde{u}) \cdot \nabla u\|_{2,2} + \|(J_k \tilde{u}) \cdot \nabla (u-\tilde{u})\|_{2,2}). \end{aligned}$$

The last term can be estimated in the same way as in (2.8); we get

$$\begin{aligned} &\|(J_k(u-\tilde{u})) \cdot \nabla u\|_{2,2} \\ &\quad \leqq c_{10} T_1^{1/2} (\|u'-\tilde{u}'\|_{2,2} + \|A_2 u - A_2 \tilde{u}\|_{2,2}) \cdot (\|A_2^{1/2} J_m u_0\|_2 + \|u'\|_{2,2} + \|A_2 u\|_{2,2}) \\ &\quad \leqq c_{11} T_1^{1/2} (\|u'-\tilde{u}'\|_{2,2} + \|A_2 u - A_2 \tilde{u}\|_{2,2}) \cdot (\|A_2^{1/2} J_m u_0\|_2 + \|u'\|_{2,2} + \|A_2 u\|_{2,2}), \\ &\|(J_k \tilde{u}) \cdot \nabla (u-\tilde{u})\|_{2,2} \\ &\quad \leqq c_{12} T_1^{1/2} (\|A_2^{1/2} J_m u_0\|_2 + \|\tilde{u}'\|_{2,2} + \|A_2 \tilde{u}\|_{2,2})(\|u'-\tilde{u}'\|_{2,2} + \|A_2 u - A_2 \tilde{u}\|_{2,2}). \end{aligned}$$

Thus we get the inequalities (2.9).

From (2.9) we conclude the applicability of the fixed point theorem with $R := 2c_1(\|A_2^{1/2} J_m u_0\|_2 + \|f\|_{2,2})$ and some sufficiently small $T_1 > 0$; we obtain a unique strong solution u of (2.7) on the interval $[0, T_1]$. In order to repeat this procedure on a second interval etc., we need the energy inequality which prevents the blow up of the solution before it reaches the point T.

Taking the scalar product of (2.6) with u, we obtain

$$\begin{aligned}\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 d\tau &= \|J_m u_0\|_2^2 + 2\int_0^t \langle f, u\rangle d\tau \\ &\leqq c_1\|u_0\|_2^2 + 2\int_0^t \|f\|_2\|u\|_2 d\tau \\ &\leqq c_1\|u_0\|_2^2 + c_2\varepsilon^{-2}\int_0^t \|f\|_2^2 d\tau + c_3\varepsilon^2\int_0^t \|u\|_2^2 d\tau \\ &\leqq c_1\|u_0\|_2^2 + c_2\varepsilon^{-2}\int_0^t \|f\|_2^2 d\tau + c_4\varepsilon^2\int_0^t \|\nabla u\|_2^2 d\tau\end{aligned}$$

for arbitrary $\varepsilon>0$. For some appropriate $\varepsilon>0$ we obtain

$$\|u(t)\|_2^2 + c_5\int_0^t \|\nabla u\|_2^2 d\tau \leqq c_6\|u_0\|_2^2 + c_7\int_0^t \|f\|_2^2 d\tau, \tag{2.10}$$

where $c_5, c_6, c_7>0$ depend only on Ω.

Using this energy inequality and (1.10) we obtain

$$\begin{aligned}\|A_2^{1/2}u(T_1)\|_2 = \|A_2^{1/2}(Fu)(T_1)\|_2 &\leqq c_1\|A_2^{1/2}J_m u_0\|_2 \\ &\quad + c_2\left(\int_0^{T_1} (\|f\|_2^2 + \|(J_k u)\cdot\nabla u\|_2^2)d\tau\right)^{1/2} \\ &\leqq c_1\|A_2^{1/2}J_m u_0\|_2 + c_3\left(\int_0^{T_1} \|f\|_2^2 d\tau\right)^{1/2} \\ &\quad + c_4\left(\int_0^{T_1} \|J_k u\|_\infty^2\|\nabla u\|_2^2 d\tau\right)^{1/2},\end{aligned}$$

$$\|J_k u\|_\infty \leqq c_5\|A_2 J_k u\|_2 \leqq c_6\|u\|_2,$$

$$\begin{aligned}\left(\int_0^{T_1} \|J_k u\|_\infty^2\|\nabla u\|_2^2 d\tau\right)^{1/2} &\leqq c_7(\sup_{0\leqq t\leqq T_1}\|u(t)\|_2^2)^{1/2}\left(\int_0^{T_1} \|\nabla u\|_2^2 d\tau\right)^{1/2} \\ &\leqq c_8\left(\|u_0\|_2^2 + \int_0^{T_1} \|f\|_2^2 d\tau\right).\end{aligned}$$

