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Abstract Cauchy problems for second order linear differential equations in a Banach space

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1. Introduction

Let X be a Banach space with norm $\|\cdot\|$ and we denote by $B(X)$ the set of all bounded linear operators on X to X. A one-parameter family $C=\{C(t);\ t\in\mathbf{R}=(-\infty,\infty)\}$ in $B(X)$ is called a *cosine family* on X if it satisfies the following three conditions:

(1.1) $$C(t+s)+C(t-s)=2C(t)C(s) \text{ for all } t, s\in\mathbf{R};$$

(1.2) $$C(0)=I \text{ (the identity operator)};$$

(1.3) $$C(t) \text{ is strongly continuous in } t.$$

The associated *sine family* $S=\{S(t);\ t\in\mathbf{R}\}$ is the one-parameter family given by

$$S(t)=\int_0^t C(s)ds.$$

The (infinitesimal) *generator* A of a cosine family C is defined by

(1.4) $$Ax=\lim_{h\to 0}2h^{-2}(C(h)-I)x$$

whenever the limit exists. Hence the set of elements x for which $\lim_{h\to 0}2h^{-2}\cdot(C(h)-I)x$ exists is the domain of A and is denoted by $D(A)$.

The following theorem was established by Sova [12], Da Prato-Giusti [1] and Fattorini [2]. It is analogous to the Hille-Yosida theorem on the generation of semigroups of class (C_0).

THEOREM 1.1. *Let A be a closed and densely defined linear operator in X. Then A is the generator of a cosine family C satisfying*

$$\|C(t)\|\leqq Me^{\omega|t|} \quad \text{for} \quad t\in\mathbf{R},$$

if and only if for all λ with $\lambda>\omega$,

(1.5) $$\lambda^2\in\rho(A) \quad (\text{the resolvent set of } A),$$

(1.6) $$\|(d/d\lambda)^n[\lambda R(\lambda^2;A)]\|\leqq Mn!(\lambda-\omega)^{-n-1}$$

for $n \in \mathbf{N} = \{0, 1, 2, \dots\}$, *where* $R(\lambda^2; A) = (\lambda^2 - A)^{-1}$ (*the resolvent of* A).

Let A be a linear operator in X and consider the Cauchy problem for the evolution equation of second order in time

$$u''(t) = Au(t), \quad u(0) = x, \quad u'(0) = y. \tag{1.7}$$

The problem (1.7) is uniformly well-posed (see [2] and [13]) if and only if A generates a cosine family C; in this case the unique solution of (1.7) is given by $u(t; x, y) = C(t)x + S(t)y$. For first order evolution equations of the form

$$u'(t) = Au(t), \quad u(0) = x, \tag{1.8}$$

several authors have treated the case where $\rho(A) = \emptyset$ (see e.g. [7], [8], [9] and [10]). However, it seems that there has been no attempt to consider the corresponding problems in the second order case.

In this paper we make an attempt to treat the Cauchy problem (1.7) in the case in which $\rho(A) = \emptyset$. We proceed with our argument as follows: Let A be a closed linear operator in X and let Y be a linear manifold of X. We then impose on them the following conditions:

(a) Y is a normed space under a certain norm $|||\cdot|||$ which is stronger than the original norm $\|\cdot\|$ of X;

(b) there exists a real ω such that for each $\lambda > \omega$, the range $R(\lambda^2 - A)$ contains Y, $R(\lambda^2) = (\lambda^2 - A)^{-1}$ exists, and such that Y is invariant under $R(\lambda^2)$;

(c) there exists a constant $M > 0$ such that

$$\left\| \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda^{n-k} (J_\lambda - I)^k x \right\| \leqq M \left(\frac{\lambda}{\lambda - \omega} \right)^n |||x|||,$$

for $x \in Y$, $\lambda > \omega$ and $n \in \mathbf{N}$, where $J_\lambda = \lambda^2 R(\lambda^2)$.

Under these conditions and an assumption on the denseness of Y_2 in Y (see §3), there is a one-parameter family $\{C(t); t \in \mathbf{R}\}$ of linear operators defined on Y such that $C(t)x$ is a solution of (1.7) with $x \in Y$ and $y = 0$.

In addition, we give another proof for the "if" part of Theorem 1.1. It should be noted that we do not make use of the Laplace transform in this proof. In §4 we shall construct approximation schemes for a cosine family C in terms of the resolvent of its generator.

2. Preliminaries

Let C be a cosine family on X and S the associated sine family. Then by condition (1.1), we have

$$C(t) = C(-t), \quad S(t) = -S(-t), \quad t \in \mathbf{R}. \tag{2.1}$$

Let A be the generator of C. Then we have for $x \in D(A)$ and $t \in \mathbf{R}$,

$$AC(t)x = C(t)Ax, \quad C'(t)x = AS(t)x = S(t)Ax. \tag{2.2}$$

Moreover, under the assumpition of Theorem 1.1, we have

$$\|C(t)\| \leqq Me^{\omega|t|}, \quad t \in \mathbf{R}, \tag{2.3}$$

$$\lambda R(\lambda^2; A) = \int_0^\infty e^{-\lambda t} C(t)dt, \quad \lambda > \omega; \tag{2.4}$$

and hence

$$\|S(t)\| \leqq M|t|e^{\omega|t|}, \quad t \in \mathbf{R}, \tag{2.5}$$

$$R(\lambda^2; A) = \int_0^\infty e^{-\lambda t} S(t)dt, \quad \lambda > \omega. \tag{2.6}$$

Now let A be a closed linear operator in X and let Y be a linear manifold of X. We impose the following conditions on A and Y:

(a) Y is a normed space under a certain norm $|||\cdot|||$ which is stronger than the original norm $\|\cdot\|$ of X;

(b) there exists a real ω such that for each $\lambda > \omega$, $R(\lambda^2 - A)$ contains Y, $R(\lambda^2) = (\lambda^2 - A)^{-1}$ exists, and such that Y is invariant under $R(\lambda^2)$;

(c) there exists a constant $M > 0$ such that

$$\left\| \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda{}^{n-k}(J_\lambda - I)^k x \right\| \leqq M\left(\frac{\lambda}{\lambda - \omega}\right)^n |||x|||$$

for $x \in Y$, $\lambda > \omega$ and $n \in \mathbf{N}$, where $J_\lambda = \lambda^2 R(\lambda^2)$.