Thus it follows

$$\|A_2^{1/2}u(T_1)\|_2 \leqq c_9\left(\|u_0\|_2 + \left(\int_0^{T_1} \|f\|_2^2 d\tau\right)^{1/2} + \|u_0\|_2^2 + \int_0^{T_1} \|f\|_2^2 d\tau\right). \tag{2.11}$$

Now we can repeat the above construction of the strong solution for the next interval $[T_1, T_2]$ with the initial value $u(T_1)$ instead of $J_m u_0$, and so forth. This is possible because the right hand side of (2.10) depends only on the data f, u_0. Therefore in (2.9) we may insert $u(T_1)$ instead of $J_m u_0$, and we see the following: T_1, $T_2, \ldots$ may be chosen so that all the intervals $[T_{\nu-1}, T_\nu]$ have the same length. In this way, we get a unique strong solution of (2.7) on the whole interval $[0, T]$. Let u_m be this solution for $m=1, 2, \ldots$ and fixed k.

In the next step we show $u_m \in L^p(0, T; D(A_p))$ and $u'_m \in L^p(0, T; L^p(\Omega)^3)$. For this purpose we have only to give a bound for $\|u'_m\|_{p,p} + \|A_p u_m\|_{p,p}$ on $[0, T]$. Moreover, we show that this bound is independent of m. This enables us to let $m \to \infty$, and in this way we obtain a strong solution of (2.2).

To find such a bound, we give another estimate of $\|(J_k u_m)\cdot \nabla u_m\|_{p,p}$. We can choose r and a with $p<r<\infty$, $1/2<a<1$ and with $a(1/p-2/3)+(1-a)/2=1/r-1/3$, and we get from Sobolev's embedding theorem [4; p. 24] the estimate $\|\nabla u_m\|_r \leqq c_1 \|\Delta u_m\|_p^a \|u_m\|_2^{1-a}$. Using $(2/3)(3/2)(1/2-(1/p-1/r))-1/2=-(1/p-1/r)$ and $(3/2)(1/2-(1/p-1/r)) \leqq 1$, we get from (1.5) the inequality $\|J_k u_m\|_{1/p-1/r} \leqq c_2 \|A_2^{(3/2)(1/2-(1/p-1/r))} J_k u_m\|_2 \leqq c_3 \|u_m\|_2$ where $c_3 = c_3(p, r, \Omega) > 0$. Therefore we obtain

$$\begin{aligned}\|(J_k u_m)\cdot \nabla u_m\|_p &\leqq c_4 \|J_k u_m\|_{1/p-1/r} \|\nabla u_m\|_{1/r} \\ &\leqq c_5 \|u_m\|_2^{2-a} \|A_p u_m\|_p^a,\end{aligned}$$

and for any $\varepsilon>0$ it follows

$$\|(J_k u_m)\cdot \nabla u_m\|_p^p \leqq c_6 \varepsilon^{1/a} \|A_p u_m\|_p^p + c_7 \varepsilon^{-1/(1-a)} \| u_m\|_2^{(2-a)p/(1-a)}.$$

Applying (1.9) to (2.6) and using the last estimate, we obtain for some sufficiently small $\varepsilon>0$ the inequalities

$$\|u'_m\|_{p,p} + \|A_p u_m\|_{p,p} \leqq c_1(\|A_p^{1-(1/p)+\delta} J_m u_0\|_p + \|f\|_{p,p} + \|(J_k u_m)\cdot \nabla u_m\|_{p,p})$$

and

$$\|u'_m\|_{p,p} + \|A_p u_m\|_{p,p} \leqq c_2(\|A_p^{1-1/p+\delta} u_0\|_p + \|f\|_{p,p} + \|u_m\|_{2,\infty}^{(2-a)/(1-a)})$$

where $c_2 = c_2(p, k, \Omega, T) > 0$ is independent of m. Together with (2.10), we get from the last inequality a bound for $\|u'_m\|_{p,p} + \|A_p u_m\|_{p,p}$ which does not depend on m.