Now for the problem (1.7) with $y = 0$ and the problem (1.7) with $x = 0$ and $y = x$ we consider the following two approximate schemes:

$$\begin{cases} \lambda^2(u_{n+1} - 2u_n + u_{n-1}) = Au_{n+1}, & n \in \mathbf{N}, \\ u_0 = x, \quad u_1 = (I - \lambda^{-2}A)^{-1}x, & \lambda > \omega, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \lambda^2(v_{n+1} - 2v_n + v_{n-1}) = Av_{n+1}, & n \in \mathbf{N}, \\ v_0 = 0, \quad v_1 = \lambda^{-1}(I - \lambda^{-2}A)^{-1}x, & \lambda > \omega, \end{cases} \tag{2.8}$$

where $x \in Y$. For $\lambda > \omega$, we put $J_\lambda = (I - \lambda^{-2}A)^{-1}$. Then we can rewrite (2.7) and (2.8) as follows:

$$u_{n+1} = 2J_\lambda u_n - J_\lambda u_{n-1}, \quad v_{n+1} = 2J_\lambda v_n - J_\lambda v_{n-1}, \quad n \in \mathbf{N}. \tag{2.9}$$

LEMMA 2.1. *For $\lambda > \omega$ and $x \in Y$ the following equalities hold*:

$$u_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda^{n-k}(J_\lambda - I)^k x, \quad n \in \mathbf{N}, \tag{2.10}$$

$$v_n = \frac{1}{\lambda} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} J_\lambda^{n-k}(J_\lambda - I)^k x, \quad n \in \mathbf{N}. \tag{2.11}$$

PROOF. It is clear that (2.10) and (2.11) hold for $n=2$. Assume that (2.10) holds for $n \leqq 2m$. Then the application of (2.9) implies

$$u_{2m+1} = 2 \sum_{k=0}^{m} \binom{2m}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k x - \sum_{k=0}^{m-1} \binom{2m-1}{2k} J_\lambda^{2m-k}(J_\lambda - I)^k x.$$

Using $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we have

$$\sum_{k=0}^{m} \binom{2m}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k x - \sum_{k=0}^{m-1} \binom{2m-1}{2k} J_\lambda^{2m-k}(J_\lambda - I)^k x$$

$$= \sum_{k=0}^{m-1} \binom{2m-1}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k x + \sum_{k=1}^{m} \binom{2m-1}{2k-1} J_\lambda^{2m+1-k}(J_\lambda - I)^k x$$

$$- \sum_{k=0}^{m-1} \binom{2m-1}{2k} J_\lambda^{2m-k}(J_\lambda - I)^k x$$

$$= \sum_{k=0}^{m-1} \binom{2m-1}{2k} J_\lambda^{2m-k}(J_\lambda - I)^{k+1} x + \sum_{k=0}^{m-1} \binom{2m-1}{2k+1} J_\lambda^{2m-k}(J_\lambda - I)^{k+1} x$$

$$= \sum_{k=0}^{m-1} \binom{2m}{2k+1} J_\lambda^{2m-k}(J_\lambda - I)^{k+1} x = \sum_{k=1}^{m} \binom{2m}{2k-1} J_\lambda^{2m+1-k}(J_\lambda - I)^k x.$$

Hence we obtain

$$u_{2m+1} = \sum_{k=0}^{m} \binom{2m}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k x + \sum_{k=1}^{m} \binom{2m}{2k-1} J_\lambda^{2m+1-k}(J_\lambda - I)^k x$$

$$= \sum_{k=0}^{m} \binom{2m+1}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k x.$$

Next assume that (2.10) holds for $n \leqq 2m+1$. Then we can derive (2.10) with n replaced by $2m+2$ in the same manner as above.

Next assume that (2.11) holds for $n \leqq 2m$. In a way similar to the proof of (2.10) we have

$$\lambda v_{2m+1} = 2 \sum_{k=0}^{m-1} \binom{2m}{2k+1} J_\lambda^{2m+1-k}(J_\lambda - I)^k$$

$$- \sum_{k=0}^{m-1} \binom{2m-1}{2k+1} J_\lambda^{2m-k}(J_\lambda - I)^k x = \sum_{k=0}^{m} \binom{2m+1}{2k+1} J_\lambda^{2m+1-k}(J_\lambda - I)^k x.$$

We may omit the proof of the rest part. q.e.d.

LEMMA 2.2. *Let A be a closed linear operator in X and let Y be a linear* *manifold of X. Assume that conditions* (a)–(c) *are satisfied. Then we have*

$$\|J_\lambda{}^n\| \leqq 2^{(n-1)^2} M\left(\frac{\lambda}{\lambda-\omega}\right)^n \|\!|x|\!\|, \tag{2.12}$$

that is

$$\|[R(\lambda^2)]^n x\| \leqq 2^{(n-1)^2} M \lambda^{-n}(\lambda-\omega)^{-n} \|\!|x|\!\|. \tag{2.13}$$

PROOF. First we note that $u_1 = J_\lambda x$. Furthermore, we see from Lemma 2.1 that

$$u_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda{}^{n-k}(J_\lambda - I)^k x = \sum_{j=0}^{[n/2]} \sum_{k=j}^{[n/2]} (-1)^j \binom{n}{2k}\binom{k}{j} J_\lambda{}^{n-j} x.$$

Let a_{n-j} be the coefficient of $J_\lambda{}^{n-j}x$, namely

$$a_{n-j} = \sum_{k=j}^{[n/2]} (-1)^j \binom{n}{2k}\binom{k}{j} = (-1)^j \sum_{k=0}^{[n/2]-j} \binom{n}{2k+2j}\binom{k+j}{j}.$$

Using a well known formula, we see that a_{n-j} can be written as

$$a_{n-j} = 2^{n-2j-1}(-1)^j \frac{n}{j}\binom{n-j-1}{j-1}.$$

But since

$$\frac{(n-1)!}{(n-j)!} \geqq \frac{(n-j-1)!}{(n-2j)!},$$

we have

$$\frac{n}{j}\binom{n-j-1}{j-1} = \frac{n(n-j-1)!}{j!(n-2j)!} \leqq \frac{n!}{j!(n-j)!} = \binom{n}{j}.$$

It then follows that

$$|a_{n-j}| \leqq 2^{n-2j-1}\binom{n}{j}.$$

Noting that

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} = 2^{n-1},$$

we obtain

$$2^{n-1} J_\lambda{}^n x = u_n - \sum_{j=1}^{[n/2]} a_{n-j} J_\lambda{}^{n-j} x.$$

We now assume that (2.12) with n replaced by k holds for $1 \leqq k \leqq n-1$. Then we have

$$\|2^{n-1}J_\lambda{}^n\| \leqq M\left(\frac{\lambda}{\lambda-\omega}\right)^n |||x||| + M\sum_{j=1}^{[n/2]} 2^{n-2j-1+(n-j-1)^2}\binom{n}{j}\left(\frac{\lambda}{\lambda-\omega}\right)^{n-j} |||x|||$$

and hence

$$\|J_\lambda{}^n\| \leqq 2^{(n-2)^2-2}M\left(\frac{\lambda}{\lambda-\omega}\right)^n\left\{1+\sum_{j=1}^{[n/2]}\binom{n}{j}\right\} |||x|||$$

$$\leqq 2^{n-3+(n-2)^2}M\left(\frac{\lambda}{\lambda-\omega}\right)^n |||x||| \leqq 2^{(n-1)^2}M\left(\frac{\lambda}{\lambda-\omega}\right)^n |||x||| .$$

q.e.d.

For λ, $\mu>\omega$ and $x\in Y$, we have

$$R(\lambda^2)x - R(\mu^2)x = -(\lambda^2-\mu^2)R(\lambda^2)R(\mu^2)x.$$

Therefore, $R(\lambda^2)x$ is continuous in λ with respect to the norm $\|\cdot\|$. Hence $R(\lambda^2)x$ is continuously differentiable with respect to λ.

In what follows we shall make use of the notations:

$$F_\lambda^n x = (d/d\lambda)^n R(\lambda^2)x, \quad G_\lambda^n x = (d/d\lambda)^n[\lambda R(\lambda^2)x].$$

To see that $F_\lambda^n x$ and $G_\lambda^n x$ are well-defined for $x\in Y$, we prepare the next lemmas.

LEMMA 2.3. *Assume that $R(\lambda^2)x$ is n times continuously differentiable with respect to the norm $\|\cdot\|$. Then the following relations hold*:

$$\lambda F_\lambda^n x + nF_\lambda^{n-1}x = G_\lambda^n x, \tag{2.14}$$

$$\lambda G_\lambda^n x + nG_\lambda^{n-1}x = AF_\lambda^n x, \tag{2.15}$$

$$F_\lambda^n x + 2\lambda nF_\lambda^{n-1}R(\lambda^2)x + n(n-1)F_\lambda^{n-2}R(\lambda^2)x = 0, \tag{2.16}$$

$$G_\lambda^n x + 2\lambda nG_\lambda^{n-1}R(\lambda^2)x + n(n-1)G_\lambda^{n-2}R(\lambda^2)x = 0. \tag{2.17}$$

PROOF. First, we note that $(\lambda^2-A)R(\lambda^2)x=x$. Differentiate both sides n times in λ we have (2.14), (2.15) and

$$\lambda^2F_\lambda^n x + 2\lambda nF_\lambda^{n-1}x + n(n-1)F_\lambda^{n-2}x = AF_\lambda^n x. \tag{2.18}$$

From (2.14) we have

$$\lambda^2G_\lambda^n x + 2\lambda nG_\lambda^{n-1}x + n(n-1)G_\lambda^{n-2}x = \lambda AF_\lambda^n x + nAF_\lambda^{n-1}x = AG_\lambda^n x. \tag{2.19}$$

Therefore, (2.16) and (2.17) are direct consequences of (2.18) and (2.19). q.e.d.

LEMMA 2.4. *Let $R(\lambda^2)x$ be as in Lemma 2.3. Then we have for $\lambda>\omega$ and $x\in Y$*

(2.20) $$G_\lambda^n x = (-1)^n n! \lambda^{-n-1} \sum_{k=0}^{[(n+1)/2]} \binom{n+1}{2k} J_\lambda{}^{n+1-k}(J_\lambda - I)^k x,$$

(2.21) $$F_\lambda^n x = (-1)^n n! \lambda^{-n-2} \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} J_\lambda{}^{n+1-k}(J_\lambda - I)^k x.$$

PROOF. By virtue of Lemma 2.1 it is enough to prove that for $x \in Y$ we have

$$G_\lambda^n x = (-1)^n n! \lambda^{-n-1} u_{n+1} \quad \text{and} \quad F_\lambda^n x = (-1)^n n! \lambda^{-n-1} v_{n+1}.$$

We shall prove them by induction. Using (2.9) and (2.17), we have

$$\begin{aligned} G_\lambda^n x &= -2\lambda n G_\lambda^{n-1} R(\lambda^2) x - n(n-1) G_\lambda^{n-2} R(\lambda^2) x \\ &= (-1)^n n! \lambda^{-n-1} (2J_\lambda u_n - J_\lambda u_{n-1}) = (-1)^n n! \lambda^{-n-1} u_{n+1}. \end{aligned}$$

Relation (2.21) can be derived in a way similar to the proof of (2.20). q. e. d.

Lemmas 2.3 and 2.4 together imply that $F_\lambda^n x$ is differentiable. Therefore, $F_\lambda^n x$ and $G_\lambda^n x$ are well-defined for $x \in Y$. Let A be a closed linear operator in X and consider the differential equation in X

(2.22) $$(d^2/dt^2) u(t) = A u(t), \quad t \in \mathbf{R}.$$

By an *abstract Cauchy problem for A* we mean the following:

ACP. Given an element $x \in X$, find an X-valued function $u(t) = u(t; x, 0)$ defined on $\mathbf{R}$ such that

(i) $u(t)$ is twice continuously differentiable in t,

(ii) for each $t \in \mathbf{R}$, $u(t) \in D(A)$ and $u(t)$ satisfies (2.22), and

(iii) $u(0) = x$, $u'(0) = 0$.

A function $u(t)$ satisfying (i)–(iii) is called a *solution* of *ACP*.

DEFINITION 2.5. Let D be a linear manifold in X such that

(2.23) there is a norm $|||\cdot|||$ under which D is a normed space,

(2.24) there are seminorms $p(\cdot)$ and $q(\cdot)$ on D.

Let $\{U(t);\, t \in \mathbf{R}\}$ be a family of operators on D into $D(A)$ satisfying

(2.25) for every $x \in D$, $u(t) = U(t)x$ is a solution of *ACP*,

(2.26) there exists a positive constant M such that

$$\|U(t)x\| \leqq M e^{\omega|t|} |||x|||, \quad \|U'(t)x\| \leqq M|t| e^{\omega|t|} p(x),$$
$$\text{and} \quad \|AU(t)x\| \leqq M e^{\omega|t|} q(x) \quad \text{for} \quad x \in D, \quad t \in \mathbf{R}.$$

We call $\{U(t);\, t \in \mathbf{R}\}$ a *family of solution operators* of *ACP* on D with type

ω. It follows from condition (ii) that $D \subset D(A)$. Also, by (2.25), the norm $|||\cdot|||$ is stronger than the original norm $\|\cdot\|$ on D.

3. Construction of solution operators

Let A be a closed linear operator in a Banach space and Y be a linear manifold in X satisfying conditions (a)–(c) which have been introduced in the preceding section. We further introduce two linear subspaces of Y:

$$Y_1 = \{x \in Y;\ Ax \in Y\},$$

$$Y_2 = \{x \in Y_1;\ Ax \in Y_1\}.$$

From the relation

$$R(\lambda^2)x - R(\mu^2)x = -(\lambda^2-\mu^2)R(\lambda^2)R(\mu^2)x \qquad \text{for } \lambda, \mu > \omega \text{ and } x \in Y,$$

we see that

$$\lambda^2 R(\lambda^2)x - x = R(\lambda^2)Ax\,(\in Y) \qquad \text{for } x \in Y_1 \text{ and } \lambda > \omega. \tag{3.1}$$

We need the following lemmas:

LEMMA 3.1. *Let A be a closed linear operator in X and let Y be a linear manifold of X satisfying* (a)–(c). *Then we have for $\lambda > \omega$ and $x \in Y$,*

$$\|G_\lambda^n x\| \leqq Mn!(\lambda-\omega)^{-n-1}|||x|||, \tag{3.2}$$

$$\|F_\lambda^n x\| \leqq M(n+1)!(\lambda-\omega)^{-n-2}|||x|||. \tag{3.3}$$

PROOF. By Lemma 2.4, it is clear that (3.2) follows from (c). From this and (2.14), we obtain (3.3) by induction. q.e.d.