We can now choose a subsequence (u_{m_j}) of (u_m) such that

$$\begin{aligned} u'_{m_j} &\longrightarrow u' \quad \text{in} \quad L^p(0, T; L^p(\Omega)^3), \\ A_p u_{m_j} &\longrightarrow A_p u \quad \text{in} \quad L^p(0, T; L^p(\Omega)^3). \end{aligned}$$

As for the nonlinear term we get $(J_k u_m)\cdot \nabla u_{m_j} \rightharpoonup v$ in $L^p(0, T; L^p(\Omega)^3)$. If $\varphi \in C_0^\infty((0, T)\times\Omega)^3$, we have

$$\int_0^T \int_\Omega ((J_k u_{m_j})\cdot \nabla u_{m_j})\varphi \, dxdt \longrightarrow \int_0^T \int_\Omega ((J_k u)\cdot \nabla u)\varphi dxdt$$

since $\nabla u_{m_j} \rightharpoonup \nabla u$ in $L^2(0, T; L^2(\Omega)^3)$ and $J_k u_{m_j} \to J_k u$ in $L^2(0, T; L^2(\Omega)^3)$ by Rellich's theorem. Thus $v = J_k u \cdot \nabla u$ and, in particular, $P_p(J_k u_{m_j} \cdot \nabla u_{m_j}) \rightharpoonup P_p(J_k u \cdot \nabla u)$ in $L^p(0, T; L^p(\Omega)^3)$. Finally we arrive at $u' + A_p u + P_p(J_k u \cdot \nabla u) = f$, $u' \in L^p(0, T; L^p(\Omega)^3)$, $A_p u \in L^p(0, T; L^p(\Omega)^3)$, $u(0) = u_0$. The u of course obeys the same bound as the u_m. Let us remark that without loss of generality we have always chosen the same subsequence of (u_m).

To show the uniqueness, we consider two strong solutions u and $\tilde{u}$ with the same data u_0, f. Then we get

$$(u - \tilde{u})' + A_p(u - \tilde{u}) = P_p[(J_k(\tilde{u} - u)) \cdot \nabla u] + P_p[(J_k \tilde{u}) \cdot \nabla(\tilde{u} - u)].$$

Using (1.9), it follows

$$\begin{aligned} \|u' - \tilde{u}'\|^p_{L^p(0,t;L^p)} + \|A_p(u - \tilde{u})\|^p_{L^p(0,t;L^p)} \leqq c_1(&\|(J_k(\tilde{u} - u)) \cdot \nabla u\|^p_{L^p(0,t;L^p)} \\ &+ \|(J_k \tilde{u}) \cdot \nabla(\tilde{u} - u)\|^p_{L^p(0,t;L^p)}) \end{aligned}$$

for $0 \leqq t \leqq T$, where c_1 is independent of t.

We set $y(t) := \|u' - \tilde{u}'\|^p_{L^p(0,t;L^p)} + \|A_p(u - \tilde{u})\|^p_{L^p(0,t;L^p)}$.