LEMMA 3.2. *Let A and Y be as in Lemma* 3.1. *Moreover we assume that Y_1 is dense in Y with respect to the norm $|||\cdot|||$. Then we have for $x \in Y$*

$$\lim_{\lambda\to\infty} \frac{(-1)^n}{n!}\lambda^{n+1}F_\lambda^n x = 0, \tag{3.4}$$

$$\lim_{\lambda\to\infty} \frac{(-1)^n}{n!}\lambda^{n+1}G_\lambda^n x = x. \tag{3.5}$$

PROOF. We prove the above formulae by induction with respect to n. From (3.1) and (3.2), we have

$$\|\lambda^2 R(\lambda^2)x - x\| \leqq \frac{M}{(\lambda-\omega)^2}|||Ax|||.$$

Since Y_1 is dense in Y, we obtain (3.4) and (3.5) in the case of $n=0$. Assume then that (3.4) and (3.5) are valid for $n \leqq k-1$. By (2.15) we have

$$\frac{(-1)^k}{k!} \lambda^{k+1} G_\lambda^k x = \frac{(-1)^{k-1}}{(k-1)!} \lambda^k G_\lambda^{k-1} x + \frac{(-1)^{k-1}}{k!} \lambda^k A F_\lambda^k x .$$

The first term on the right side is convergent to x as $\lambda \to \infty$ by the induction hypotheses. For $x \in Y_1$, we have

$$\left\| \frac{(-1)^k}{k!} \lambda^k F_\lambda^k A x \right\| \leqq M \frac{(k+1)\lambda^k}{(\lambda-\omega)^{k+2}} |||Ax||| .$$

Therefore the second term is covergent to 0. Since Y_1 is dense in Y by assumption, (3.5) is obtained for any $x \in Y$. Next, (2.14) yields

$$\frac{(-1)^k}{k!} \lambda^{k+1} F_\lambda^k x = \frac{(-1)^{k-1}}{(k-1)!} \lambda^{k-1} F_\lambda^{k-1} x + \frac{(-1)^k}{k!} \lambda^k G_\lambda^k x .$$

The first term on the right side is convergent to 0 by the assumption of induction. For $x \in Y$, we have

$$\left\| \frac{(-1)^k}{k!} \lambda^k G_\lambda^k x \right\| \leqq \frac{M\lambda^k}{(\lambda-\omega)^{k+1}} |||x||| .$$

From this (3.4) follows. q.e.d.

For $x \in Y$ and $t \geqq 0$, we set

(3.6) $$C_n(t)x = \frac{(-1)^{n-1}}{(n-1)!} (n/t)^n G_{n/t}^{n-1} x \quad \text{for} \quad t>0, \quad C_n(0)x = x,$$

(3.7) $$S_n(t)x = \frac{(-1)^n}{n!} (n/t)^{n+1} F_{n/t}^n x \quad \text{for} \quad t>0, \quad S_n(0)x = 0,$$

(3.8) $$W_n(t)x = \frac{(-1)^{n+1}}{(n+1)!} (n/t)^{n+2} G_{n/t}^{n+1} x \quad \text{for} \quad t>0, \quad W_n(0)x = x.$$

By virtue of Lemma 3.2, $C_n(t)x$, $S_n(t)x$ and $W_n(t)x$ are continuous in $t \geqq 0$. From (3.2) and (3.3), we have for $n \geqq 2\omega t$, $t \geqq 0$ and $x \in Y$,

(3.9) $$\|C_n(t)x\| \leqq Me^{2\omega t} |||x||| ,$$

(3.10) $$\|S_n(t)x\| \leqq 8M|t|e^{2\omega t} |||x||| ,$$

(3.11) $$\|W_n(t)x\| \leqq 4Me^{2\omega t} |||x||| .$$

Differentiating $C_n(t)x$ and $S_n(t)x$ in t, we see from (2.14) and (2.15) that for $x \in Y_1$

$$(3.12)\qquad (d/dt)C_n(t)x = \frac{(-1)^n}{n!}(n/t)^{n+1}\{nG_{n/t}^{n-1}x+(n/t)G_{n/t}^{n}x\}$$
$$= AS_n(t)x = S_n(t)Ax,$$

$$(3.13)\qquad (d/dt)S_n(t)x = \frac{n+1}{n}\frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+2}G_{n/t}^{n+1}x = \frac{n+1}{n}W_n(t)x\,.$$

Noting that

$$W_n(t)x - C_n(t)x = \frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+1}AF_{n/t}^{n+1}x + \frac{(-1)^n}{n!}(n/t)^nAF_{n/t}^{n}x$$
$$= \frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+1}AF_{n/t}^{n+1}x + \frac{t}{n}AS_n(t)x\,,$$

we have

$$(3.14)\qquad \|W_n(t)x-C_n(t)x\| \leqq \frac{32t^2}{n}Me^{2\omega t}|||Ax||| \qquad \text{for} \quad x\in Y_1\,.$$

LEMMA 3.3. *Let A be a closed linear operator in X and let Y be a linear manifold of X satisfying* (a)–(c). *Moreover we assume that Y_2 is dense in Y with respect to $|||\cdot|||$. Then for any $x\in Y$, $C_n(t)x$ and $S_n(t)x$ defined respectively by* (3.6) *and* (3.7) *both converge as $n\to\infty$ with respect to $\|\cdot\|$. In each case the convergence is uniform with respect to t on bounded intervals of* $[0, \infty)$.

PROOF. Let $\varepsilon>0$ and $x\in Y_1$. It follows from (3.12) and (3.13) that

$$C_m(\varepsilon)C_n(t-\varepsilon)x - C_m(t-\varepsilon)C_n(\varepsilon)x$$
$$+ S_m(\varepsilon)S_n(t-\varepsilon)x - S_m(t-\varepsilon)S_n(\varepsilon)Ax$$
$$= \int_\varepsilon^{t-\varepsilon}\frac{d}{ds}[C_m(t-s)C_n(s)x]ds + \int_\varepsilon^{t-\varepsilon}\frac{d}{ds}[S_m(t-s)S_n(s)Ax]ds$$
$$= \int_\varepsilon^{t-\varepsilon}[C_m(t-s)AS_n(s)x - S_m(t-s)AC_n(s)x]ds$$
$$+ \int_\varepsilon^{t-\varepsilon}\left[\frac{n+1}{n}S_m(t-s)A\,W_n(s)x - \frac{m+1}{m}W_m(t-s)\,AS_n\,(s)x\right]ds,$$

and hence

$$C_n(t)x - C_m(t)x = \int_0^t[C_m(t-s)-W_m(t-s)]S_n(s)Axds$$
$$+ \int_0^t S_m(t-s)[W_n(s)-C_n(s)]Axds$$
$$+\int_0^t\left[\frac{1}{n}S_m(t-s)\,W_n(s) - \frac{1}{m}W_m(t-s)S_n(s)\right]Axds.$$

Therefore (3.14) implies that for each $x \in Y_2$

$$\|C_n(t)x - C_m(t)x\| \leqq \frac{64}{3}\left(\frac{1}{n} + \frac{1}{m}\right) M^2 t^4 e^{2\omega t} |||A^2 x|||$$

$$+16\left(\frac{1}{n} + \frac{1}{m}\right) M^2 t^2 e^{2\omega t} |||Ax|||.$$

Since Y_2 is dense in Y, $C_n(t)x$ converges uniformly for bounded t for each $x \in Y$. Moreover we see from (3.14) that $W_n(t)x$ converges too. Since

$$S_n(t)x = \int_0^t \frac{n+1}{n} W_n(s)xds,$$

$S_n(t)x$ also converges uniformly for bounded t. q.e.d.

For $x \in Y$ and $t \geqq 0$ we define $C(t)x$ and $S(t)x$ as the $\|\cdot\|$-limit of $C_n(t)x$ and $S_n(t)x$, respectively:

$$\text{(3.15)} \qquad C(t)x = \lim_{n\to\infty} C_n(t)x = \lim_{n\to\infty} W_n(t)x,$$

$$\text{(3.16)} \qquad S(t)x = \lim_{n\to\infty} S_n(t)x = \int_0^t C(s)xds.$$

We can extend $C(t)x$ and $S(t)x$ for $t<0$ as follows. For $t \geqq 0$, we set $C(-t)x = C(t)x$ and $S(-t)x = -S(t)x$. We denote the extensions again by $\{C(t)x;\, t \in \mathbf{R}\}$ and $\{S(t)x;\, t \in \mathbf{R}\}$. Then $C(t)x$ and $S(t)x$ satisfy

$$\text{(3.17)} \qquad \|C(t)x\| \leqq Me^{\omega|t|} |||x|||,$$

$$\text{(3.18)} \qquad \|S(t)x\| \leqq M|t|e^{\omega|t|} |||x|||.$$

From (3.12) and (3.13) we have for $x \in Y_1$

$$\text{(3.19)} \qquad C_n(t)x = x + \int_0^t \frac{n+1}{n} (t-s) W_n(s)Axds.$$

Passing to the limit as $n \to \infty$, we obtain

$$\text{(3.20)} \qquad C(t)x = x + \int_0^t (t-s)C(s)Axds$$

and we infer that $C(t)x$ is twice continuously differentiable in t for $x \in Y_1$. Noting that $AC_n(t)x = C_n(t)Ax$ for $x \in Y_1$ and A is closed, we see that $C(t)Y_1 \subset D(A)$. Moreover, we obtain

$$\text{(3.21)} \qquad (d/dt)C(t)x = AS(t)x, \quad (d^2/dt^2)C(t)x = AC(t)x = C(t)Ax,$$

for $x \in Y_1$ and

$$(3.22)\qquad C(0)x = x, \quad x \in Y, \quad C'(0)x = 0, \quad x \in Y_1 .$$

We are now in a position to state the main result of this section.

THEOREM 3.4. *Let A be a closed linear operator in X and let Y be a linear manifold of X satisfying* (a)–(c). *Furthermore we assume that Y_2 is dense in Y with respect to the norm $|||\cdot|||$. Then $\{C(t); t \in \mathbf{R}\}$ has the following properties*:

(i) $\|C(t)x\| \leqq Me^{\omega|t|}|||x|||$ *for* $x \in Y$ *and* $t \in \mathbf{R}$,

(ii) *for each $x \in Y$, $C(t)x$ is $\|\cdot\|$-continuous in $t \in \mathbf{R}$,*

(iii) $C(t)x - x = \int_0^t S(s)Axds$ *for* $x \in Y_1$ *and* $t \in \mathbf{R}$,

(iv) $\lambda R(\lambda^2)x = \int_0^\infty e^{-\lambda t}C(t)xdt$ *for* $\lambda > \omega$ *and* $x \in Y$,

(v) $R(\lambda^2)x = \int_0^\infty e^{-\lambda t}S(t)xdt$ *for* $\lambda > \omega$ *and* $x \in Y$.

PROOF. It remains to prove (iii)–(v). For $x \in Y_1$, we obtain $(d/ds)C(s)x = AS(s)x = S(s)Ax$. Integrating both sides of this equality from $s=0$ to $s=t$, we get (iii). For $x \in Y_1$

$$(d/ds)e^{-\lambda s}S(s)x = -\lambda e^{-\lambda s}S(s)x + e^{-\lambda s}C(s)x.$$

Integrating both sides of this identity from $s=0$ to $s=t$, we have

$$e^{-\lambda t}S(t)x = -\lambda\int_0^t e^{-\lambda s}S(s)xds + \int_0^t e^{-\lambda s}C(s)xds.$$

Since $\|S(t)x\| \leqq Mte^{\omega t}|||x|||$, letting $t \to \infty$ gives

$$(3.23)\qquad \int_0^\infty e^{-\lambda s}S(s)xds = \frac{1}{\lambda}\int_0^\infty e^{-\lambda s}C(s)xds.$$

Similarly, for $x \in Y_1$ and $\lambda > \omega$

$$\begin{aligned}(d/ds)e^{\lambda s}C(s)R(\lambda^2)x &= -\lambda e^{-\lambda s}C(s)R(\lambda^2)x + e^{-\lambda s}S(s)AR(\lambda^2)x \\ &= -\frac{1}{\lambda}e^{-\lambda s}C(s)x - \frac{1}{\lambda}e^{-\lambda s}C(s)R(\lambda^2)Ax + e^{-\lambda s}S(s)AR(\lambda^2)x.\end{aligned}$$

Hence integrating this from $s=0$ to $s=t$ yields

$$\begin{aligned}e^{-\lambda t}C(t)R(\lambda^2)x - R(\lambda^2)x &= -\frac{1}{\lambda}\int_0^t e^{-\lambda s}C(s)xds \\ &- \frac{1}{\lambda}\int_0^t e^{-\lambda s}C(s)R(\lambda^2)Axds + \int_0^t e^{-\lambda s}S(s)R(\lambda^2)Axds.\end{aligned}$$

Assertion (iv) follows from (i) and assertion (v) is deduced from (iv) and (3.23).
q. e. d.

THEOREM 3.5. *Let A, Y, Y_1 and Y_2 be as in Theorem 3.4. Then $\{C(t); t\in\mathbf{R}\}$ is a unique family of solution operators of ACP on Y_1 with type ω.*

PROOF. We have already proved the assertion except the claim for the uniqueness. Let $\{U(t)\}$ be a family of solution operators of ACP on Y_1 with type ω. Then for $x\in Y_1$, we have

$$(d/ds)[e^{-\lambda s}U'(s)x] = -\lambda e^{-\lambda s}U'(s)x + e^{-\lambda s}AU(s)x$$

and hence

$$e^{-\lambda t}U'(t)x = -\lambda\int_0^t e^{-\lambda s}U'(s)xds + A\int_0^t e^{-\lambda s}U(s)xds.$$

Letting $t\to\infty$, we have

$$\lambda\int_0^\infty e^{-\lambda s}U'(s)xds = A\int_0^\infty e^{-\lambda s}U(s)xds.$$

Since the left-hand side can be written as

$$-\lambda x + \lambda^2\int_0^\infty e^{-\lambda s}U(s)xds$$

and $R(\lambda^2-A)\supset Y$, we have

$$\lambda R(\lambda^2)x = \int_0^\infty e^{-\lambda s}U(s)xds.$$

By Theorem 3.4 (iv) there can not exist more than one family of solution operators satisfying conditions (2.25) and (2.26) stated in Definition 2.5. q. e. d.

In the remainder of this section we consider a particular case and establish a second-order analogue of the first order case which was treated by Oharu [9].

Let A be a densely defined and closed linear operator in X and let k be a positive integer. Then we may regard $D(A^k)$ as a Banach space with respect to the norm

$$\|x\|_k = \|x\| + \|Ax\| + \cdots + \|A^k x\|.$$

We write $[D(A^k)]$ for this Banach space.