The same estimates which we have used for (2.8) yield the inequality $y(t) \leqq c \int_0^t y(\tau) d\tau$. In order to show this we use the same notation as in the proof of (2.8) and obtain:

$$\begin{aligned} \|(J_k(\tilde{u} - u)) \cdot \nabla u\|^p_{L^p(0,t,;L^p)} &\leqq c_2 \int_0^t \|J_k(\tilde{u} - u)\|^p_{1/p - 1/r} \|\nabla u\|^p_{1/r} d\tau \\ &\leqq c_3 \int_0^t \|\tilde{u} - u\|^p_p \|\nabla u\|^p_r d\tau. \end{aligned}$$

Using (1.7) and (1.8), the last expression is

$$\begin{aligned} &\leqq c_4 \int_0^t y(\tau) \|\nabla u\|^p_r d\tau \\ &\leqq c_5 \left(\int_0^t y(\tau) d\tau\right)\left(\|A_p^{1-1/p} u_0\|^p_p + \int_0^T (\|u'\|^p_p + \|A_p u\|^p_p) d\tau\right) \\ &\leqq c_6 \int_0^t y(\tau) d\tau. \end{aligned}$$

In the same way it follows

$$\|(J_k \tilde{u}) \cdot \nabla(\tilde{u} - u)\|^p_{L^p(0,t;L^p)} \leqq c_7 \int_0^t y(\tau) d\tau$$

and thus we obtain the inequality $y(t) \leqq c \int_0^t y(\tau) d\tau$. Together with $y(0) = 0$ we get that $y(t)=0$ for all $t\in[0, T]$; it follows $u=\tilde{u}$ by Gronwall's inequality.

The energy equality (2.4) follows by taking the scalar product of (2.1) with u. Lemma 2.3 is proved.

Let us make two remarks: First we want to explain why we have used regularized initial values $J_m u_0$. The reason is simply that in the second part of the preceding proof we need an initial value in $D(A_p^{1-(1/p)+\delta})$, whereas in the first part it is sufficient to have $u_0\in D(A_2^{1/2})$. Secondly, it follows from the linear theory that the solution in Lemma 2.3 is in $C^0((0, T), D(A_2^{1/2}))$.

PROOF OF THEOREM 1.3. Let f, u_0 and s, q be as in 1.3, and let u_k be the strong solution of (2.2) for $k=1, 2,\dots$. We write (2.2) in the form

$$u_k' + A_p u_k + P_p[u_k\cdot\nabla u_k] = P_p f + P_p[(I-J_k)u_k\cdot\nabla u_k]$$

and show that $g_k := P_k[(I-J_k)u_k\cdot\nabla u_k]$ belongs to $L^p(0, T; L^p(\Omega)^3)$ and tends to zero in $L^s(0, T; L^q(\Omega)^3)$ as $k\to\infty$. Then we have proved the last assertion of 1.3. However, because $2/s+3/q>4$, we see that $s<2$ and $q<2$; it follows that $s\leqq p$, $q\leqq p$ and therefore, that $L^p(0, T; L^p(\Omega)^3)$ is contained in $L^s(0, T; L^q(\Omega)^3)$ as a dense subset. Thus we obtain the first assertion of 1.3 too.

As for the main part we show first that $g_k\in L^p(0, T; L^p(\Omega)^3)$. To prove this we choose $r=2$ in case $p=2$ and $r>p$ with $2-5/p>1-3/r$ in case $2<p$. Then from (1.8) we obtain

$$\|\nabla u_k\|_{r,\infty} \leqq c_1(\|A_p^{1-1/p}u_0\|_p + \|u_k'\|_{p,p} + \|A_p u_k\|_{p,p})$$

and using $2/3-1/p>-(1/p-1/r)$ and (1.5), we arrive at

$$\|u_k\|_{1/p-1/r} \leqq c_2\|A_p u_k\|_p .$$

Therefore we get

$$\begin{aligned}\|g_k\|_{p,p} &\leqq c_3\Big(\int_0^T \|(I-J_k)u_k\cdot\nabla u_k\|_p^p dt\Big)^{1/p}\\ &\leqq c_4\Big(\int_0^T \|u_k\|_{1/p-1/r}^p\|\nabla u_k\|_{1/r}^p dt\Big)^{1/p}\\ &\leqq c_5(\|A_p^{1-1/p}u_0\|_p + \|u_k'\|_{p,p} + \|A_p u_k\|_{p,p})\Big(\int_0^T \|A_p u_k\|_p^p dt)\Big)^{1/p}.\end{aligned}$$

Because u_k is a strong solution of (2.2) it follows $g_k\in L^p(0, T; L^p(\Omega)^3)$.