We consider the following conditions:

(1) there is $\omega\in\mathbf{R}$ such that $\{\lambda^2;\ \lambda>\omega\}\subset\rho(A)$,

(2) there exists a constant $M>0$ such that

$$\|G_\lambda^n x\| \leqq Mn!(\lambda-\omega)^{-n-1}\|x\|_k \qquad \text{for}\quad \lambda>\omega \quad\text{and}\quad x\in D(A^k).$$

Then the pair of A and $[D(A^k)]$ satisfies the conditions (a)–(c). Furthermore, $D(A^n)$ is dense in $[D(A^k)]$ for $n \geqq k$ (see Oharu [9; Lemma 2.7]). Consequently, there is a family $\{C(t);\, t \in \mathbf{R}\}$ of solution operators of ACP on $[D(A^k)]$ with type ω:

$$\|C(t)x\| \leqq Me^{\omega|t|}\|x\|_k \qquad \text{for} \quad x \in D(A^k) \quad \text{and} \quad t \in \mathbf{R}.$$

THEOREM 3.6. *Let A be a densely defined and closed linear operator in X. Assume that the above conditions* (1) *and* (2) *are satisfied. Then for each $x \in D(A^{k+1})$, $C(t)x$ becomes a unique solution of ACP. Moreover, for $x \in D(A^{2k})$, $\{C(t)\}$ has the cosine property*

$$C(t+s)x + C(t-s)x = 2C(t)C(s)x.$$

PROOF. Let $u(t)$ be any solution of ACP with the same initial value x. Putting

$$v(t) = C(t)x - u(t), \quad t \in \mathbf{R},$$

we see that $v(0)=0$, $v'(0)=0$ and $v''(t)=Av(t)$. Now, let $\lambda_0 \in \rho(A)$. Then $R(\lambda_0{}^2;\, A)^k v(t) \in D(A^{k+1})$ and

$$AC(t-s)R(\lambda_0{}^2;\, A)^k v(s) = C(t-s)R(\lambda_0{}^2;\, A)^k Av(s).$$

By integration by parts we have

$$\begin{aligned}\int_0^t AC(t-s)R(\lambda_0{}^2;\, A)^k v(s)ds &= \int_0^t C''(t-s)R(\lambda_0{}^2;\, A)^k v(s)ds \\ &= \int_0^t C'(t-s)R(\lambda_0{}^2;\, A)^k v'(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^t C(t-s)R(\lambda_0{}^2;\, A)^k Av(s)ds &= \int_0^t C(t-s)R(\lambda_0{}^2;\, A)^k v''(s)ds \\ &= R(\lambda_0{}^2;\, A)^k v'(t) + \int_0^t C'(t-s)R(\lambda_0{}^2;\, A)^k v'(s)ds.\end{aligned}$$

Therefore, $v'(t)=0$ for $t \in \mathbf{R}$ and $v(t)$ must be constant. But $v(0)=0$, and $v(t)\equiv 0$.

Next, let $x \in D(A^{2k+1})$. Then $C(t)x \in D(A^{k+1})$. We set for $x \in D(A^{2k+1})$

$$w(t) = C(t+s)x + C(t-s)x - 2C(t)C(s)x.$$

Then, in a way similar to the above argument, we can show that $w(t)\equiv 0$, i.e.,

$$C(t-s)x + C(t-s)x = 2C(t)C(s)x \qquad \text{for} \quad x \in D(A^{2k+1}).$$

Here $C(t)C(s)$ is a bounded linear operator on $[D(A^{2k})]$ to X and so are $C(t+s)$ and $C(t-s)$. Noting that $D(A^{2k+1})$ is dense in $[D(A^{2k})]$ (see Oharu [9; Lemma 2.7]), we obtain the desired assertion. q. e. d.

4. Approximation of cosine families

Let A be a closed and densely defined linear operator in X satisfying

$$\lambda^2 \in \rho(A), \quad \lambda > \omega, \tag{4.1}$$

$$\|(d/d\lambda)^n[\lambda R(\lambda^2; A)]\| \leqq Mn!(\lambda-\omega)^{-n-1}. \tag{4.2}$$

We write $s\text{-}\lim_{n\to\infty} C_n = C$, if $\{C_n\}$ converges to some $C \in B(X)$ in the sense of the strong operator topology. We set for $\lambda > \omega$

$$G_{n,\lambda} = (d/d\lambda)^n[\lambda R(\lambda^2; A)], \tag{4.3}$$

$$F_{n,\lambda} = (d/d\lambda)^n R(\lambda^2; A). \tag{4.4}$$

By Lemma 2.4, we obtain

$$G_{n-1,\lambda}x = (-1)^{n-1}(n-1)!\lambda^{-n}u_n \qquad \text{for} \quad x \in X.$$

Therefore, we have

$$\frac{(-1)^{n-1}}{(n-1)!}\lambda^n G_{n-1,\lambda} = \sum_{k=0}^{[n/2]}\binom{n}{2k} J_\lambda{}^{n-k}(J_\lambda - I)^k, \tag{4.5}$$

$$\frac{(-1)^n}{n!}\lambda^{n+1}F_{n,\lambda} = \frac{1}{\lambda}\sum_{k=0}^{[n/2]}\binom{n+1}{2k+1} J_\lambda{}^{n+1-k}(J_\lambda - I)^k. \tag{4.6}$$

We then set for $n/t > \omega$

$$C_n(t) = \sum_{k=0}^{[n/2]}\binom{n}{2k} J_{n/t}{}^{n-k}(J_{n/t} - I)^k,$$

$$S_n(t) = \frac{t}{n}\sum_{k=0}^{[n/2]}\binom{n+1}{2k+1} J_{n/t}{}^{n+1-k}(J_{n/t} - I)^k.$$

Then, in view of the argument developed in Section 3, one finds two families $\{C(t)\}$ and $\{S(t)\}$ obtained by

$$C(t) = s\text{-}\lim_{n\to\infty} C_n(t) = s\text{-}\lim_{n\to\infty}\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{n}{t}\right)^n G_{n-1,n/t}, \tag{4.7}$$

$$S(t) = s\text{-}\lim_{n\to\infty} S_n(t) = s\text{-}\lim_{n\to\infty}\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} F_{n,n/t}. \tag{4.8}$$

According to Theorem 3.6 (see also [5]; Lemma 2.4), $\{C(t)\}$ has the cosine property. Furthermore, it satisfies the following growth condition:

$$\|C(t)\| \leqq Me^{\omega|t|}.$$

The generator of $\{C(t)\}$ is precisely equal to A. This is seen by employing the method which was established by Kisynski [4] or Sova [12]. Moreover, $\{C(t)\}$ is uniquely determined by A; this is easily seen from the next lemma.

LEMMA 4.1. *Let $C_1(t)$ and $C_2(t)$ be cosine families with generators A_1 and A_2, respectively. Assume that $D(A_1)\subset D(A_2)$. Then we have for $x \in D(A_1)$*

$$C_1(t)x - C_2(t)x = \int_0^t S_2(t-s)(A_1-A_2)C_1(s)xds.$$

PROOF. For $x \in D(A_1)$, we have

$$\begin{aligned}[C_1(t)-C_2(t)]x &= \int_0^t \frac{d}{ds}[C_2(t-s)C_1(s)x]ds + \int_0^t \frac{d}{ds}[S_2(t-s)S_1(s)A_1x]ds \\ &= \int_0^t [C_2(t-s)S_1(s)A_1x - S_2(t-s)A_2C_1(s)x]ds \\ &\quad + \int_0^t [S_2(t-s)C_1(s)A_1x - C_2(t-s)S_1(s)A_1x]ds \\ &= \int_0^t S_2(t-s)(A_1-A_2)C_1(s)xds. \end{aligned}$$ q. e. d.

Consequently, we have obtained another proof for the "if" part of Theorem 1.1. Namely ,we have

THEOREM 4.2. *Let C be a cosine family and let S be the associated sine family. Then we have*

$$C(t) = s\text{-}\lim_{n\to\infty} \sum_{k=0}^{[n/2]} \binom{n}{2k} J_{n/t}{}^{n-k}(J_{n/t}-I)^k, \tag{4.9}$$

$$S(t) = s\text{-}\lim_{n\to\infty} \frac{t}{n} \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} J_{n/t}{}^{n+1-k}(J_{n/t}-I)^k, \tag{4.10}$$

where the convergence is uniform with respect to t in any bounded interval of R.

REMARK 4.3. The representations (4.9) and (4.10) in terms of the resolvent were first obtained by Webb [14]. But he used the representation theorem for strongly continuous groups and required the result of Kisynski [4] in which second order differential equations are converted into first order systems. In [6], Lutz announced (4.5) and proved (4.9) by using a different method. In his proof it is shown that the rate of convergence of (4.9) is $O(1/\sqrt{m})$ for $n=2m$. Moreover, we have the following representation:

$$AF_{n,\lambda} = (d/d\lambda)^n[\lambda^2 R(\lambda^2;\, A)] = \lambda G_{n,\lambda} + nG_{n-1,\lambda}$$
$$= (-1)^n n!\lambda^{-n} \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} J_\lambda{}^{n-k} (J_\lambda - I)^{k+1}.$$

We assume that $\|AF_{n,\lambda}\| \leqq Mn!(\lambda-\omega)^{-n}$ for $\lambda > \omega$, then we obtain $\|G_{n,\lambda}\| \leqq M(n+1)!(\lambda-\omega)^{-n-1}$ for $\lambda > \omega$.

Next we establish another type of approximation formula of a cosine family C. We define for $\lambda > \omega$

$$U_{n,\lambda} = \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda{}^{n-k}(J_\lambda - I)^k, \tag{4.11}$$

$$V_{n,\lambda} = \frac{1}{\lambda} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} J_\lambda{}^{n-k}(J_\lambda - I)^k, \quad V_{0,\lambda} = 0. \tag{4.12}$$

We start with the following

LEMMA 4.4. *Let* $U_{n,\lambda}$ *and* $V_{n,\lambda}$ *be defined by* (4.11) *and* (4.12), *respectively. Then we have the following relations*:

$$U_{n,\lambda} = J_\lambda U_{n-1,\lambda} + \lambda(J_\lambda - I)V_{n-1,\lambda}, \tag{4.13}$$

$$\lambda V_{n,\lambda} = \lambda J_\lambda V_{n-1,\lambda} + J_\lambda U_{n-1,\lambda}, \tag{4.14}$$

$$U_{n,\lambda} = \lambda V_{n,\lambda} - \lambda V_{n-1,\lambda}, \tag{4.15}$$

$$\lambda V_{n,\lambda} = \sum_{k=1}^{n} U_{k,\lambda}. \tag{4.16}$$

PROOF. For $n = 2m$ we have

$$U_{2m,\lambda} = \sum_{k=0}^{m} \binom{2m}{2k} J_\lambda{}^{2m-k}(J_\lambda - I)^k$$
$$= \binom{2m}{0} J_\lambda{}^{2m} + \sum_{k=1}^{m-1} \left[\binom{2m-1}{2k} + \binom{2m-1}{2k-1}\right] J_\lambda{}^{2m-k} (J_\lambda - I)^k$$
$$+ \binom{2m}{2m} J_\lambda{}^m (J_\lambda - I)^m$$
$$= \left[\sum_{k=0}^{m-1} \binom{2m-1}{2k} + \sum_{k=1}^{m} \binom{2m-1}{2k-1}\right] J_\lambda{}^{2m-k}(J_\lambda - I)^k$$
$$= J_\lambda U_{2m-1,\lambda} + \sum_{k=0}^{m-1} \binom{2m-1}{2k-1} J_\lambda{}^{2m-1-k}(J_\lambda - I)^{k+1}$$
$$= J_\lambda U_{2m-1,\lambda} + \lambda(J_\lambda - I)V_{2m-1,\lambda}.$$

For $n = 2m+1$, we have

$$U_{2m+1,\lambda} = \sum_{k=0}^{m}\binom{2m+1}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k$$

$$= \sum_{k=1}^{m}\left[\binom{2m}{2k} + \binom{2m}{2k-1}\right] J_\lambda^{2m+1-k}(J_\lambda - I)^k + \binom{2m+1}{0} J_\lambda^{2m+1}$$

$$= \sum_{k=0}^{m}\binom{2m}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k + \sum_{k=1}^{m}\binom{2m}{2k-1} J_\lambda^{2m+1-k}(J_\lambda - I)^k$$

$$= J_\lambda U_{2m,\lambda} + \sum_{k=0}^{m-1}\binom{2m}{2k+1} J_\lambda^{2m-k}(J_\lambda - I)^{k+1}$$

$$= J_\lambda U_{2m,\lambda} + \lambda(J_\lambda - I)V_{2m,\lambda}.$$

Therefore we obtain (4.13). Similarly, for $n=2m$ we have

$$\lambda V_{2m,\lambda} = \sum_{k=0}^{m-1}\binom{2m}{2k+1} J_\lambda^{2m-k}(J_\lambda - I)^k$$

$$= J_\lambda \sum_{k=0}^{m-1}\binom{2m-1}{2k+1} J_\lambda^{2m-1-k}(J_\lambda - I)^k + J_\lambda \sum_{k=0}^{m-1}\binom{2m-1}{2k} J_\lambda^{2m-1-k}(J_\lambda - I)^k$$

$$= \lambda J_\lambda V_{2m-1,\lambda} + J_\lambda U_{2m-1,\lambda}.$$

For $n=2m+1$, we get

$$\lambda V_{2m+1,\lambda} = \sum_{k=0}^{m}\binom{2m+1}{2k+1} J_\lambda^{2m+1-k}(J_\lambda - I)^k$$

$$= \binom{2m+1}{2m+1} J_\lambda^{m+1}(J_\lambda - I)^m + J_\lambda \sum_{k=0}^{m-1}\binom{2m}{2k+1} J_\lambda^{2m-k}(J_\lambda - I)^k$$

$$+ \sum_{k=0}^{m-1}\binom{2m}{2k} J_\lambda^{2m+1-k}(J_\lambda - I)^k$$

$$= \lambda J_\lambda V_{2m,\lambda} + J_\lambda \sum_{k=0}^{m}\binom{2m}{2k} J_\lambda^{2m-k}(J_\lambda - I)^k = \lambda J_\lambda V_{2m,\lambda} + J_\lambda U_{2m,\lambda}.$$

Therefore we obtain (4.14). Combining (4.13) and (4.14), we obtain

$$U_{n,\lambda} = (J_\lambda U_{n-1,\lambda} + \lambda J_\lambda V_{n-1,\lambda}) - \lambda V_{n-1,\lambda} = \lambda V_{n,\lambda} - \lambda V_{n-1,\lambda}.$$

Thus, we have (4.15). Moreover we have

$$\lambda V_{n,\lambda} = \lambda \sum_{k=0}^{n}(V_{k,\lambda} - V_{k-1,\lambda}) = \sum_{k=0}^{n} U_{k,\lambda}.$$ q.e.d.