In order to show that $g_k\to 0$ in $L^s(0, T; L^q(\Omega)^3)$, we take an $r>q$ with $4=2/s+3/r$; this is possible because $4<2/s+3/q$. Then we can choose $\alpha\in(0, 1)$ with $(3/2)(1-1/r)-\alpha\geqq(3/2)(1-1/q)$ and we get

$$(2/3)((3/2)(1-1/r)-\alpha) - 1/2 \geqq (2/3)(3/2)(1-1/q) - 1/2 = -(1/q-1/2)$$

Using $I-J_k=(1/k)A_pJ_k$, (2.1) and (1.5), we obtain the following estimates:

$$\begin{aligned}\|g_k\|_q &= \|P_p[((I-J_k)u_k\cdot\nabla u_k]\|_q \leqq c_1\|((I-J_k)u_k)\cdot\nabla u_k\|_q\\ &= c_1\|((1/k)A_pJ_kA_p^{-\alpha}A_p^{\alpha}u_k)\cdot\nabla u_k\|_q = c_1\|((1/k)A_p^{1-\alpha}J_kA_p^{\alpha}u_k)\cdot\nabla u_k\|_q\\ &\leqq c_2\|(1/k)A_p^{1-\alpha}J_kA_p^{\alpha}u_k\|_{1/q-1/2}\|\nabla u_k\|_{1/2}\\ &\leqq c_3\|(1/k)A_p^{1-\alpha}J_k\|\,\|A_p^{\alpha}u_k\|_{1/q-1/2}\|\nabla u_k\|_{1/2}\\ &\leqq c_4k^{-\alpha}\|A_p^{\alpha}u_k\|_{1/q-1/2}\|\nabla u_k\|_{1/2}\\ &\leqq c_5k^{-\alpha}\|A_2^{(3/2)(1-1/r)-\alpha}A_2^{\alpha}u_k\|_2\|A_2^{1/2}u_k\|_2\\ &= c_5k^{-\alpha}\|A_2^{(3/2)(1-1/r)}u_k\|_2\|A_2^{1/2}u_k\|_2\\ &\leqq c_6k^{-\alpha}\|A_2^{1/2}u_k\|_2^{3(1-1/r)}\|u_k\|_2^{1-3(1-1/r)}\|A_2^{1/2}u_k\|_2\\ &= c_6k^{-\alpha}\|A_2^{1/2}u_k\|_2^{1+3(1-1/r)}\|u_k\|_2^{1-3(1-1/r)}.\end{aligned}$$

Here we have used that $A_p^{\beta}v=A_2^{\beta}v$ holds for $v\in D(A_p^{\beta})\cap D(A_2^{\beta})$, $0\leqq\beta\leqq 1$.

Using $2=s(1+3(1-1/r))$ (because $4=2/s+3/r$) and $s(1+3(1-1/r))=2s-2$ we obtain

$$\begin{aligned}\|g_k\|_{q,s} &= \left(\int_0^T \|g_k\|_q^s dt\right)^{1/s} \leqq c_6k^{-\alpha}\left(\int_0^T \|A_2^{1/2}u_k\|_2^{s(1+3(1-1/r))}\|u_k\|_2^{s(1-3(1-1/r))}dt\right)^{1/s}\\ &\leqq c_6k^{-\alpha}\left(\int_0^T \|A_2^{1/2}u_k\|_2^2 dt\right)^{1/s}\|u_k\|_{2,\infty}^{2-2/s}\\ &= c_6k^{-\alpha}\|A_2^{1/2}u_k\|_{2,2}^{2/s}\|u_k\|_{2,\infty}^{2-2/s}.\end{aligned}$$

The energy inequality (2.5) shows that

$$\sup_k(\|A_2^{1/2}u_k\|_{2,2}^{2/s}\|u_k\|_{2,\infty}^{2-2/s}) = \sup_k(\|\nabla u_k\|_{2,2}^{2/s}\|u_k\|_{2,\infty}^{2-2/s})$$

has a bound which is independent of k. Thus we see that $\|g_k\|_{q,s}\to 0$ as $k\to\infty$, and Theorem 1.3 is proved.

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