We now assume that $U_{n,\lambda}$ satisfies the boundedness condition

(4.17) $$\|U_{n,\lambda}\| \leqq M\left(\frac{\lambda}{\lambda-\omega}\right)^n.$$

Under this assumption (4.16) implies

$$(4.18)\qquad \|V_{n,\lambda}\| \leqq \frac{1}{\lambda} \sum_{k=0}^{n} \|U_{k,\lambda}\| \leqq \frac{M}{\lambda} \sum_{k=0}^{n} \left(\frac{\lambda}{\lambda-\omega}\right)^{k} \leqq \frac{Mn}{\lambda}\left(\frac{\lambda}{\lambda-\omega}\right)^{n}.$$

To formulate an approximation formula for a cosine family, we put for $t \geqq 0$:

$$(4.19)\qquad C_{\lambda}(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} U_{n,\lambda},$$

$$(4.20)\qquad S_{\lambda}(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} V_{n,\lambda}.$$

Then, by (4.17) and (4.18), $C_{\lambda}(t)$ and $S_{\lambda}(t)$ are estimated as

$$\|C_{\lambda}(t)\| \leqq Me^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!}\left(\frac{\lambda}{\lambda-\omega}\right)^{n} = M \exp\left(\frac{\lambda \omega t}{\lambda-\omega}\right),$$

and

$$\|S_{\lambda}(t)\| \leqq Me^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n}}{n!} \frac{n}{\lambda}\left(\frac{\lambda}{\lambda-\omega}\right)^{n} = M \frac{\lambda t}{\lambda-\omega} \exp\left(\frac{\lambda \omega t}{\lambda-\omega}\right).$$

Therefore $C_{\lambda}(t)$ and $S_{\lambda}(t)$ are well-defined. Noting that $\lambda^{2}(J_{\lambda}-I)=AJ_{\lambda}$ and $V_{0,\lambda}=0$, we have by Lemma 4.4

$$(d/dt)C_{\lambda}(t) = -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} U_{n,\lambda} + \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} U_{n,\lambda}$$

$$= \lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} \{U_{n+1,\lambda} - U_{n,\lambda}\}$$

$$= \lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} \{(J_{\lambda}-I)U_{n,\lambda} + \lambda(J_{\lambda}-I)V_{n,\lambda}\}$$

$$= AJ_{\lambda}S_{\lambda}(t) + \lambda(J_{\lambda}-I)C_{\lambda}(t).$$

Moreover, we have

$$(d/dt)S_{\lambda}(t) = -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} V_{n,\lambda} + \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} V_{n,\lambda}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} \{\lambda V_{n+1,\lambda} - \lambda V_{n,\lambda}\}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} \{\lambda(J_{\lambda}-I)V_{n,\lambda} + J_{\lambda}U_{n,\lambda}\} = J_{\lambda}C_{\lambda}(t) + \lambda(J_{\lambda}-I)S_{\lambda}(t).$$

Thus we obtain the relation

$$(d^{2}/dt^{2})C_{\lambda}(t) = AJ_{\lambda}{}^{2}C_{\lambda}(t) + 2\lambda(\lambda_{\lambda}-I)AJ_{\lambda}S_{\lambda}(t) + \lambda^{2}(J_{\lambda}-I)^{2}C_{\lambda}(t).$$

THEOREM 4.5. *Let A be a closed and densely defined linear operator in X*

satisfying (4.1) *and* (4.17). *Then A determines a unique cosine family C which is represented as*

$$C(t) = s\text{-}\lim_{\lambda\to\infty} C_\lambda(t) \qquad \text{for} \quad t\in\mathbf{R}, \tag{4.21}$$

and the convergence is uniform with respect to t in each bounded interval of $\mathbf{R}$.

PROOF. For $x\in D(A)$, we have

$$\|\lambda(J_\lambda - I)x\| = \frac{1}{\lambda}\|AJ_\lambda x\| \leqq \frac{M}{\lambda-\omega}\|Ax\|.$$

We here derive the conclusion in a way similar to the proof of Lemma 3.3. For $x\in D(A)$ we have

$$\begin{aligned}
C_\lambda(t)x - C(t)x &= \int_0^t \frac{d}{ds}[C(t-s)C_\lambda(s)x]ds + \int_0^t \frac{d}{ds}[S(t-s)S_\lambda(s)AJ_\lambda x]ds \\
&= \int_0^t C(t-s)\{AJ_\lambda S_\lambda(s) + \lambda(J_\lambda - I)C_\lambda(s)\}xds - \int_0^t S(t-s)AC_\lambda(s)xds \\
&\quad + \int_0^t S(t-s)\{J_\lambda C_\lambda(s) + \lambda(J_\lambda - I)S_\lambda(s)\}AJ_\lambda xds - \int_0^t C(t-s)AJ_\lambda S_\lambda(s)xds \\
&= \int_0^t S(t-s)(AJ_\lambda{}^2 - A)C_\lambda(s)xds \\
&\quad + \int_0^t [C(t-s)\lambda(J_\lambda - I)C_\lambda(s)x + S(t-s)\lambda(J_\lambda - I)S_\lambda(s)x]ds \\
&= \int_0^t S(t-s)C_\lambda(s)(J_\lambda{}^2 - I)Axds \\
&\quad + \int_0^t [C(t-s)C_\lambda(s) + S(t-s)S_\lambda(s)]\lambda(J_\lambda - I)xds.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|C_\lambda(t)x - C(t)x\| &\leqq M^2e^{\omega t}\int_0^t (t-s)\exp\left(\frac{\omega^2 s}{\lambda-\omega}\right)ds\|(J_\lambda{}^2 - I)Ax\| \\
&\quad + M^2e^{\omega t}\int_0^t\left\{1 + (t-s)\frac{\lambda s}{\lambda-\omega}\right\}\exp\left(\frac{\omega^2 s}{\lambda-\omega}\right)ds\|\lambda^{-1}AJ_\lambda x\|.
\end{aligned}$$

q. e. d.

REMARK 4.6. The family $\{C_\lambda(t)\}$ defined by (4.19) is not a cosine family generated by AJ_λ, but it is closely related to the Yosida-approximation of semi-groups.

We see from (4.5) that (4.17) is equivalent to (4.2) for $\lambda^2\in\rho(A)$. Therefore, (1.6) in Theorem 1.1 may be replaced by (4.17):

THEOREM 4.7. *Let A be a closed and densely defined linear operator in X.*

Then A is the generator of a cosine family C if and only if for all λ with $\lambda > \omega$,

(I) $\lambda^2 \in \rho(A)$,

(II) $$\left\| \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda^{n-k}(J_\lambda - I)^k \right\| \leqq M\left(\frac{\lambda}{\lambda-\omega}\right)^n, \quad n \in \mathbf{N}.$$

REMARK 4.8. We do not know whether (II) is equivalent to the boundedness condition for the powers of J_λ. Assume that for $\lambda > \omega$ and $n \in \mathbf{N}$

$$\|J_\lambda^n\| \leqq \frac{M}{2^{n-1}}\left(\frac{\lambda}{\lambda-\omega}\right)^n \quad \text{and} \quad \|J_\lambda - I\| < 1/2.$$

Then we have

$$\left\| \sum_{k=0}^{[n/2]} \binom{n}{2k} J_\lambda^{n-k}(J_\lambda - I)^k \right\| \leqq \frac{M}{2^{n-1}} \sum_{k=0}^{[n/2]} \binom{n}{2k} \left(\frac{\lambda}{\lambda-\omega}\right)^{n-k}$$
$$\leqq \frac{M}{2^{n-1}} \left(\frac{\lambda}{\lambda-\omega}\right)^n \sum_{k=0}^{[n/2]} \binom{n}{2k} \leqq M\left(\frac{\lambda}{\lambda-\omega}\right)^n.$$

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